CONTROL SYSTEMS 1

Zusammenfassung

Inhalt

Diese Formelsammlung wurde für den Kurs Control Systems 1 im 3. Semester (HS 2016) erstellt. Die Theorie stammt von folgenden Quellen:
- Vorlesungsfolien 2016 von Prof. E. Frazzoli
- Vorlesungsfolien 2015 von Dr. G. Ochsner
- Buch: Analysis and Synthesis of Single-Input Single-Output Control System, L. Guzzella

Da die Theorie in der Reglungstechnik teilweise sehr komplex ist, haben wir in dieser Formelsammlung grossen Wert daraufgelegt, zu jedem Thema Beispiele (grau gefärbt) und Plots zur Veranschaulichung hinzuzufügen.

Für die Vollständigkeit und Korrektheit können wir keine Garantie übernehmen. Falls ihr Fehler findet, bzw. falls es Unklarheiten gibt, könnt ihr uns ein Mail schreiben.

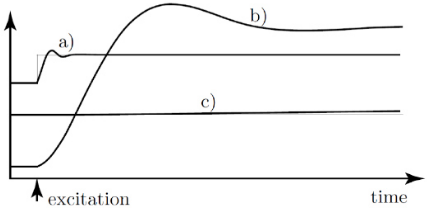
Matthias Wieland; Mario Millhäusler
wielandm@student.ethz.ch; milmario@student.ethz.ch

Control Systems I

Inhalt

General

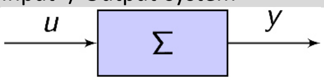
Relevant dynamics



a) Fast/algebraic variables
b) Relevant/dynamic variables
c) Slow/static variables (can be approximated as constnt)

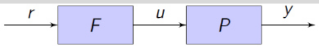
Definitions

Input- / Output-system



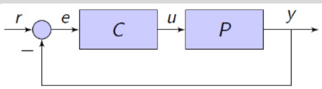
$y = \Sigma \cdot u$
$u = input$
$y = output$

Feed-forward



Relies on e precise knowledge of the plant, and does not change its dynamics.

Feedback



$C = controller$
$P = plant$

Feedback control allows us to:

- Stabilize an unstable System
- Handle uncertainties in the System
- Reject external disturbances

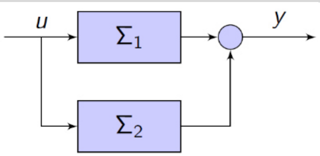
But can also

- Introduce instability, even in an otherwise stable system.
- Feed sensor noise into the system.

Serial interconnection

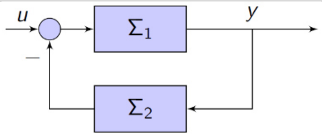
$\Sigma = \Sigma_2 \cdot \Sigma_1$

Parallel interconnection



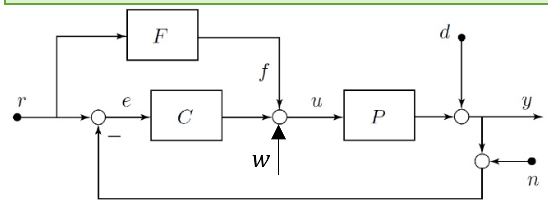
$\Sigma = \Sigma_1 + \Sigma_2$

Feedback interconnection



$$\Sigma = \frac{\Sigma_1}{1 + \Sigma_2 \Sigma_1}$$

Standard Control System



$r = input$
$e = deviation/abweichung$
$u = Stellgrösse$
$d = disturbance$
$n = noise$
$f = feedthrough$
$y = output\ /\ Ausgangsgrösse$
$C = Controller/Regler$
$P = Plant/Strecke$
$w = Steuerfehler$

Two degrees of freedom (feed-forward & feedback) allow better transient behavior for example good tracking of rapidly-changing reference inputs.

Example

Racecar:

- r = velocity trajectory (in general: “Sollwert”)
- u =pedal position α
- y = velocity (in general: “Grösse auf die r geregelt werden soll → nur indirekt beinflussbar)
- C = Driver
- P = Car
- x = $\begin{pmatrix} velocity \\ driving\ force \end{pmatrix}$

Systemeigenschaften

- **Siso** (Single input, single output)
  u,y are one-dimensional
- **Mimo** (Multiple input, multiple output)
  u,y are not one-dimensional
- **Linear**
  x, y, u have maximum exponent 1
  $\Sigma(\alpha \cdot u_1 + \beta \cdot u_2) = \alpha \cdot \Sigma(u_1) + \beta \cdot \Sigma(u_2)$
- **Not linear**
  x can be exponential/quadratic/trigonometric, …
- **Static**
  System has no memory, no derivatives (e.g. $\dot{x}$).
- **Dynamic**
  System has a memory, $\dot{x}$ exists. $\dot{x} = \frac{1}{m} \cdot u$
- **Time-variant**
  Parameter change over time. $\dot{x} = \frac{1}{m(t)} \cdot u$
- **Time-invariant**
  Parameter are constant, independent of time t
- **Order/Dimension of a system**
  Number of state variables in your system. This corresponds to the highest derivative of your ODE or the number of ODE’s. Because one equation of n^{th} order can be rewritten as n equations of 1^{st} order.
- **Causal**
  An input-output system Σ is causal if, for any $t \in \mathbb{T}$,the output at time t depends only on the values of the input on$(-\infty, t]$.
- **Strictly causal**
  An input-output system Σ is strictly causal if, for any $t \in \mathbb{T}$, the output at time t depends only on the values of the input on $(-\infty, t)$.

Examples

$\frac{d}{dt} y(t) = \sin(u(t))$

→ Time-invariant, Dynamic, SISO, not linear

$\Sigma(s) = e^{-sT}$

→ Time-invariant, Dynamic, SISO, linear

$y(t) = 2tu_1(t) + u_2(t)$

→ Time-variant, Static, MIMO, linear

Modeling

We would like to find a model for our plant P, which tells us how the system’s output reacts to a change in the input. This model is used to synthesize the controller C. The model of the plant is not a part of the final control system.

How to model

1. Identify the system boundaries (Systemgrenzen).
2. Identify the relevant reservoirs and the corresponding level variables.
   a. Simplify fast / algebraic variables.
   b. Identify relevant / dynamic variables.
   c. Make slow / static variables constant.
3. Formulate the ODE (Ordinary Differential Equation)
   $$\frac{d}{dt}(reservoir\ content) = \sum inflows - \sum outflows$$
4. Formulate the algebraic relations for the flows between the reservoirs.
5. Identify the system parameters using experiments.
6. Validate the model with experiments other than those used for the identification.

Equilibrium

A system is in Equilibrium if

$$\frac{d}{dt}z(t) = f(z(t), v(t)) = f(z_e, v_e) = 0$$

and

$$w(t) = g(z(t), v(t)) = g(z_e, v_e) = w_e$$

While the pair (z_e, v_e) form an equilibrium of the system.

Normalization

The goal is to replace the physical variables $z(t), v(t)$ and $w(t)$ in the form

$$\frac{d}{dt}z(t) = f(z(t), v(t))$$
$$w(t) = g(z(t), v(t))$$

by the normalized variables $x(t), u(t)$ and $y(t)$, which have a magnitude of ≈ 1. Each variable is normalized by a constant z_0, v_0 and w_0.

$$z_i(t) = z_{i,0} \cdot x_i(t)$$
$$v(t) = v_0 \cdot u(t)$$
$$w(t) = w_0 \cdot y(t)$$

so that

$$\boxed{x_i(t) = \frac{z_i(t)}{z_{i,0}}, u(t) = \frac{v(t)}{v_0}, y(t) = \frac{w(t)}{w_0}}$$

Whereby the normalization for $z(t)$ can be compactly expressed in vector notation:

$$z = T_0 \cdot x\ , T_0 = \begin{pmatrix} z_{1,0} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_{n,0} \end{pmatrix}, z_{i,0} \in \mathbb{R}\backslash\{0\}$$

By inserting this, we get a new set of differential equations:

$$\frac{d}{dt}x(t) = T_0^{-1} \cdot f(T_0 \cdot x(t), v_0 \cdot u(t)) =: f_0(x(t), u(t))$$
$$y(t) = w_0^{-1} \cdot g(T_0 \cdot x(t), v_0 \cdot u(t)) =: g_0(x(t), u(t))$$

$$\boxed{\begin{aligned} \frac{d}{dt}x(t) &= f_0(x(t), u(t)) \\ y(t) &= g_0(x(t), u(t)) \end{aligned}}$$

Linearization

We start with de normalized Differential Equations from above:

$$\begin{cases} \frac{d}{dt}x(t) = f_0(x(t), u(t)) \\ y(t) = g_0(x(t), u(t)) \end{cases}$$

We linearize the system around an equilibrium point (x_e, u_e) which is either given or can be easily calculated.

By neglecting the higher order terms, the linearized system is given by:

$$\begin{cases} \frac{d}{dt}\delta x(t) = A \cdot \delta x(t) + b \cdot \delta u(t) \\ \delta y(t) = c \cdot \delta x(t) + d \cdot \delta u(t) \end{cases}$$

where

$$A = \left.\frac{\partial f_0}{\partial x}\right|_{x=x_e, u=u_e} = \begin{bmatrix} \left.\frac{\partial f_{0,1}}{\partial x_1}\right|_{x=x_e,u=u_e} & \cdots & \left.\frac{\partial f_{0,1}}{\partial x_n}\right|_{x=x_e,u=u_e} \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f_{0,n}}{\partial x_1}\right|_{x=x_e,u=u_e} & \cdots & \left.\frac{\partial f_{0,n}}{\partial x_n}\right|_{x=x_e,u=u_e} \end{bmatrix}$$

$$b = \left.\frac{\partial f_0}{\partial u}\right|_{x=x_e, u=u_e} = \begin{bmatrix} \left.\frac{\partial f_{0,1}}{\partial u}\right|_{x=x_e,u=u_e} \\ \vdots \\ \left.\frac{\partial f_{0,n}}{\partial u}\right|_{x=x_e,u=e} \end{bmatrix}$$

$$c = \left.\frac{\partial g_0}{\partial x}\right|_{x=x_e, u=u_e} = \begin{bmatrix} \left.\frac{\partial g_0}{\partial x_1}\right|_{x=x_e,u=u_e} & \cdots & \left.\frac{\partial g_0}{\partial x_n}\right|_{x=x_e,u=u_e} \end{bmatrix}$$

$$d = \left.\frac{\partial g_0}{\partial u}\right|_{x=x_0, u=u_0} = \begin{bmatrix} \left.\frac{\partial g_0}{\partial u}\right|_{x=x_0,u=u_0} \end{bmatrix}$$

Beispiel Modellieren

We're looking at one single wheel, where a Spring and a Damper are acting. The forces are as follows:

$$\boldsymbol{F_{Spring}(t) = -k(x(t) - l_0)^3, k > 0}$$
$$\boldsymbol{F_{Damper}(t) = -b\left(\frac{d}{dt}x(t)\right), b > 0}$$

a) Determine the State-Space Description of the form

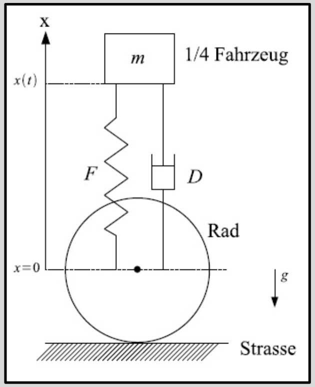
$$\begin{cases} \dot{z}_1 = f_1(z) \\ \dot{z}_2 = f_2(z) \end{cases}$$

For the vector $\boldsymbol{z(t)} = [z_1\ z_2]^T = [x(t)\ \dot{x}(t)]^T$

b) Determine the Equilibrium $\boldsymbol{z_0} = [z_{1,0}\ z_{2,0}]^T$ of the wheel without disturbances.

c) If the Equilibrium of the system is at $\boldsymbol{x = x_0}$, and the current velocity is $|\dot{x}(t)| \leq c_1$, normalize the system for these points in the normalized variables $q_1(t), q_2(t)$, as well as the equilibrium $\boldsymbol{q_0} = [q_{1,0}\ \ q_{2,0}]^T$.

d) Linearize the normalized system around the equilibrium $\boldsymbol{q_0}$ and indicate the A Matrix of the Form $\boldsymbol{\delta\dot{q} = A \cdot \delta q}$.



Solution

With the linear momentum principle, it follows:

$$m\ddot{x} = -g + F_S + F_D \rightarrow \ddot{x} = -g + \frac{F_S + F_D}{m}$$

$$\underline{\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -g - \frac{1}{m}(k(z_1 - l_0)^3 + bz_2) \end{cases}}$$

Equilibrium

$\dot{z}_{1,0} = \dot{z}_{2,0} = 0$

$$z_{2,0} = 0$$
$$z_{1,0} = \sqrt[3]{-\frac{gm}{k}} + l_0$$

Normalization

$$\begin{cases} z_1 = q_1 x_0 \\ z_2 = q_2 c_1 \end{cases} \rightarrow \begin{cases} \dot{q}_1 = \frac{\dot{z}_1}{x_0} \\ \dot{q}_2 = \frac{\dot{z}_2}{c_1} \end{cases} = \begin{cases} \frac{c_1}{x_0}q_2 \\ \frac{1}{c_1}\left(-g - \frac{1}{m}(k(q_1x_0 - l_0)^3 + q_2c_1b)\right) \end{cases}$$

$$\begin{cases} q_{1,0} = \frac{z_{1,0}}{x_0} = 1 \\ q_{2,0} = \frac{z_{2,0}}{c_1} = 0 \end{cases}$$

Linearization

$$A = \left.\frac{\partial f_0}{\partial x}\right|_{x=x_e, u=u_e} = \underline{\underline{\begin{pmatrix} 0 & \frac{c_1}{x_0} \\ -\frac{3x_0k(x_0 - l_0)^2}{mc_1} & -\frac{b}{m} \end{pmatrix}}}$$

State-Space Description

From here on, the prefix δ is omitted and the State-Space Description of a System is defined as

$$\begin{cases} \frac{d}{dt}x(t) = A\cdot x(t) + b\cdot u(t) \\ y(t) = c\cdot x(t) + d\cdot u(t) \end{cases}$$

where

$$\begin{cases} A\in\mathbb{R}^{n\times n} \\ b\in\mathbb{R}^{n\times 1} \\ c\in\mathbb{R}^{1\times n} \\ d\in\mathbb{R}^{1\times 1} \end{cases} \rightarrow \text{"Jacobian Matrices"}$$

A: How does the system affect itself?
b: How does the input affect the System?
c: How does the system affect the output?
d: How does the input affect the output?

Coordinate Transformations

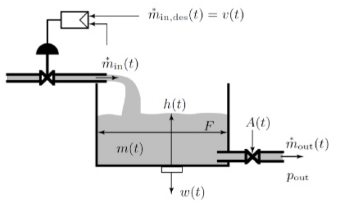
A state-space description can be transformed into another coordinates frame:

$$x = T\cdot\tilde{x}, T\in\mathbb{R}^{n\times n}, \det(T)\neq 0$$

$$\begin{cases} \frac{d}{dt}\tilde{x}(t) = T^{-1}\cdot A\cdot T\cdot\tilde{x}(t) + T^{-1}\cdot b\cdot u(t) \\ y(t) = c\cdot T\cdot\tilde{x}(t) + d\cdot u(t) \end{cases}$$

While the columns of T are the new unit vectors.

Overview



Modeling

$$\frac{d}{dt}z(t) = f(z(t), v(t))$$
$$w(t) = g(z(t), v(t))$$

Normalization

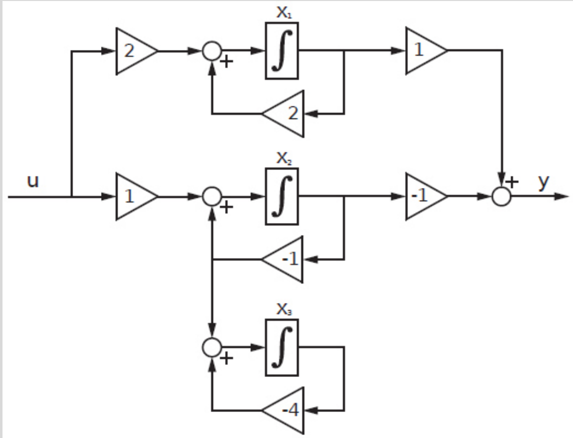
$$\frac{d}{dt}x(t) = f_0(x(t), u(t))$$
$$y(t) = g_0(x(t), u(t))$$

Linearization

$$\frac{d}{dt}x(t) = A\cdot x(t) + b\cdot u(t)$$
$$y(t) = c\cdot x(t) + d\cdot u(t)$$

Example State--Space

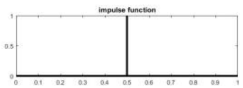
Determine the State-space description of the following model:



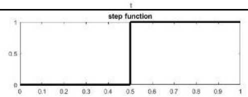
It can be seen directly:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -4 \end{pmatrix};\ b = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$
$$c = (1\ \ -1\ \ 0);\ d = (0)$$

Test Signals

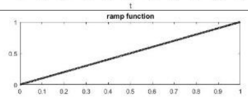


$\delta(t) = impulse\ function$

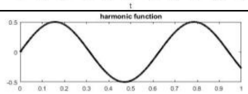


$h(t) = step\ function$

$$h(t) = \begin{cases} 0;\ for\ t < k \\ y_0;\ for\ t \geq k \end{cases}$$

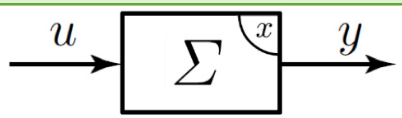


$p(t) = ramp\ function$
$p(t) = t\cdot h(t)$



$c(t) = harmonic\ function$
$c(t) = \cos(\omega t)\cdot h(t)$

First order systems



$$\begin{cases} \frac{d}{dt}x(t) = -\frac{1}{\tau}\cdot x(t) + \frac{k}{\tau}\cdot u(t) \\ y(t) = x(t) \end{cases}$$

where

$$\tau > 0, time\ constant$$
$$k > 0, gain$$

and

$$\Sigma(s) = \frac{Y(s)}{U(s)} = \frac{k}{1+\tau s}$$

where

$$\Sigma(s) = Transfer\ function$$
$$Y(s) = Response$$
$$U(s) = Input$$

Responses of first order systems

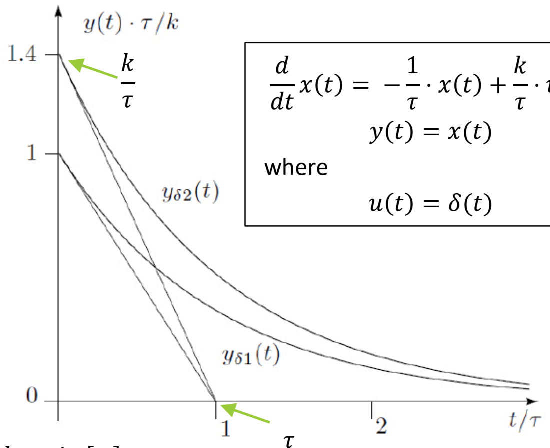
$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\cdot d\tau$$

$$y(t) = Ce^{At}x_0 + C\int_0^t \phi(t-\tau)Bu(\tau)\cdot d\tau + Du(t)$$

Where:

- A,B,C,D are the matrices calculated in “Linearization”

Impulse response



$$\frac{d}{dt}x(t) = -\frac{1}{\tau}\cdot x(t) + \frac{k}{\tau}\cdot u(t)$$
$$y(t) = x(t)$$

where

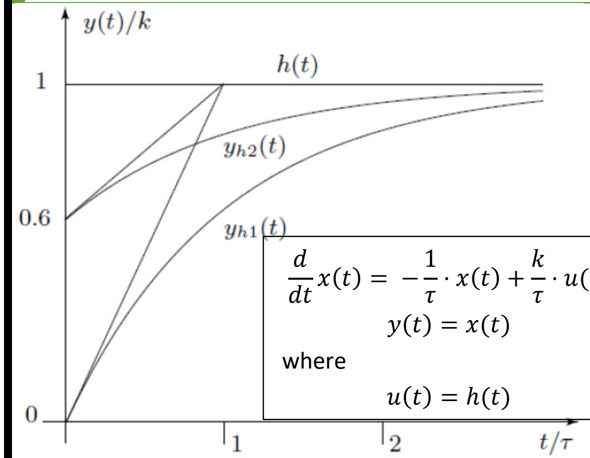
$$u(t) = \delta(t)$$

$k: gain\ [-]$

This leads to the response:

$$y_\delta(t) = e^{-\frac{t}{\tau}}\cdot\left(x_0 + \frac{k}{\tau}\right)$$

Step response



$$\frac{d}{dt}x(t) = -\frac{1}{\tau}\cdot x(t) + \frac{k}{\tau}\cdot u(t)$$
$$y(t) = x(t)$$
where
$$u(t) = h(t)$$

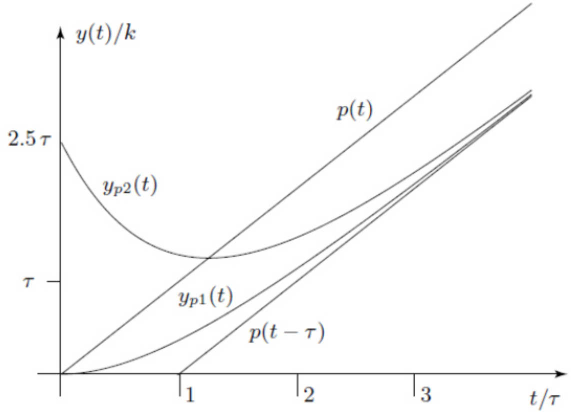
This leads to the response:

$$y_h(t) = e^{-\frac{t}{\tau}} \cdot x_0 + \int_0^t e^{\left(\frac{\rho - t}{\tau}\right)} \cdot \frac{k}{\tau}\, d\rho$$

$$\boxed{y_h(t) = e^{-\frac{t}{\tau}} \cdot x_0 + k \cdot \left(1 - e^{-\frac{t}{\tau}}\right)}$$

- Inclination (Steigung) in $y(0) = k/\tau$

Ramp response



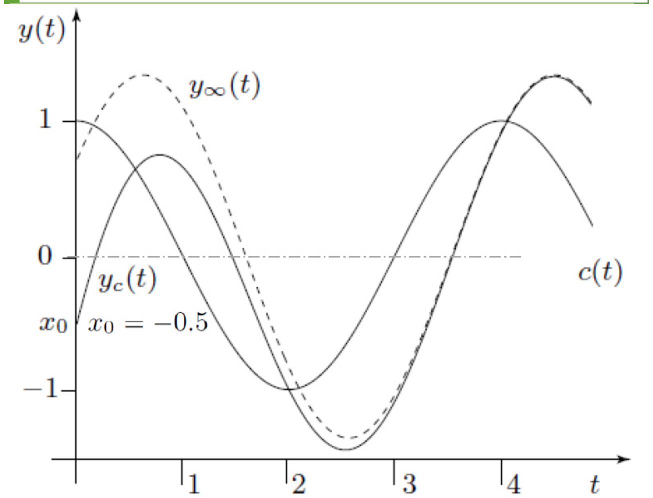
- Periodic vibrations: Input and output have the same frequency

$$\frac{d}{dt}x(t) = -\frac{1}{\tau}\cdot x(t) + \frac{k}{\tau}\cdot u(t)$$
$$y(t) = x(t)$$
where
$$u(t) = p(t)$$

This leads to the response:

$$\boxed{y_p(t) = E^{-\frac{t}{\tau}} \cdot x_o + k \cdot \left(t + \left(e^{-\frac{t}{\tau}} - 1\right) \cdot \tau\right)}$$

Harmonic response



$$\frac{d}{dt}x(t) = -\frac{1}{\tau}\cdot x(t) + \frac{k}{\tau}\cdot u(t)$$
$$y(t) = x(t)$$
where
$$u(t) = c(t)$$

This leads to the respose:

$$\boxed{y_c(t) = e^{-\frac{t}{\tau}} \cdot x_0 + (\cos(\omega \cdot t) + \omega \cdot \tau \cdot \sin(\omega \cdot t) - e^{-\frac{t}{\tau}} \cdot \frac{k}{1 + \omega^2 \cdot \tau^2}}$$

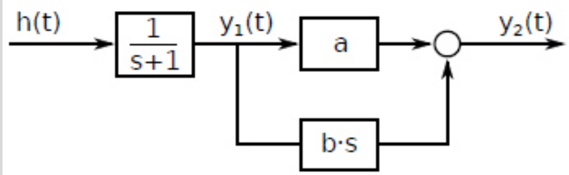
$y_\infty(t) = m(\omega) \cdot \cos\big(\omega t + \varphi(\omega)\big)$

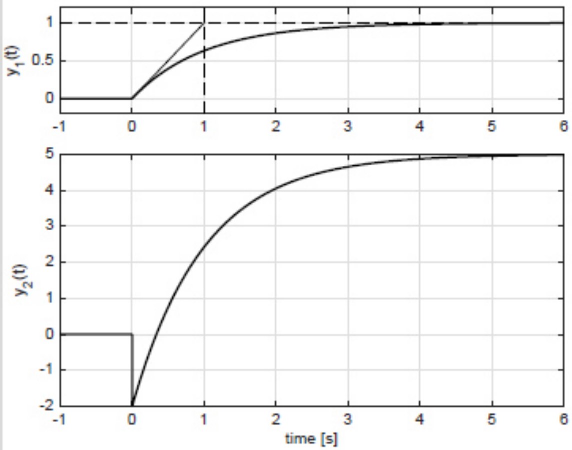
$m(\omega) = \dfrac{k}{\sqrt{1 + \omega^2 \tau^2}}$

$\varphi(\omega) = -\arctan(\omega\tau)$

Example response

Given is the following system





Determine the parameters a and b.

Solution

We can write the system as follows:

$$y_2(t) = a \cdot y_1(t) + b \cdot \dot{y}_1(t)$$

And we can calculate / read in the diagram

$$\lim_{t \to \infty} y_2(t) = a \cdot y_1(t) + b \cdot 0$$
$$5 = a \cdot 1 \to \underline{a = 5}$$
$$\lim_{t \to 0^+} y_2(t) = a \cdot 0 + b \cdot \dot{y}_1(t)$$
$$-2 = b \cdot 1 \to \underline{b = -2}$$

Example step responses

Question:

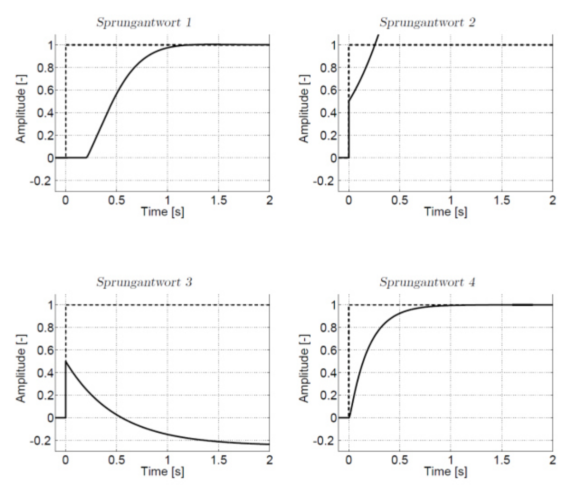
Which transfer function leads to which step response?

a) $L_a(s) = \frac{s-1}{s+5}$

b) $L_b(s) = \frac{s+1}{s-5}$

c) $L_c(s) = \frac{5}{s(0.01s+1)}$

d) $L_d(s) = \frac{5}{s(0.01s+1)} \cdot e^{-0.2s}$



Solution

Tf	$L_a(s)$	$L_b(s)$	$L_c(s)$	$L_d(s)$
Step r.	3	2	4	1

Explanation:

- Differentiate between systems with (converge towards 1) and without Integrators
- Calculate the closed loop transfer function for each system: $T_{abcd}(s) = \frac{L_{abcd}(s)}{1+L_{abcd}(s)}$
- Check whether these T(s) have unstable Poles: $T_a(s)$ has a stable Pole, $T_b(s)$ hasn't
- Check which system has a time delay

Example Impulse response

Question:

Given the following plot of a 2nd order system. Find the transfer function $\Sigma(s)$.

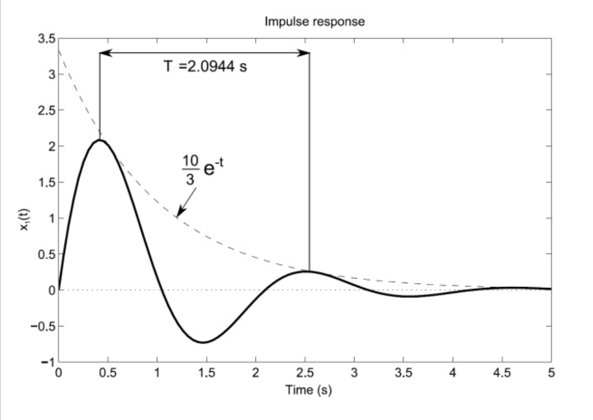


Abbildung 3: Impulsantwort von $\Sigma_2(s)$

Solution:

Describe the plot with the following equation:

$$x(t) = \frac{10}{3} \cdot e^{-t} \cdot \sin(\omega t)$$

Find the Laplace transform:

$$\sin(\omega t) = \frac{\omega}{s^2 + \omega^2}$$

Recall s-shifting:

$$x(t)e^{at} = X(s-a)$$

This gives you the transfer function

$$\frac{10}{3} e^{-t} \sin(\omega t) = \frac{10}{3} \cdot \frac{\omega}{(s+1)^2 + \omega^2}$$

Insert $\omega = \frac{2\pi}{T_0} \approx 3 \frac{rad}{s}$

$$X(s) = \frac{10}{s^2 + 2s + 10}$$

Remark: This exercise is also solvable by pole-reconstruction

Stability

We differentiate between 2 different "Stability-Concepts":

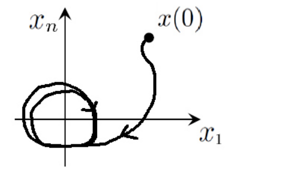
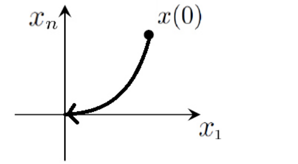
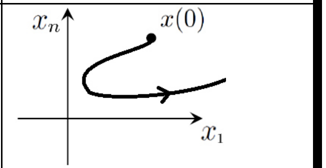
1. Time Domain / State Space Description (Lyapunov):
   a. $x(0) \neq 0 \;\; and \;\; u(t) = 0$
2. Frequency Domain / Input-Output Description (BIBO):
   a. $x(0) = 0 \;\; and \;\; u(t) \neq 0$

Time Domain

Lyapunov Stability

The Lyapunov Stability analyzes the behavior of the state trajectory $\boldsymbol{x(t)}$ around an equilibrium point x_e when $u(t) = 0$ and $x(0) \neq 0$.

We differ the Lyapunov stability in three categories:

Lyapunov stable $if \; \lVert x(t) \rVert < \infty$ $\forall \, t \in [0, \infty]$ $and \; Re(\lambda_i) \leq 0 \; \forall i$ →if only one $Re(\lambda_i) = 0$, or A is diagonalizable it's always stable, otherwise we can't tell	
Asymptotically stable $if \; \lim_{t\to\infty} \lVert x(t) \rVert = 0$ $and \; Re(\lambda_i) < 0 \; \forall i$	
Unstable $if \; \lim_{t\to\infty} \lVert x(t) \rVert = \infty$ $\rightarrow Re(\lambda_i) > 0 \; \forall i$	

The Lyapunov stability can be detected by the following rules:

Eigenvalues of the Linearized System Matrix A: $\lambda_i = \sigma_i + j \cdot \omega_i$	Linearized System	Nonlinear System
All $\sigma_i < 0$	Asymptotically stable	Asymptotically stable
Any $\sigma_i > 0$	Unstable	Unstable
One single $\sigma = 0$ and all other $\sigma_i < 0$	Stable	No statement possible
Two or more $\sigma = 0$ and all other $\sigma_i < 0$	If A diagonalizable: stable Otherwise no statement possible	No statement possible

If all eigenvalues are real, there is no overshoot possible.

- Due to possible Zero Pole Cancellation is the Lyapunov Stability "unsuitable" to analyze frequency responses: The only thing it tells: If a frequency response diverges → it's unstable

Example

A given System has the following eigenvalues:

$\lambda_{1,2} = -2 \pm 3i$

$\lambda_{3,4} = 0$

$\lambda_5 = -10$

$\lambda_6 = -0.1$

What can you say about the stability of the linearized and the nonlinear system?

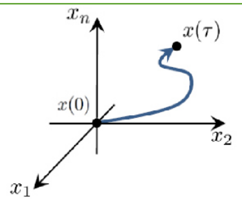
Solution

The linearized system is Lyapunov stable

No statement possible about the nonlinear system

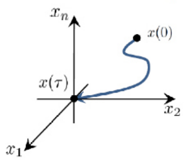
Reachability

In finite time all points in $\mathbb{R}^n$ are reachable from x(0)=0 then system is completely controllable.



Controllability

If all points in $\mathbb{R}^n$ starting from $x(0) = x_e$ can be forced in finite time to 0, then the system is completely controllable.



If zeros and poles can be cancelled out in the transfer function, the system is neither controllable nor observable.

Reachability & Controllability

For LTI (linear time invariant) systems, the reachable and the controllable subspace are identical.

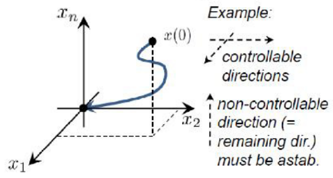
A system {A,b} is completely reachable/controllable iff $R_n = \{b,\ A \cdot b,\ A^2 \cdot b,\ A^3 \cdot b, \dots, A^{n-1} \cdot b\} \in \mathbb{R}^{nxn}$ has full Rank n, where n is also the dimension of the A Matrix

Column vectors of R_n span the reachable subspace → rank of R_n indicates dimension of reachable subspace.

→ All Points in $\mathbb{R}^{nxn}$ can be visited using input u

Stabilizability

An (unstable) system is said to be potentially stabilizable if those state variables that are not controllable are asymptotically stable. To stabilize an unstable system the unstable but controllable state variables must be observable as well



$$rank[\lambda_i \mathbb{I} - A \quad b] = n \quad \forall i \ mit \ Re(\lambda_i) \geq 0$$

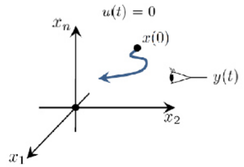
$$rank \begin{bmatrix} \lambda_i \mathbb{I} - A \\ c \end{bmatrix} = n \quad \forall i \ mit \ Re(\lambda_i) \geq 0$$

- No Pole/Zero cancellation with unstable Pole, otherwise not controllable

Observability

A system {A,c} is completely observable iff

$$O_n = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} c \\ c \cdot A \\ \vdots \\ c \cdot A^{n-1} \end{bmatrix}$$

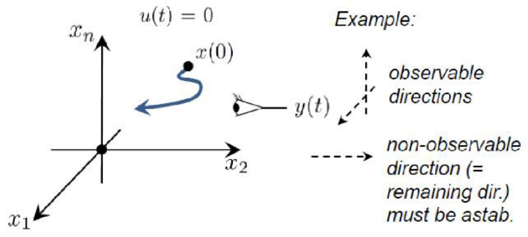


Has full rank or $\det(O_n) \neq 0$

Row vectors of O_n span the observable subspace → rank of O_n indicates dimension of observable subspace

If zeros and poles can be cancelled out in the transfer function, the system is neither controllable nor observable.

Detectability



A system is detectable iff all of its unobservable modes are asymptotically stable.

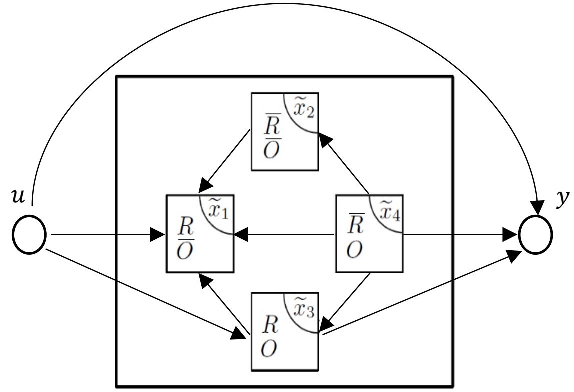
An (unstable) system is stabilizable, if the system is potentially stabilizable and detectable.

In general again, we need a coordinate transformation to find out which state carriable (or modes) are observable/ non-observable/ asymptotically stable/ etc.

State space decomposition

In general, the state space can be subdivided into four subspaces:

1. $RO: controllable \ \& \ observable$



2. $\bar{R}O: not \ controllable \ \& \ observable$
3. $R\bar{O}: controllable \ \& \ not \ observable$
4. $\overline{RO}: not \ controllable \ \& \ not \ observable$

The rules to derive that structure are:

- The input u may only act on the reachable subspaces
- The output y may only be influenced by the observable subspaces

- The unreachable subspaces may not be influenced by a subspace that is influenced by the input u
- The unobservable subspaces may not influence a
- subspace that eventually influences y

Frequency domain

Minimal Realization

$$\Sigma(s) = \frac{b(s)}{a(s)} = \frac{(s-\xi_1)(s-\xi_2)\cdots}{(s-\pi_1)(s-\pi_2)\cdots}$$

Definition:

1. $\min\{Rank(O_n), Rankt(R_n)\} = n$, iff completely reachable and controllable
2. No Zero/Pole cancellation (Remember: if no pole-zero cancellations → the eigenvalues and the poles are the same)

There are ∞ possible ways for the minimal realization (da die Matrizen nicht eindeutig der TF zugeordnet werden können)

To get to the minimal realization state space description:

1. Cancel out the non-reachable and non-controllable part of the matrices
2. If unclear:
   a) Calculate transfer function.
   $\Sigma(s) = \frac{Y(s)}{U(s)} = c(s\mathbb{I} - A)^{-1} \cdot b + d$
   b) Pole-Zero cancellation
   c) Translate to state space description

→ see also p.7: "*I/O or State-Space, FD → State-Space, TD*"

Example minimal realization

Determine the minimal realization of the following system:

$$A = \begin{pmatrix} -2 & 5 & 0 \\ 0 & -4 & 0 \\ 3 & 2 & 1 \end{pmatrix}; \quad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$c = (0.5 \quad 0 \quad 0); \quad d = (0)$$

Solution

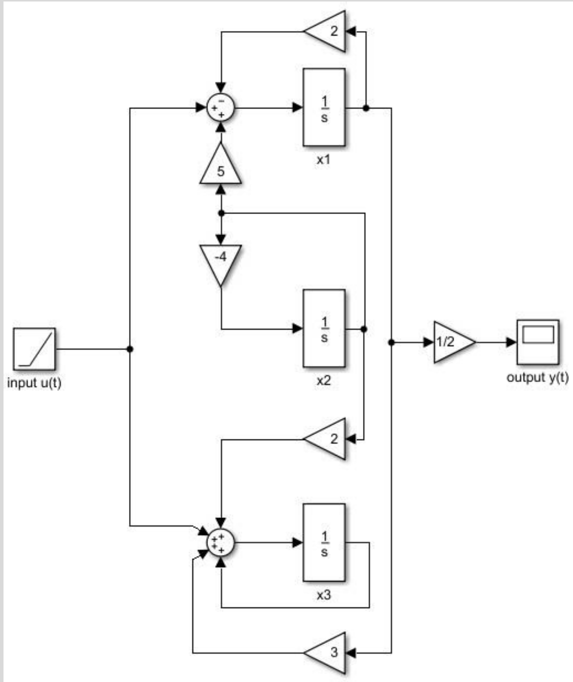
If we draw the Signal-flow graph, we get the following:

The output is only affected by x_1, which itself is affected by $x_1$and x_2, therefore, $x_3$is **not observable.**

The input only affects x_1 and x_3, therefore, x_2 is **not controllable.**

For the minimal realization, we only want the variables, that are **observable and controllable,** therefore $x_2$and x_3 cut out and we get

$$A = -2\,; b = 1; c = 0.5; d = 0$$



BIBO-Stability

BIBO = Bounded Input Bounded Output

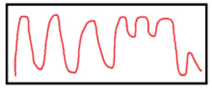
A system is BIBO stable, iff all finite inputs $|u(t)| < M_1$ result in finite outputs $|y(t)| < M_2$

For linear systems, this property is satisfied when

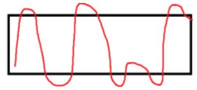
$$\int_0^\infty |\sigma(t)| \cdot dt \; < \infty$$

→the integral must converge to 0

→the real part of all Poles π_i must be negative: $Re(\pi_i) < 0$



Bibo-stable



Not Bibo-stable

→ BIBO stable systems are not affected by Zero Pole Cancellations (unlike Lyapunov): $\Sigma(s) = \frac{s-1}{(s-1)(s+2)} = \frac{1}{s+2}$

Examples

$$Y(s) = \left(\frac{1}{s} - \frac{13}{9} \cdot \frac{1}{s+8} + \frac{1}{s^2}\right) U(s)$$

Not BIBO-stable (two poles at 0)

$$y(t) = 3e^{-\alpha t} \cdot u(t); \quad \alpha, t > 0$$

BIBO-stable ($u(t) = finite \rightarrow y(t) = finite$)

$$y(t) = u(t) \cdot \cos(\omega t)$$

BIBO-stable ($u(t) = finite \rightarrow y(t) = finite$)

$$y(t) = \frac{12}{u^2(t)}$$

Not BIBO-stable ($u(t) = 0 \rightarrow y(t) = \infty$)

Comparison: BIBO / Lyapunov

For a system in **minimal realization** (= completely controllable an observable) holds:

- Lyapunov asymptotically stable ←→ BIBO stable
- Lyapunov stable → Not BIBO stable
- Lyapunov unstable → Not BIBO stable

For a system in **with uncontrollable or unobservable modes** holds:

- Lyapunov asymptotically stable → BIBO stable
- Lyapunov stable → ?
- Lyapunov unstable → ?
- BIBO stable → ?
- Not BIBO stable → Lyapunov stable or unstable

Input/Output System Description

The state-space contains a lot of information about the *internal* behavior of a system. These informations are often not required, so we can use a simpler model description, the input/output-description:

$$y^{(n)}(t) + a_{n-1} \cdot y^{(n-1)}(t) + \cdots + a_1 \cdot y^{(1)}(t) + a_0 \cdot y(t) = b_m \cdot u^{(m)}(t) + \cdots + b_1 \cdot u^{(1)}(t) + b_0 \cdot u(t)$$

Where the initial conditions are chosen to be zero:

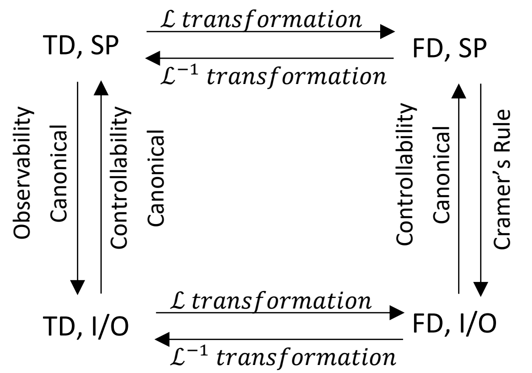
$$y(0) = y^{(1)}(0) = \cdots = 0$$

- The order n of the I/O-description corresponds to the number of observable and controllable state variables of the SP description
- If $m = n$, the system has a feedthrough ($d \neq 0$).

Domains

	Time-Domain (TD)	Frequency-Domain (FD)
SP	$\frac{d}{dx}x(t) = A \cdot x(t) + b \cdot u(t)$ $y(t) = c \cdot x(t) + d \cdot u(t)$	$Y(s) = (c(s\mathbb{I} - A)^{-1} \cdot b + d) \cdot U(s)$
I/O	$y^{(n)}(t) + \cdots + a_1 \cdot y^{(1)}(t) + a_0 \cdot y(t) =$ $b_m \cdot u^{(m)}(t) + \cdots + b_1 \cdot u^{(1)}(t) + b_0 \cdot u(t)$	$Y(s) = \frac{b_m s^m + \cdots + b_0}{s^n + \cdots + a_1 s^1 + a_0} U(s)$

Conversions



State-Space, Time-Domain → State-Space, Freq.-Domain

$$\Sigma(s) = \frac{Y(s)}{U(s)} = c(s\mathbb{I} - A)^{-1} \cdot b + d = c \cdot \frac{adj(s\mathbb{I} - A)}{det(s\mathbb{I} - A)} \cdot b + d$$

In case $\{A, b, c, d\}$ are just scalar Numbers: $\Sigma(s) = \frac{c \cdot b}{s - A} + d$

Where

$\Sigma(s) = Transfer\ function$

$U(s) = frequency\ input$

$Y(s) = frequency\ response$

And

$$adj\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$adj\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} det\begin{pmatrix} e & f \\ h & i \end{pmatrix} & -det\begin{pmatrix} d & f \\ g & i \end{pmatrix} & det\begin{pmatrix} d & e \\ g & h \end{pmatrix} \\ -det\begin{pmatrix} b & c \\ h & i \end{pmatrix} & det\begin{pmatrix} a & c \\ g & i \end{pmatrix} & -det\begin{pmatrix} a & b \\ g & h \end{pmatrix} \\ det\begin{pmatrix} b & c \\ e & f \end{pmatrix} & -det\begin{pmatrix} a & c \\ d & f \end{pmatrix} & det\begin{pmatrix} a & b \\ d & e \end{pmatrix} \end{pmatrix}^T$$

Input/output or State-Space, FD → State-Space, TD

With the given Frequency Domain in the form

$$Y(s) = \frac{b_m s^m + \cdots + b_0}{s^n + \cdots + a_1 s^1 + a_0} U(s)$$

We can calculate the State-Space Description as follows with the **Controller Canonical Form**:

<u>Fall $m < n$</u>

$$\begin{cases} A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \\ c = [b_0 \quad \cdots \quad b_m \quad 0] \quad d = [0] \end{cases}$$

If necessary, fill up c with 0's to match the dim.

<u>Fall $m = n$</u>

$$\begin{cases} A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \\ c = [(b_0 - b_n \cdot a_0) \quad \cdots \quad (b_{n-1} - b_n \cdot a_{n-1})] \quad d = [b_n] \end{cases}$$

The calculated system is a *minimal realization* and can be written in one matrix:

$$\left(\begin{array}{c|c} A & b \\ c & d \end{array}\right)$$

Example

Transfer function: $g(s) = \frac{2s+3}{s^2+s+3}$

$$A_{min} = \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix}; b_{min} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; c_{min} = [3 \quad 2]; d = 0$$

A minimal realization of this system is:

$$\dot{x}(t) = A_{min}x(t) + b_{min}u(t)$$
$$y(t) = c_{min}x(t)$$

Example Domains (FS 2008)

From the plant $P(s)$ we know all the Poles and Zeros:

$$\begin{cases} \pi_1 = -\frac{1}{2} \\ \pi_{2,3} = -1 \pm i \\ \zeta_1 = \frac{1}{2} \end{cases}$$

The static gain is given by 2.

a) Determine the Transfer function $P(s)$.

b) Determine the ODE of $P(s)$.

c) Determine the state-space description of $P(s)$.

Solution

The Poles lead to the function $P_1(s)$

$$P_1(s) = \frac{k}{\left(s + \frac{1}{2}\right)(s + 1 + i)(s + 1 - i)}$$

The zero leads to the function $P_2(s)$

$$P_2(s) = \left(s - \frac{1}{2}\right)$$

The solution is therefore given by

$$P(s) = P_1(s)P_2(s) = \frac{k \cdot \left(s - \frac{1}{2}\right)}{\left(s + \frac{1}{2}\right) \cdot (s + 1 + i)(s + 1 - i)}$$

$$P(0) = 2 \rightarrow k = -4$$

$$P(s) = \frac{-4s + 2}{(s^2 + 2s + 2)\left(s + \frac{1}{2}\right)}$$

For the ODE, we expand

$$P(s) = \frac{-4s + 2}{s^3 + \frac{5}{2}s^2 + 3s + 1} = \frac{b_m s^m + \cdots + b_0}{s^n + \cdots + a_1 s^1 + a_0}$$

The ODE is therefore given by

$$y'''(t) + \frac{5}{2}y''(t) + 3y'(t) + y(t) = -4u'(t) + 2u(t)$$

The State-space description can be directly written as

$$\begin{cases} \frac{d}{dx}x(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2.5 \end{pmatrix} \cdot x(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot u(t) \\ y(t) = (2 \quad -4 \quad 0) \cdot x(t) + (0) \cdot u(t) \end{cases}$$

Laplace Transform (Frequency Domain)

In the time domain the output of a system is calculated by the *convolution* of the input signal with de response of the system. A concatenation (Verkettung) of systems is extremely difficult to analyze, so we convert it into the frequency domain by the *Laplace transform*.

$$\mathcal{L}(x(t)) = X(s) = \int_0^\infty e^{-st} \cdot x(t)dt$$

And the inverse Laplace transform

$$\mathcal{L}^{-1}(X(s)) = \frac{1}{2\pi j} \cdot \oint X(s) \cdot e^{st}\, ds, t \geq 0$$

Properties

- $\mathcal{L}(\alpha f(t) + \beta(g(t)) = \alpha \cdot \mathcal{L}(f(t)) + \beta \cdot \mathcal{L}(g(t)) = \alpha \cdot F(s) + \beta \cdot G(s)$
- $\mathcal{L}(e^{at} f(t)) = F(s-a)$

T-shifting

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}\,\mathcal{L}(f(t))$$
$$u(t-a)f(t-a) = \mathcal{L}^{-1}\left(e^{-as}\mathcal{L}(f(t))\right)$$

S-shifting

$$\mathcal{L}(e^{at} f(t))(s) = F(s-a)$$

→ s-shifting corresponds to damping in that formulas

Calculate $F(s)$, then replace s with $(s-a)$.

$$\mathcal{L}(f(t-a))(s) = e^{-as} \cdot F(s)$$

→ s-shifting corresponds to a delay in that formula

T- and S-shifting

$$\mathcal{L}\left[e^{a(t-b)} f(t-b) u(t-b)\right] = e^{-bs} F(s-a)$$

Derivation t

$$\mathcal{L}\left(f^{(n)}(t)\right) = s^n \mathcal{L}(f(t)) - \sum_{j=0}^{n-1} s^{n-1-j} f^{(j)}(0), n \geq 1$$

For example:

- $\mathcal{L}(f'(t)) = s \cdot \mathcal{L}(f(t)) - f(0)$
- $\mathcal{L}(f''(t)) = s^2 \cdot \mathcal{L}(f(t)) - s \cdot f(0) - f'(0)$

Derivation s

$$\mathcal{L}(t \cdot f(t)) = -\frac{d}{ds} F(s)$$

Integration t

$$\mathcal{L}\left(\int_0^t f(x)\, dx\right) = \frac{1}{s} \mathcal{L}(f(t)), t > 0, s > 0$$

Integration s

$$\mathcal{L}\left(\frac{1}{t} \cdot f(t)\right) = \int_s^\infty F(\tau) d\tau$$

Convolution

$$f(t) * g(t) = \int_0^t f(\tau) \cdot g(t-\tau)\, d\tau = F(s) \cdot G(s)$$
$$\mathcal{L}(f(t) \cdot g(t)) = F(s) * G(s)$$
$$\mathcal{L}(f(t) * g(t)) = F(s) \cdot G(s)$$

Similarity

$$\mathcal{L}\left(\frac{1}{a} \cdot f\left(\frac{t}{a}\right)\right) = F(s \cdot a)$$

Known Laplace Transforms

$x(t)$	$X(s)$
$\boldsymbol{impulse: \delta(t)}$	1
$\boldsymbol{step: h(t)}$	$\frac{1}{s}$
$\boldsymbol{h(t) \cdot t^n}$	$\frac{n!}{s^{n+1}}$
$\boldsymbol{h(t) \cdot e^{at}}$	$\frac{1}{s-a}$
$\boldsymbol{h(t) \cdot t^n \cdot e^{a \cdot t}}$	$\frac{n!}{(s-a)^{n+1}}$
$\boldsymbol{h(t) \cdot \sin(\omega \cdot t)}$	$\frac{\omega}{s^2+\omega^2}$
$\boldsymbol{h(t) \cdot \cos(\omega \cdot t)}$	$\frac{s}{s^2+\omega^2}$
$\boldsymbol{h(t) \cdot \sinh(\omega \cdot t)}$	$\frac{\omega}{s^2-\omega^2}$
$\boldsymbol{h(t) \cdot \cosh(\omega \cdot t)}$	$\frac{s}{s^2-\omega^2}$
$\boldsymbol{k \cdot u(t-a)}$	$k \cdot \frac{e^{-as}}{s}, a > 0$
$\boldsymbol{\delta(t-a)}$	$e^{-as}, a > 0$

Initial and final value theorem

Initial value theorem:

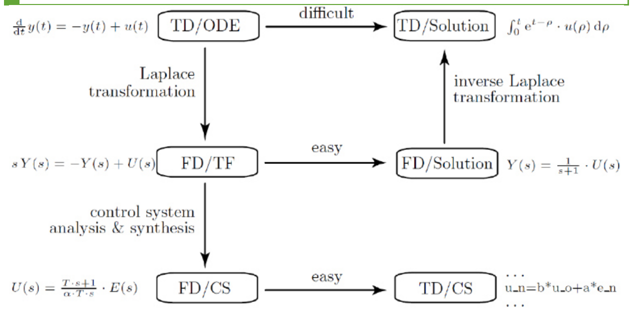
$$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} s \cdot X(s)$$

Final value theorem:

$$\lim_{x \to \infty} x(t) = \lim_{s \to 0^+} s \cdot X(s)$$

These two theorems only hold, if $X(s)$ is a stable function.

Overview



Analysis of linear Systems in the frequency domain

Since the transfer function Σ is given by

$$\Sigma(s) = \frac{b_m s^m + \dots + b_0}{s^n + \dots + a_1 s^1 + a_0}$$

We can make several statements about the system:

Poles

The poles of the transfer function of a system define its impulse response in the time domain and thereby its dynamics.

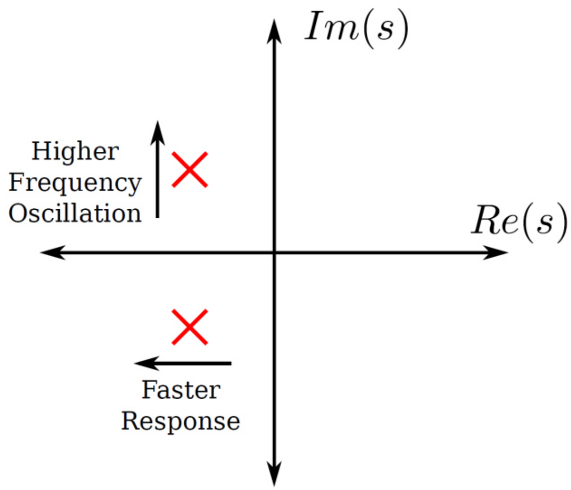
$$\Sigma(s) = \frac{\xi(s)}{(s-\pi_1)^{\phi_1} \cdot (s-\pi_2)^{\phi_2} \cdot \ldots \cdot (s-\pi_n)^{\phi_n}}$$

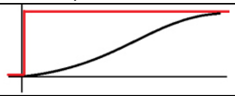
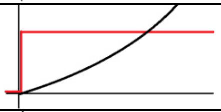
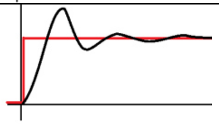
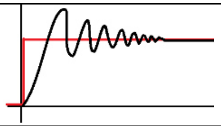
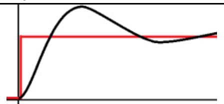
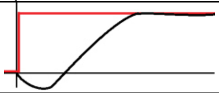
where

$\pi_i = Poles\ of\ the\ transfer\ function$

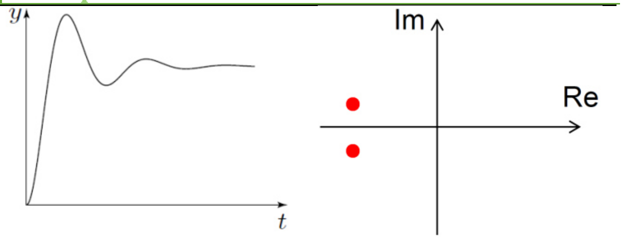
$\phi_i = order\ of\ the\ pole$

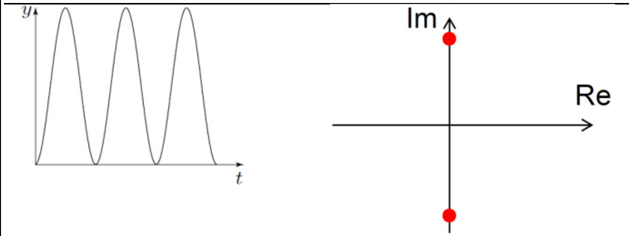
General rule:

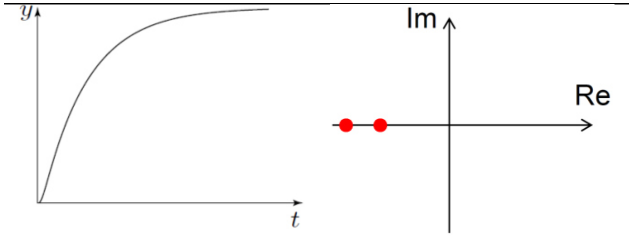


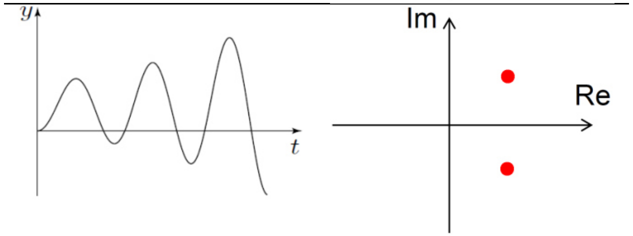
Descr.	Criteria	Consequence	If near 0
Pole stable	$Re(Pole) < 0$		Slow increase
Pole unstable	$Re(Pole) > 0$		Slow diversion
Pole complex	$Im(Pole) \neq 0$		
Pole big complex	$\lvert Im(Pole) \rvert$ $bigger$		
Zero minimal phase	$Zero < 0$		Big overshoot
Zero not minimal phase	$Zero > 0$		Big undershoot

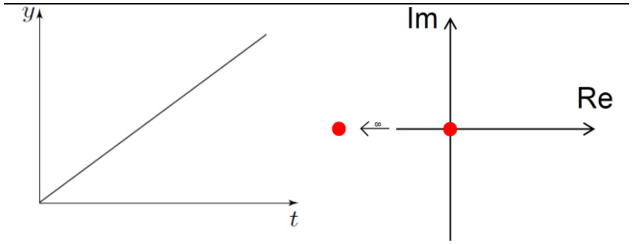
Examples

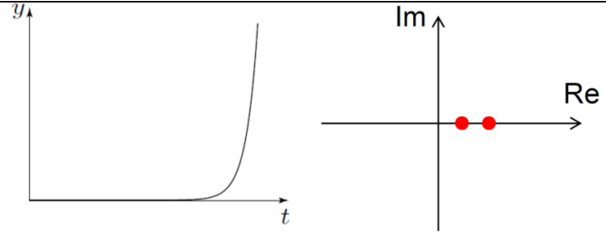












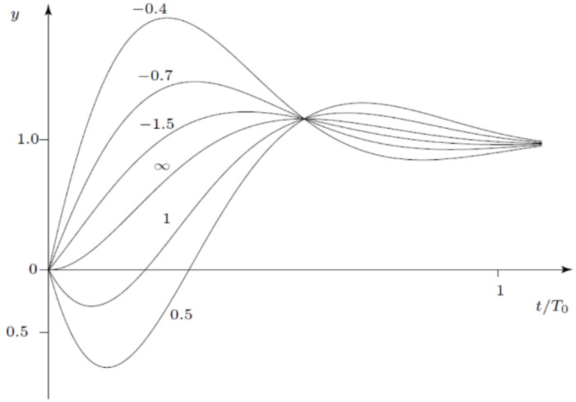
Zeros

The zeros of the transfer function of a system define the dynamics yielding an output of zero

- A zero close to a pole can reduce its influence or lead to over- or undershoot.
- If a zero and a pole happen to be at the same place, they reduce themselves.
- The closer to the origin, the more important is the influence of the zero. This influence manifests itself in an increasing overshoot of the step response.
- A nonminimumphase zero ($Re(\zeta) > 0$) poses an important limitation on feedback control.
- In contrast to system instability, nonminimumphase zeros can often be shifted by a different sensor configuration.

Minimumphase zeros

The following image shows the influence of a zero ζ_i on a step response $h(t)$.



This system is a nonminimumphase system for the zeros $\zeta = \{0.5, 1\}$ and is a minimumphase system for the zeros $\zeta = \{-0.4; -0.7; -1.5; \infty\}$.

> A System is a nonminimumphase system iff there exists at least one zero with a positive real part $Re(\zeta) > 0$. A nonminimumphase system "lies". The response initially goes in the wrong direction (=Undershoot).

Root Locus

Root locus analysis is a graphical method for examining how the roots change with variation of (mostly) the gain k. The question is: What happens to the System when the gain k goes to ∞ ?

To describe the stability of the closed-loop system T(s), we draw the Root Locus of the open-loop system L(s).

Analysis:

- $Re(closed\ loop\ poles) < 0$ → stable
- $Re(closed\ loop\ poles) > 0$ → unstable
- $Re(2\ equal\ closed\ l.poles\ ) = 0$ → unstable
- $Re(2\ distict\ closed\ l.poles) = 0$ → stable

Sketching Rules

1. Root loci start at poles → go to zeros
2. There are n lines (loci) where n is the degree of Poles or Zeros (whichever is greater).
3. As k increases from 0 to ∞, the roots move from the poles of $G(s)$ to the zeros of $G(s)$.
4. When roots are complex, they occur in conjugate pairs.
5. At no time will the same root cross over its path.
6. The portion (Anteil) of the real axis to the left of an odd number of open loop poles and zeros are part of the loci. → **Roots are always sketched from the right to the left.**
7. Lines leave and enter the real axis at 90°.
8. If there are not enough poles or zeros to make a pair, then the extra lines go to / come from infinity.
9. Lines go to infinity along asymptotes.
10. If there are at least two lines to infinity, then the sum of all roots is constant.
11. K going from 0 to $-\infty$ can be drawn by reversing rule 5 and adding 180° to the asymptote angles.

Asymptotes

Contact point / Centroid of asymptotes

$$S_{com} = \frac{\sum x_{Poles} - \sum x_{Zeros}}{\#Poles - \#Zeros}$$

$$x_i \rightarrow Coordinates\ on\ the\ Real\ axis$$

Angle of asymptotes

$$\alpha_n = \frac{2n+1}{\#Poles - \#Zeros} \cdot 180°$$

$$n = \{0; 1; \dots; (\#Poles - \#Zeros - 1)\}$$

Example 1

$$P(s) = \frac{s+6}{(s+4)(s+2)(s+5)(s+7)}$$

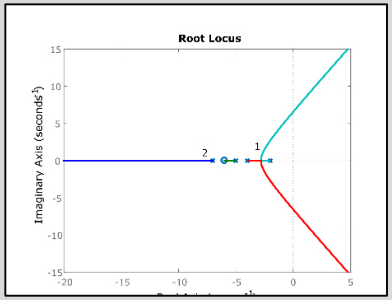
1. Draw all the Poles and Zeros
2. Connect the Points from right to left
3. In Point 1, two Poles are connected, which means that they "collide" and leave the Real axis at 90°.
4. Since the Pole at Point 2 has no "partner" the line goes to infinity.
5. $S_{com} = \frac{(-4-2-5-7)-(-6)}{4-1} = -4$

$$\alpha_0 = \frac{1}{3} \cdot 180° = 60°$$

$$\alpha_1 = \frac{3}{3} \cdot 180° = 180°$$

$$\alpha_2 = \frac{5}{3} \cdot 180° = 300° = -60°$$

The root of the asymptotes is at the point .4 and then goes into three different directions (-60°, 60°, 180°).

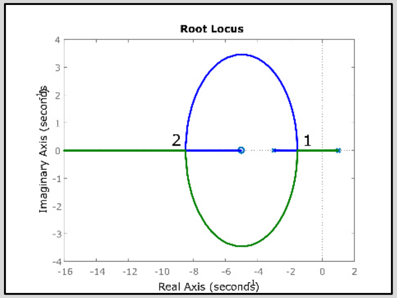


Example 2

$$P(s) = \frac{s+5}{(s-1)(s+3)}$$

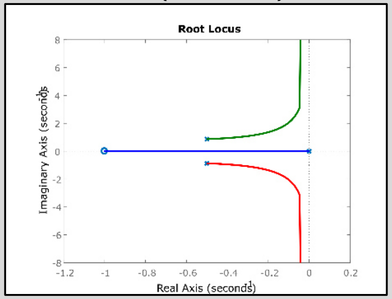
1. Draw all the Poles and Zeros
2. Connect the points from right to left
3. In Point 1, two Poles "collide" and leave the Real axis 90°.
4. $S_{com} = \frac{(1-3)-(-5)}{2-1} = 3$

   $\alpha_0 = \frac{1}{1} \cdot 180° = 180°$
5. The one asymptote must go to infinity, and the other pole must be connected to the zero. As they cannot cross or touch each other, the two lines move in a circle as shown in the picture.

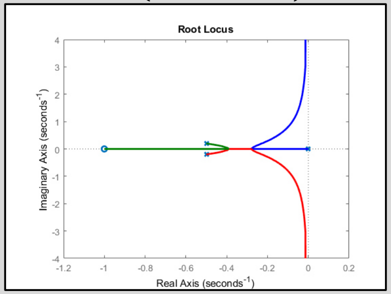


Example 3

$$P(s) = \frac{s+1}{(s^2+s+1) \cdot s}$$



$$P(s) = \frac{s+1}{(s^2+s+0.29) \cdot s}$$



These Plots are too complex to draw by hand. We see, with a small deviation of one pole, the entire plot changes. These plots should just be plotted in Matlab, as there are too many uncertainties to draw them by hand.

BIBO Stability

See BIBO-Stability on page 7.

Frequency Responses

If a asymptotically stable system has the input $u(t) = \cos(\omega t)$, the output converges to a stationary solution:

$$y_\infty(t) = |\Sigma(j\omega)| \cdot \cos(\omega t + \angle\Sigma(j\omega))$$

By substituting $\{s \rightarrow j\omega\}$, the homogenous part of the output will be eliminated and the particular (oscillating) part remains. The same goes for unstable systems (where the homogenous part goes to infinity).

Magnitude

$A, B, C \in \mathbb{C}$

$$\left|A \cdot \frac{B}{C}\right| = |A| \cdot \frac{|B|}{|C|}$$

$$\left|\frac{(a+jb)^x}{(c+jd)^y}\right| = \frac{\left(\sqrt{a^2+b^2}\right)^x}{\left(\sqrt{c^2+d^2}\right)^y}$$

$$\left|e^{-j\omega \cdot T}\right| = |\cos(\omega \cdot T) - j \cdot \sin(\omega \cdot T)| = 1$$

Phase

$$\angle\left(A \cdot \frac{B}{C}\right) = \angle A + \angle B - \angle C$$

$$\angle\left(\frac{(a+jb)^q}{(c+jd)^k}\right) = q \cdot \angle(a + j \cdot b) - k \cdot \angle(c + j \cdot d)$$

$$= q \cdot \arctan\left(\frac{b}{a}\right) - \arctan\left(\frac{d}{c}\right)$$

$$\angle(a + j \cdot b) = \begin{cases} \arctan\left(\frac{b}{a}\right) & , a > 0, b \text{ beliebig} \\ \arctan\left(\frac{b}{a}\right) + \pi & , a < 0, b \geq 0 \\ \arctan\left(\frac{b}{a}\right) - \pi & , a < 0, b < 0 \end{cases}$$

$$\phi = \begin{cases} \frac{\pi}{2} & , a > 0, b \text{ beliebig} \\ -\frac{\pi}{2} & , a = 0, b < 0 \\ \text{unbestimmt} & , a = 0, b = 0 \end{cases}$$

$$\begin{cases} \arctan(\infty) = \frac{\pi}{2} \\ \arctan(-\infty) = -\frac{\pi}{2} \end{cases}$$

$$\begin{cases} \arctan(1) = \frac{\pi}{4} \\ \arctan(-1) = -\frac{\pi}{4} \\ \arctan(0) = 0 \end{cases}$$

$$\angle(e^{-j\omega \cdot T}) = \angle\big(\cos(\omega \cdot T) - j \cdot \sin(\omega \cdot T)\big) = -\omega \cdot T$$

$$\angle(a + jb)^c = c \cdot \arctan\left(\frac{b}{a}\right)$$

$$\lim(\angle(a + j \cdot b)^c) = c \cdot \lim(\angle(a + j \cdot b))$$

dB-Scale

$$x_{dB} = 20 \cdot \log_{10}(x); \qquad x = 10^{\frac{x_{dB}}{20}}$$

$$\left(\frac{1}{X}\right)\bigg]_{dB} = -(X|_{dB})$$

$$(X \cdot Y)|_{dB} = X|_{dB} + Y|_{dB}$$

x	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	2	10	100
x_{dB}	-60	-40	-20	≈ -6	≈ -3	0	≈ 3	≈ 6	20	40

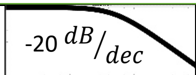
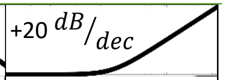
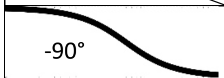
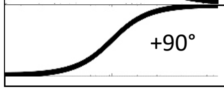
Bode-Diagram

The frequency response can be displayed by two different diagrams. The first one is the Bode-Diagram with its two separate curves.

Bode-Diagrams are frequency-explicit representations of the frequency response $\Sigma(j\omega)$ that display the magnitude function $m(\omega) = |\Sigma(j\omega)|$ and the phase function $\varphi(\omega) = \angle\Sigma(j\omega)$.

Remark: very useful for control system design, however may be misleading in determining closed-loop stability (e.g., for open-loop unstable systems).

General rules

Magnitude / Phase	-20 dB/dec 	+20 dB/dec 
-90° 	$\frac{1}{s+\omega_k} \rightarrow stable$	$(s-\omega_k)$ $non\ minimumphase$
+90° 	$\frac{1}{s-\omega_k} \rightarrow unstable$	$(s+\omega_k)$ $minimumphase$

- Time delay Phase change $= -\omega \cdot T$
  Magnitude change $= 0$

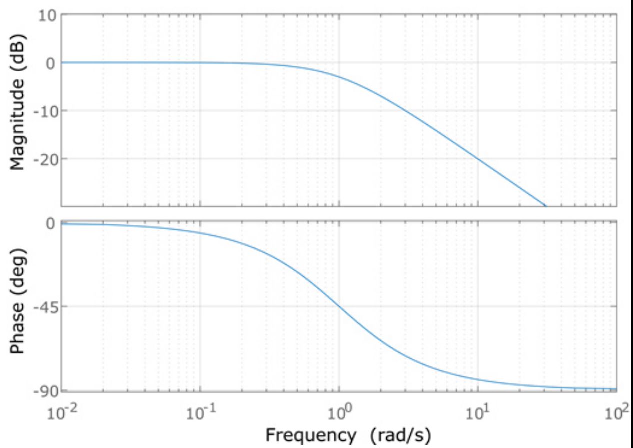
Bode-Diagram of a 1st order system

$$\Sigma(s) = k \cdot \left(\frac{1}{\tau \cdot s + 1}\right)$$

where

- Static gain $= |\Sigma(0)| = k$
- Cut-off frequency $\omega_0 = |1/\tau|$
- Magnitude change $= -20\, dB/dec \ at\ \omega_\pi$
- Phase change $= -90° \ at\ \omega_\pi$

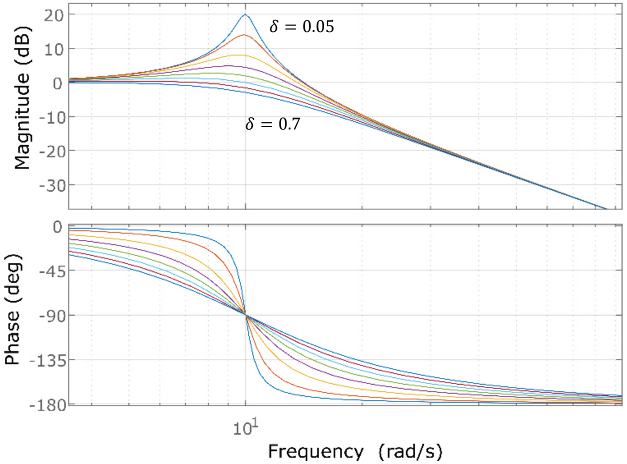
Remember: the static gain k from the transfer function does not have the unit dB. Therefore, the value from the bode plot must be converted from dB to "no unit".



Bode-Diagram of a 2nd order system

$$\Sigma(s) = k \cdot \frac{\omega_0^2}{s^2 + 2\delta\omega_0 \cdot s + \omega_0^2}$$

- $y(t) = (1 - e^{-\sigma t} \cdot \cos(\omega t))$
  Where: $\sigma = Re(-pole)$



where

- Static gain $|\Sigma(s)| = k$
- Cut-off frequency ω_0
- Magnitude change $= -40\ dB/dec\ at\ \omega_0$
- Phase change $= -180°\ at\ \omega_0$
- Peak frequency $\omega_{max} = \omega_0 \cdot \sqrt{1 - 2 \cdot \delta^2},\ \delta < \frac{1}{\sqrt{2}}$
- Peak maximum $= \frac{1}{2\delta\sqrt{1-\delta^2}}$
- Phase margin $\varphi = 71° - 117° \cdot \hat{\epsilon}$

For $\delta \ll 1$

- Peak frequency $\omega_{max} \approx \omega_0$
- Peak maximum $\approx \frac{1}{2\delta}$

Other Rules (Time domain)

$$\hat{\epsilon} = e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}, \hat{\epsilon} = overshoot$$

$$t_{90} = T_0(0.14 + 0.4\delta)$$

$$t_{90} = \frac{2\pi \cdot (0.14 + 0.4\delta)}{\omega_0} = \frac{1.7}{\omega_c}$$

$$\delta = -\frac{\ln(\hat{\epsilon})}{\sqrt{\pi^2 + \ln^2(\hat{\epsilon})}}$$

$$T_0 = \frac{2\pi}{\omega_0}$$

Bode-Diagram of higher order systems

Higher order systems can be decomposed into a series connection of low-order systems. Due to the logarithmic scale of the magnitude plot, the magnitude response of each subsystem can be added up to construct the magnitude response of the overall system:

$$20 \cdot \log_{10}(|\Sigma_1 \cdot \Sigma_2|) = 20 \cdot \log_{10}(|\Sigma_1|) + 20 \cdot \log_{10}(|\Sigma_2|)$$

Similarly, the phase response of each subsystem can be added up:

$$\angle(\Sigma_1 \cdot \Sigma_2) = \angle(\Sigma_1) + \angle(\Sigma_2)$$

Bode's Law

- In the Bode plot, the magnitude slope and the phase are not independent.
- In particular, if the slope of the Bode magnitude is $\lambda \cdot 20\ dB/dec$ over a range of more than ≈ 1 decade, the phase in that range will be approximately $\lambda \cdot 90°$

Example 1

$$G(s) = \frac{1}{2}\frac{(s+2)(s+10)}{(s^2+s+1)(s+5)}$$

1. Split it up in normal forms

$$G(s) = 2 \cdot \left(\frac{s}{2} + 1\right)\left(\frac{s}{10} + 1\right)\left(\frac{1}{s^2 + s + 1}\right)\left(\frac{1}{\frac{s}{5} + 1}\right)$$

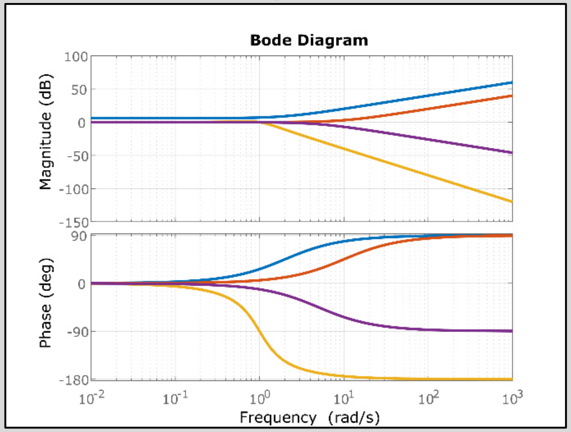
2. Draw for every single one a bode-diagram

   a. $2 \cdot \left(\frac{s}{2} + 1\right)$
   $static\ gain = 2 \approx 6dB$
   $\omega_0 = 2\ rad/s$
   $Mag.change = +20\ dB/dec$
   $Phase\ change = +90°$

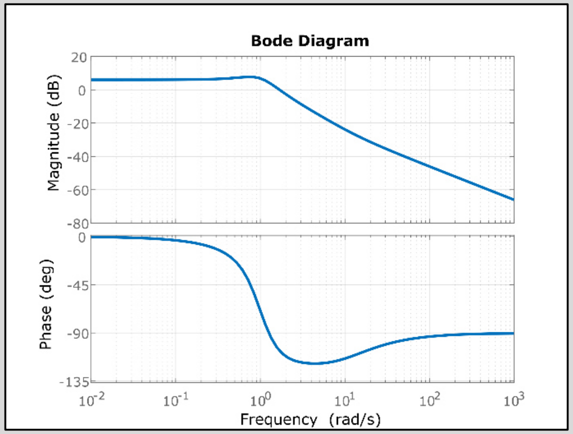
   b. $\left(\frac{s}{10} + 1\right)$
   $static\ gain = 1 = 0dB$
   $\omega_0 = 10\ rad/s$
   $Mag.change = +20\ dB/dec$
   $Phase\ change = +90°$

   c. $\left(\frac{1}{s^2 + s + 1}\right)$
   $static\ gain = 1 = 0dB$
   $\omega_0 = 1\ rad/s$
   $Mag.change = -40\ dB/dec$
   $Phase\ change = -180°$

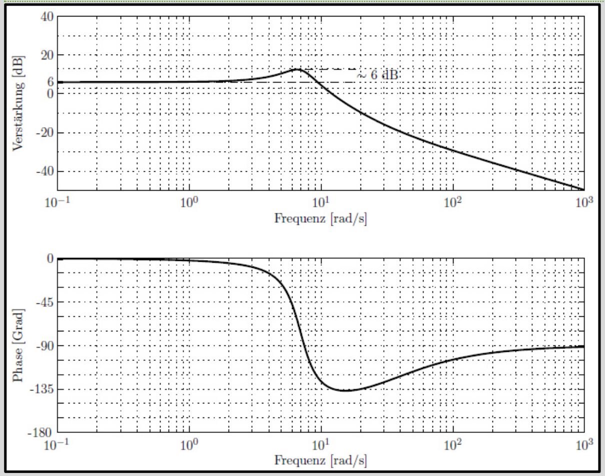
   d. $\left(\frac{1}{\frac{s}{5} + 1}\right)$
   $static\ gain = 1 = 0dB$
   $\omega_0 = 5\ rad/s$
   $Mag.change = -20\ dB/dec$
   $Phase\ change = -90°$



3. Add them together (Superposition) to create the final plot.



Example Bode Plot



a) Identify the transfer function $P(s)$.

b) To control this system, we take a PI-Controller of the form

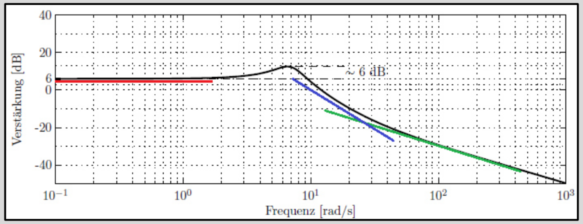
$$C(s) = k_p\left(1 + \frac{1}{T_i s}\right) = \frac{k_p T_i s + k_p}{T_i s};\quad k_p = 4, T_i = 1$$

Draw the Bode-plot for the controller $C(s)$.

c) Draw the Bode-plot of the loop-gain $L(s) = P(s) \cdot C(s)$

d) Determine (graphically) the phase margin φ and the gain margin γ.

Solution



a) Low frequencies

$$\begin{Bmatrix} slope = 0\ dB/dec \\ phase = 0° \end{Bmatrix} \rightarrow P_1(s) = k$$

Middle frequencies

The system looks like a 2nd order System, so it has to be of the form

$$P_2(s) = \frac{\omega_0^2}{s^2 + 2\delta\omega_0 s + \omega_0}$$

High frequencies

As we can see, the slope of the 2nd order system ($-40\ dB/dec$) chance into $-20\ dB/dec$, and the Phase changes in the positive direction.

$$\begin{Bmatrix} slope = +20\ dB/dec \\ phase = +90° \end{Bmatrix} \rightarrow P_3(s) = \frac{s + \omega_k}{\omega_k}$$

Numbers:

Low frequencies: the static gain $P(0) = k = 6dB \approx 2 \rightarrow k = 2$

Middle frequencies: The overshoot is about 6dB, for small δ it follows:

$$|\Sigma(j\omega_{max})| = 6dB \approx 2 \approx \frac{1}{2\delta} \rightarrow \delta = \frac{1}{4}$$

$$\omega_{max} \approx \omega_0 = 7\ rad/s$$

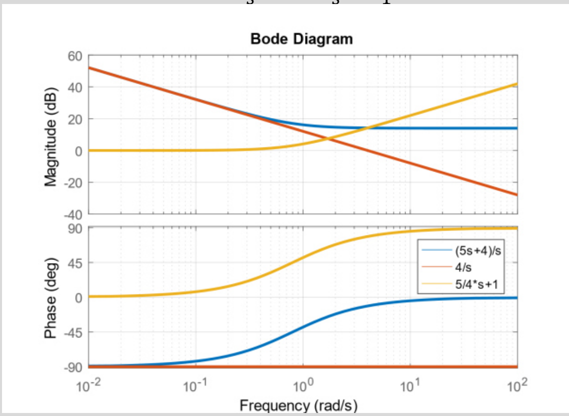
High frequencies: When we draw the two asymptotes (blue and green), we see that the are crossing at $\omega_k \approx 30\ rad/s$. The blue asymptote is drawn from the point $(7\ rad/s\,; 6dB)$ k ha no influence on the single 2nd order System.

The System is

$$P(s) = 2 \cdot \frac{7^2}{s^2 + 2 \cdot \frac{1}{4} \cdot 7 \cdot s + 7} \cdot \frac{s + 30}{30} = \frac{49}{15} \frac{s + 30}{s^2 + \frac{7}{2}s + 49}$$

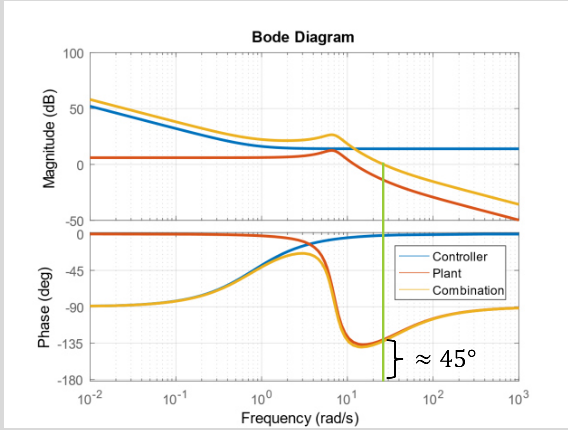
b) Bode plot of the controller

$$C(s) = \frac{5s + 4}{s} = 4 \cdot \frac{1}{s} \cdot \frac{\frac{5}{4}s + 1}{1}$$



c) Bode plot of the loop gain

For the Plot of $L(s) = C(s) \cdot P(s)$ we simple add up the two plots of $C(s)$ and $P(s)$



d) $\begin{cases} \varphi \approx 45° \\ \gamma = \infty \end{cases}$

Asymptotic properties

$$\Sigma(s) = \frac{b_m \cdot s^m + \cdots + b_1 \cdot s + b_0}{s^k \cdot (s^{n-k} + a_{n-1-k} * s^{n-1-k} + \cdots + a_1 \cdot s + a_0)}$$

$$a_0 \neq 0$$

- Order of the system $= n$
- Relative degree $r = n - m$
- Type $= k$

Type k

$$\Sigma(s) = \frac{b_m \cdot s^m + \cdots + b_1 \cdot s + b_0}{s^k \cdot (s^{n-k} + a_{n-1-k} * s^{n-1-k} + \cdots + a_1 \cdot s + a_0)}$$

$$a_0 \neq 0$$

→ Wie viele Pole hat es im Ursprung (→ #Integratoren)

For frequencies $\omega \rightarrow 0$, can determine type k

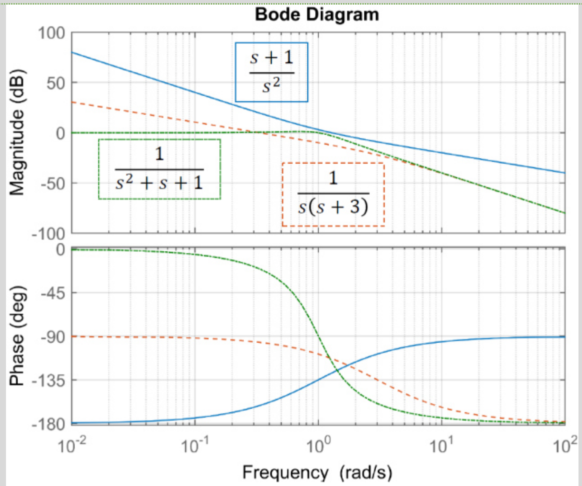
- One only needs to analyze the influence of the static gain and the pole locations "to the very left"

$$\Sigma(j\omega)|_{\omega=0} = \frac{b_o}{a_0} \cdot \lim_{\omega \to 0} \frac{1}{(j\omega)^k}$$

$$\angle\Sigma(0) = \begin{cases} -k \cdot 90° & if\ sign\left(\frac{b_o}{a_0}\right) > 0 \\ -180° - k \cdot 90° & if\ sign\left(\frac{b_o}{a_0}\right) < 0 \end{cases}$$

Example on the next page!

Example Type k



$k = 2 \rightarrow \angle\Sigma(0) = -180°$
$k = 0 \rightarrow \angle\Sigma(0) = 0°$
$k = 1 \rightarrow \angle\Sigma(0) = -90°$

Relative degree r

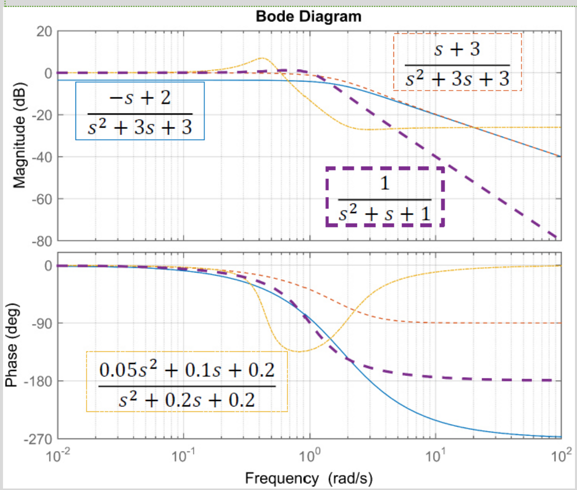
$$\Sigma(s) = \frac{b_m \cdot s^m + \cdots + b_1 \cdot s + b_0}{s^k \cdot (s^{n-k} + a_{n-1-k} * s^{n-1-k} + \cdots + a_1 \cdot s + a_0)}$$
$$a_0 \neq 0$$

For frequencies $\omega \rightarrow \infty$, one can determine the relative degree $r = n - m$

- One only needs to analyze the highest power $\Sigma(s) = \frac{b_m \cdot s^m}{s^n} + 0$
- r is never negativ

$$\frac{\partial |\Sigma(j\omega)|_{dB}}{\partial \log(\omega)} = slope = -r \cdot 20dB; \quad for\ \omega \rightarrow \infty$$

Example degree r



$r = 1 \rightarrow slope = -20\ dB/dec$
$r = 1 \rightarrow slope = -20\ dB/dec$
$r = 2 \rightarrow slope = -40\ dB/dec$
$r = 0 \rightarrow slope = 0\ dB/dec$
→ analyze the slope at high frequencies

Noise n(t)

Noise n(t) is an electrical " interference", which causes a deviation of the measured y(t) from its actual value.

➔ Noise is fast i.e. it is mainly present at higher frequencies.

Good noise rejection:

$|T(j\omega)| = \frac{|L(j\omega)|}{|1+L(j\omega)|}$ has to be small at high frequencies → $|L(j\omega)| < \frac{1}{|W_2(j\omega)|}$

The system is stable iff $|W_2(\pi^+)| < 1$

Disturbance d(t)

Disturbances d(t) are external influences, which cause a deviation of the output y(t) from the reference signal r(t).

➔ disturbances are slow i.e. they are present at lower frequencies

Good disturbance rejection:

$|S(j\omega)| = \frac{1}{|1+L(j\omega)|}$ has to be small at low frequencies → $|L(j\omega)| > |W_1(j\omega)|$

Crossover frequency ω_c

The crossover frequency ω_c is the frequency, at which L(s) crosses the 0 dB line

$$|L(j\omega)| = 1$$

Bandwidth

The bandwidth of the closed-loop system is defined as the maximum frequency ω for which $|T(j\omega)| > \frac{1}{\sqrt{2}}$, i.e., the output can track the commands to within a factor of ≈ 0.7

➔ It determines the speed of the time-domain behavior

➔ Is approximately equal to ω_c

Nyquist-Diagram

In Nyquist diagrams, the curve $\Sigma(j\omega)$ is plotted directly in the complex plane, where the real and imaginary parts of $\Sigma(j\omega) = x(\omega) + j \cdot y(\omega)$ are used as coordinates of a curve in a plane.

The frequency ω does not appear directly in thie representation, but only implicitly as the curve parameter in both $x(\omega)$ and $y(\omega)$.

The Nyquist-Diagrams are frequency-implicit representations.

Nyquist diagram identification

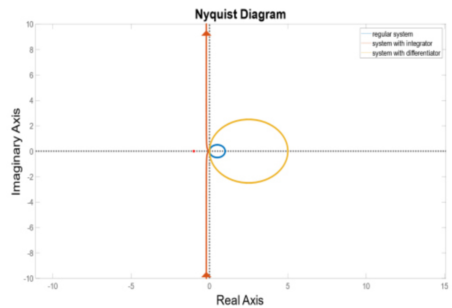
- Start: $\lim_{\omega \rightarrow 0^+} |L(j\omega)|$
- End: $\lim_{\omega \rightarrow \pm\infty} |L(j\omega)|$
- Eintrittswinkel in den Ursprung: $\lim_{\omega \rightarrow \pm\infty} \angle(L(j\omega))$
- Delay: $\angle L(j \cdot \omega T) \sim \omega T\ (Spiral)$

Symmetry: $Im(L(-j\omega)) = -Im(L(j\omega))$

Turn the Nyquist Diagram by $\varphi°$ in the negative direction means a time delay of $e^{-\varphi s}$

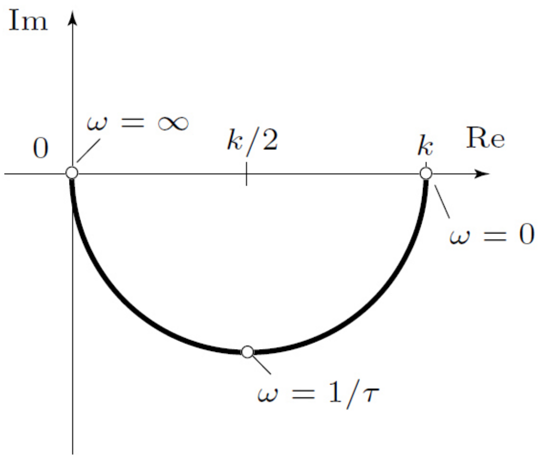
Influences:

- Integrator



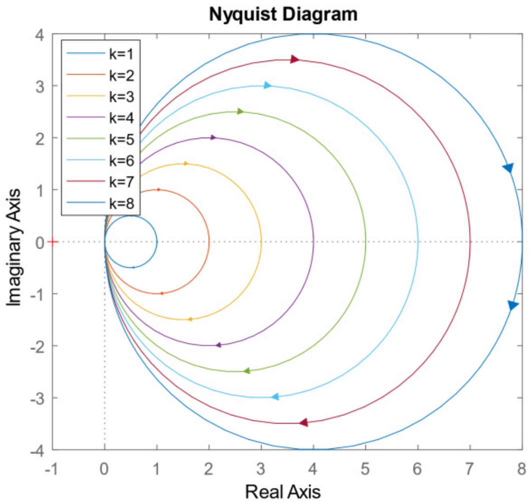
Nyquist diagram of a 1st order system

$$\Sigma(s) = \frac{k}{\tau \cdot s + 1}$$



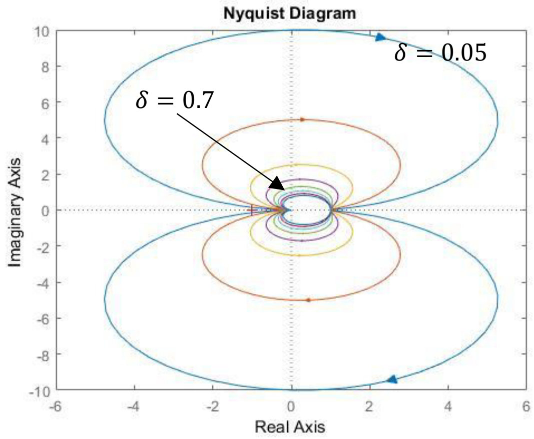
Where

- Static gain $|\Sigma(0)| = k$
- Cut-off frequency *not visible*
- Magnitude change *not visible*
- Phase change $-90°$



Nyquist diagram of a 2nd order system

$$\Sigma(s) = k \cdot \frac{\omega_0^2}{s^2 + 2 \cdot \delta \cdot \omega_0 \cdot s + \omega_0^2}$$



Where

- Static gain $|\Sigma(0)| = k$
- Cut-off frequency *not visible*
- Magnitude change *not visible*
- Phase change $-180°$

Nyquist Criterion

The goal is to determine the stability of the closed-loop system by looking at the the open loop function $L(j\omega)$.
A closed loop system is asymptotically stable iff:

Guzzella: $n_c = n_+ + \frac{n_0}{2}$

Frazzoli : $N = Z - P$

Legend : $\frac{1}{2} = 1 + \frac{1}{2}$

- n_c: Number of mathematically positive encirclements (counterclockwise) of the Point-1 of $L(j\omega)$ with ω from $-\infty$ to ∞
- n_+: Number of Poles of $L(s)$ with positive real part
- n_0 : Number of Poles of $L(s)$ with real part zero
- N: encirclements of -1 clockwise
- Z: Number of zeros / Number of closed-loop poles
- P: Number of poles / Number of unstable open-loop poles

The Nyquist criterion also applies to systems with time-delays!

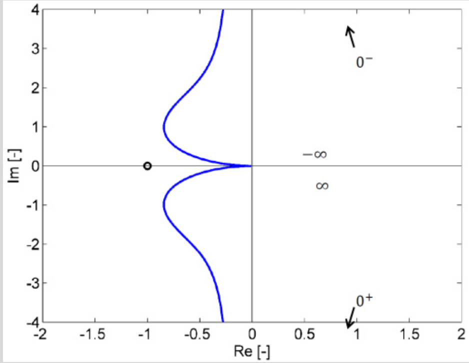
Transformation:

$$N = -n_c$$
$$Z = 0$$
$$P = n_+$$

Example Nyquist criterion

Is the closed-loop system asymptotically stable?

$$L(s) = P(s) \cdot C(s)$$
$$= \frac{100}{s^2 + 12s + 100} \cdot 0.1 \left(\frac{0.1s + 1}{0.1s}\right)$$



$$\left\{\begin{matrix} n_c = \frac{1}{2} \\ n_+ = 0 \\ n_0 = 1 \end{matrix}\right\} \to \frac{1}{2} = 0 + \frac{1}{2} \to \underline{\underline{\textit{Asymptotically stable}}}$$

Nyquist condition on Bode plots

Iff the open-loop is stable:

- Whenever $\angle(L(j\omega)) = 180°$ → $|L(j\omega)| < 1$
- The magnitude plot should be below the 0 dB line, when the phase plot crosses the $-180°$ line.

Robust closed-loop stability

The uncertain closed-loop system is asymptotically stable if the nominal closed-loop system is asymptotically stable and the following inequality is satisfied:

$$|L(j\omega) \cdot W_2(j\omega)| < |1 + L(j\omega)|$$
$$\leftrightarrow |T(j\omega)| < \left|\frac{1}{W_2(j\omega)}\right|$$

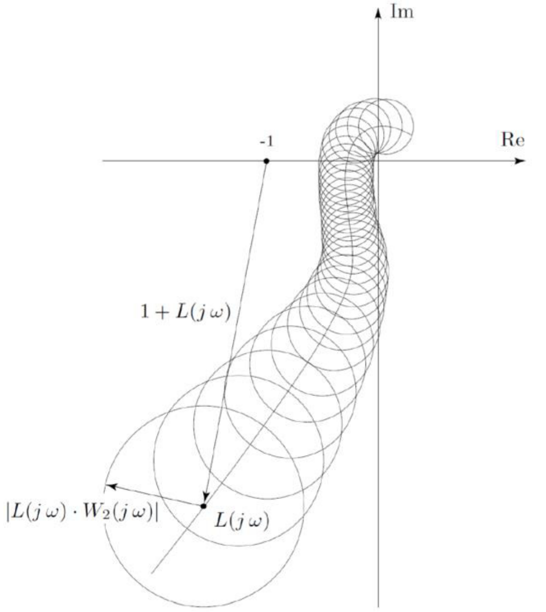


Figure 9.4. Illustration of the robust Nyquist stability theorem.

Robustness

Crossover Frequency ω_c: Is the Magnitude, where $L(j\omega)$crosses the unit circle.

Phase margin φ: Is the distance from the $-180°$ where $L(s)$ enters the unit circle in the Nyquist diagram (magnitude 1) (measured from L(s), not T(s), not S(s))

$$\varphi = \pi - \angle L(j\omega_c)$$

Gain margin γ: Inverse of the magnitude at $-180°$ (measured from L(s), not T(s), not S(s))

$$\angle L(j\omega) = \pi$$

$$Im(j\omega_\gamma) = 0$$

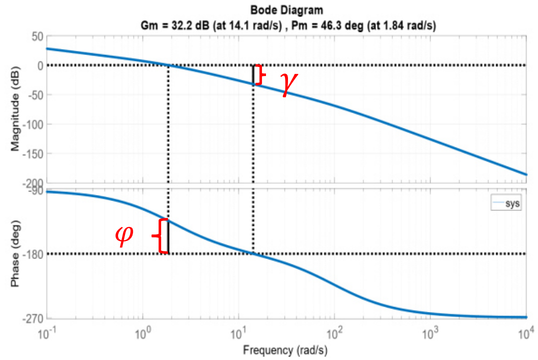
$$\frac{1}{\gamma} = |Re(j\omega_\gamma)|$$

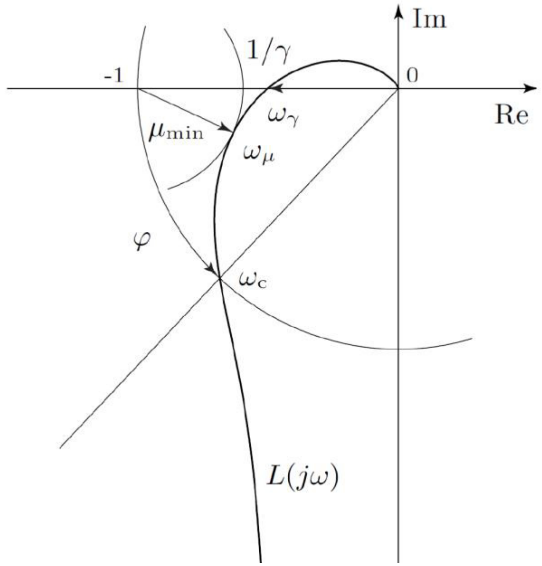
Minimum return difference μ_{min}: Minimum distance from -1

➔ If $L(j\omega)$ crosses the point -1 in the Nyquist diagram, it becomes unstable!!

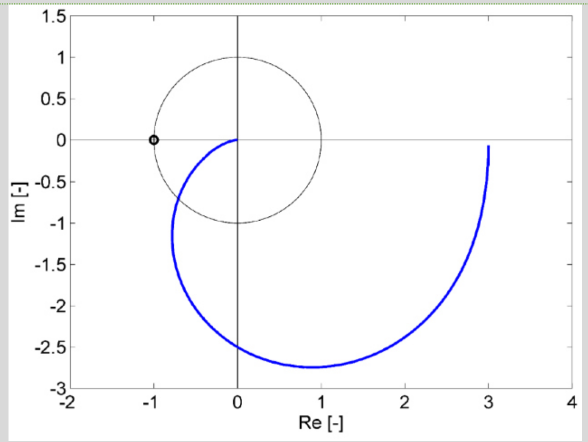
$$\mu = |1 + L(j\omega)|_{min} = \frac{1}{|S(j\omega)|_{max}}$$

⚠ The units in the Bode Diagram are in dB; in the transfer function are no units → conversion!



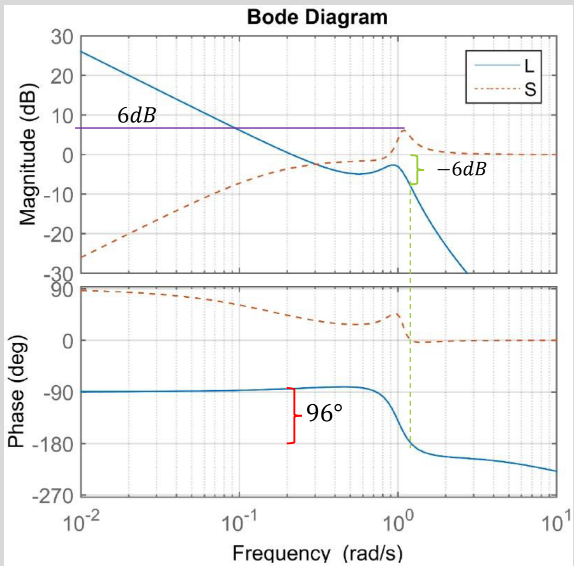


Example Robustness



$$\begin{cases} \varphi = 45° \\ \gamma = \infty \\ \mu_{min} \approx \frac{1}{2} \end{cases}$$

Example Robustness 2



$$\begin{cases} \varphi = 96° \\ \gamma = \frac{1}{-6\text{dB}} = \frac{1}{\frac{1}{2}} = 2 \\ \mu_{min} = \frac{1}{\max|S(j\omega)|} = \frac{1}{6dB} = \frac{1}{2} \end{cases}$$

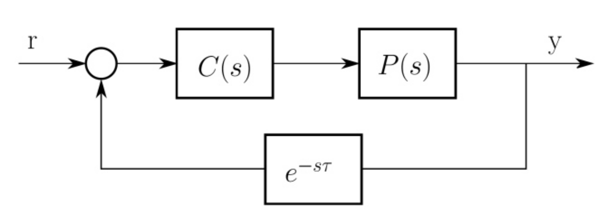
Example Robustness 3

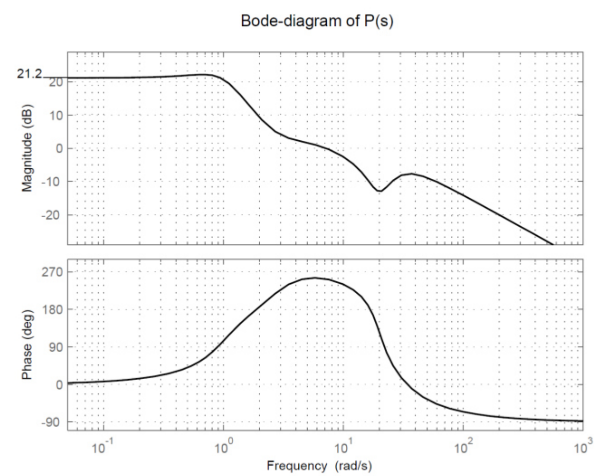
Question

Consider the following block diagram, where C(s)=1 and P(s) is depicted in the bode plot.

a) Find the delay τ?

b) Assume $\tau = 0$ and $C(s) = k_p$, for which values of the gain k_p is the system asymptotically stable?





Solution:

a) the magnitude curve is crossing the unity gain (0 dB) at approximately $\omega_\varphi = 7\frac{rad}{s}$, where the phase is around 250°. Thus, the phase margin

φ = 250° − 180° = 70° = 1.22 rad. The maximum value of delay τ for which the system remains asymptotically stable is calculated as

$$\tau_{max} = \frac{\varphi}{\omega_\varphi} = \frac{1.22}{7} \approx 0.17s$$

b) The phase curve crosses the phase 180° at two points: $\omega_{\gamma 1} \approx 1.9\frac{rad}{s}$ and $\omega_{\gamma 2} \approx 17\frac{rad}{s}$, where the corresponding magnitudes are 10 dB and −10 dB.

Therefore, a positive and a negative gain-margin is resulted:

$$\gamma_- = 0 - 10 = -10\ dB = 0.32$$
$$\gamma_+ = 0 - (-10) = 10\ dB = 3.16$$

➔ $0.32 < k_p < 3.16$

System identification

- Given → Frequency Response of a System
- Goal → Find transfer function $\Sigma(s)$

Reconstruct the frequency response by iteratively adding standard transfer function blocks

Works well if

- System is of relatively low order
- Separation of time constants is large enough

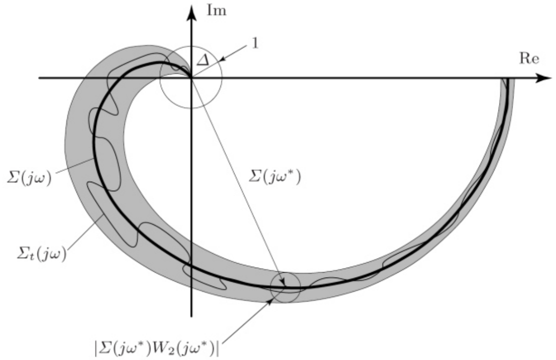
Procedure:

- Identify type and relative degree
- Start with low frequencies and move to higher ones
- Use series connections, this yields additions in the Bode diagram
- First element has gain $k \neq 1$, all other elements have gain $k = 1\ (0\ dB)$

1. Measure data
2. Identification and fitting of nominal model
3. Fitting of uncertainty bound

Model uncertainty

- Problem: Working with uncertain models can lead to unsuccessful controller designs
- Solution: Take model uncertainty into account



True system: $\Sigma_t(j\omega) = m^t \cdot e^{j\cdot\varphi^t}$

Nominal system: $\Sigma(j\omega) = m \cdot e^{j\cdot\varphi}$

Condition:

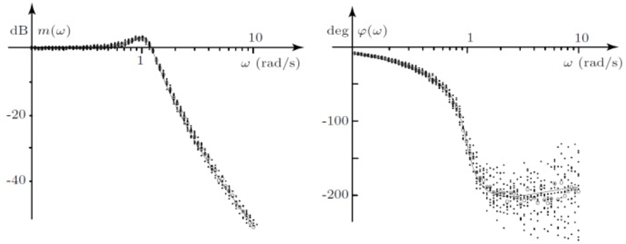
$$m \cdot e^{j\cdot\varphi^t} \in \{m \cdot e^{j\cdot\varphi} \cdot (1 + \Delta \cdot W_2(j\omega))\}$$

$$\left|\frac{m \cdot e^{j\cdot\varphi^t}}{m \cdot e^{j\cdot\varphi}} - 1\right| \leq |W\_2(j\omega)|$$

- Magnitude m
- Phase φ

1. Measure data & 2. Identify and fit nominal model

1. Perform measurements to identify the frequency response of the system under investigation: Measure the frequency response K times at each frequency $\omega_i, (i = \{1, \ldots, I\})$:
$$m_{i,k} \cdot e^{j\cdot\varphi_{i,k}}, k = \{1, \ldots, K\}$$
2. Identify and fit the nominal model $\Sigma(s)$
At each frequency ω_i, average magnitude and phase over all K measurement data points. Then apply system identification methods to identify a nominal transfer function.



3. Fitting of uncertainty bound

3. Fit the uncertainty bound $W_2(s)$.

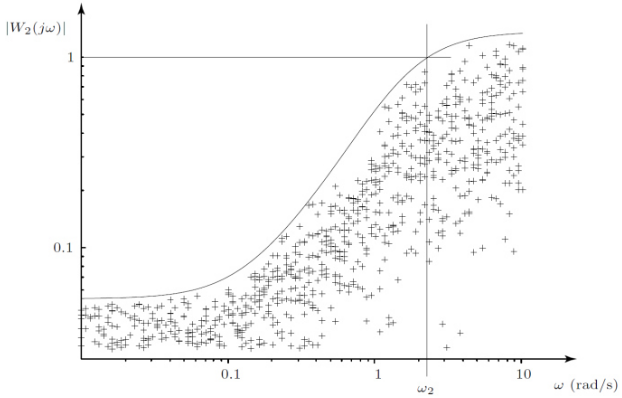
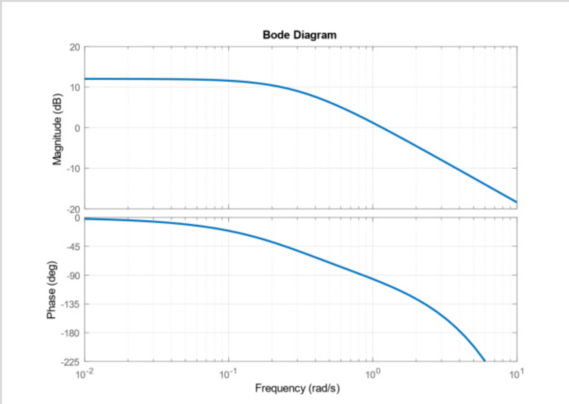


Figure 8.13. Measured differences and upper uncertainty bound $|W_2(j\omega)|$.

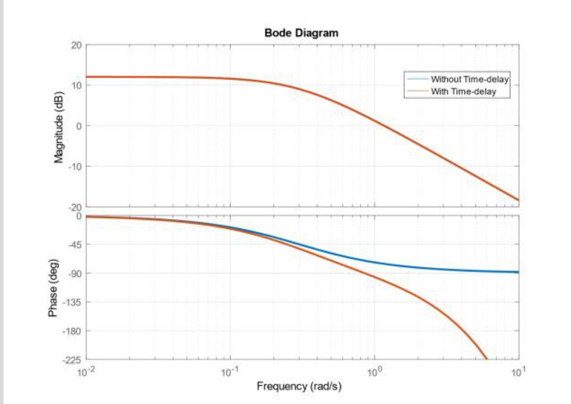
Example System identification

Determine the Transfer function of the following System with time-delay



This system is a simple 1st order system with the following parameters

$\Sigma_1(s) = k \cdot \left(\frac{1}{\tau \cdot s+1}\right); \begin{cases} k = 12dB \approx 4 \\ \tau = \frac{1}{\omega_0} = \frac{1}{0.3} \end{cases}$ Since the time delay only affects the Phase, we can see the influence of the time delay in the following plot



At $\omega = 6\,rad/s$, we see that the difference of the Phase is about 135°, so the time-delay decreases the phase at this point.

Phasechange$(e^{-Ts}) = -\omega T \rightarrow T = \frac{135°}{-\omega} = \frac{\left(\frac{\pi}{180°} \cdot 135°\right)}{-6rad/s} \approx 0.4$

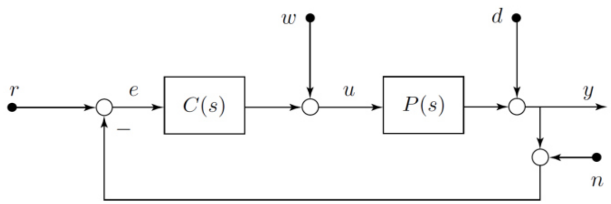
So, the system is given by

$$\Sigma(s) = \frac{4}{\frac{s}{0.3}+1} \cdot e^{-0.4s}$$

Specifications for Feedback Systems

Specifications on the closed-loop behavior are typically given using two main paradigms, plus one that can be seen both ways:

- Steady-state error
- Time-domain specifications
- Frequency-domain specifications



- **Loop gain L(s):** The loop gain is the open-loop transfer function from $e \rightarrow y$

$$L(s) := C(s) \cdot P(s) = \frac{T(s)}{S(s)}$$

$$L(s) = \frac{T(s)}{1 - T(s)}$$

- **Sensitivity S(s):** The Sensitivity is the closed-loop transfer function from $d \rightarrow y$ $(resp. r \rightarrow e)$

$$y = d + P(s)C(s) \cdot e = d + P(s)C(s)(-y) = \frac{d}{1 + L(s)}$$

$$S(s) = \frac{1}{1 + L(s)}$$

- **Complementary sensitivity T(s)**: The complementary sensitivity is the closed-loop transfer function from $r \rightarrow y$

$$y = P(s) \cdot u = P(s)C(s) \cdot e = P(s)C(s)(r - y)$$

$$= \frac{r \cdot L(s)}{1 + L(s)}$$

$$T(s) = \frac{L(s)}{1 + L(s)}$$

  - Use T(s) to determine the closed loop stability (not L(s))

- $\boldsymbol{n \rightarrow y}$:

$$y = P(s)C(s)(-n - y) = -\frac{L(s) \cdot n}{1 + L(s)} = -T(s)$$

General Properties:

- For very large and very small L(s), the following approximations hold:

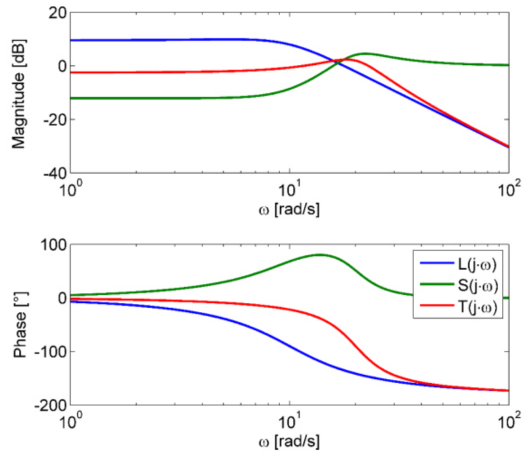
$$|L(s)| \gg 1 \rightarrow S(s) \approx \frac{1}{L(s)} \text{ and } T(s) \approx 1$$

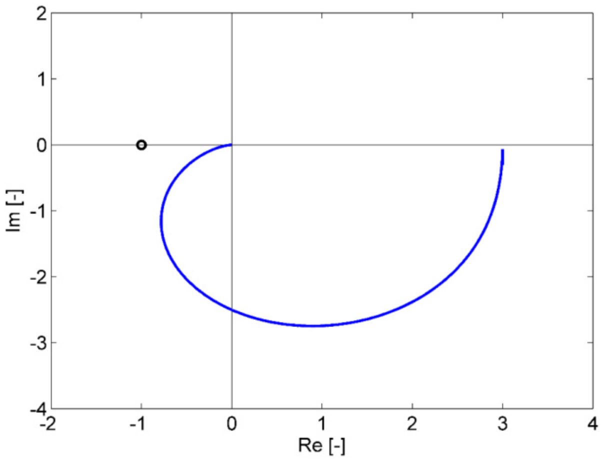
$$|L(s)| \ll 1 \rightarrow T(s) \approx L(s) \text{ and } S(s) \approx 1$$

- $T(s) + S(s) = \frac{L(s)}{1+L(s)} + \frac{1}{1+L(s)} = 1$

  → Only one of them can be substantially smaller than 1!

Graphical Interpretation:





Closed-loop dynamics:

- The entire closed-loop dynamics can be compactly expressed by:
$$Y(s) = S(s) \cdot \big(D(s) + P(s) \cdot W(s)\big) + T(s) \cdot \big(R(s) - N(s)\big)$$
- $Y(s)$ should follow $R(s)$ as precise as possible
- $D(s)$ & $N(s)$ should be as little as possible → $T(s) = 0$

Example

Calculate the transfer function of the noise $n(t)$ to $u(t)$ as a function of $C(s)$ and $P(s)$

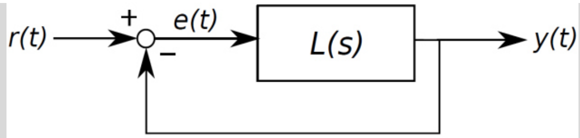
$$U(s) = E(s) \cdot C(s) = \big(-P(s)U(s) - N(s)\big) \cdot C(s)$$
$$U(s) \cdot \big(1 + P(s)C(s)\big) = -N(s)C(s)$$
$$U(s) = -\frac{N(s)C(s)}{1 + P(s)C(s)}$$

The solution is

$$\frac{U(s)}{N(s)} = -\frac{C(s)}{1 + P(s)C(s)}$$

Example

Determine the function of the error $e(t)$ over time t for the input $r(t) = h(t)$ and $L(s) = \frac{1}{s}$



Solution

We calculate

$$E(s) = R(s) - Y(s) = R(s) - E(s)L(s)$$
$$E(s) \cdot \big(1 + L(s)\big) = R(s)$$
$$E(s) = \frac{R(s)}{1 + L(s)} = \frac{\frac{1}{s}}{1 + \frac{1}{s}} = \frac{1}{s+1}$$
$$e(t) = \mathcal{L}^{-1}\big(E(s)\big) = \underline{h(t) \cdot e^{-t}}$$

Internal stability

For internal stability, all nine transfer functions

$$\begin{bmatrix} U(s) \\ Y(s) \\ E(s) \end{bmatrix} = \begin{bmatrix} S(s) & -S(s) \cdot C(s) & S(s) \cdot C(s) \\ S(s) \cdot P(s) & S(s) & T(s) \\ -S(s) \cdot P(s) & -S(s) & S(s) \end{bmatrix} \cdot \begin{bmatrix} W(s) \\ D(s) \\ R(s) \end{bmatrix}$$

Must be asymptotically stable.

Another way to check whether a closed-loop system is asymptotically stable, is to require that $S(s)$ is asymptotically stable and to check whether the following interpolation conditions are satisfied

$$S(\zeta_i^+) = 1 \quad and \quad S(\pi_i^+) = 0$$

A third method to check for closed-loop stability is to show that $1 + L(s)$ has no zeros in the right halt of the complex plane.

Steady-state error

$$e_\infty = \lim_{t \to \infty} e(t) = \lim_{s \to 0^+} s \cdot E(s) = r - y_\infty$$

Where: $E(s) = S(s) \cdot \big(R(s) - D(s) - N(s) - P(s) \cdot W(s)\big)$

1. Set irrelevant terms = 0
2. Laplace transform of relevant terms
3. Solve limes

	$w(t) = 0$	$w(t) \neq 0$
Static error is tolerable $\lvert e_\infty \rvert \le e_{max}$	$\left\lvert \frac{1}{1 + P(0) \cdot C(0)} \right\rvert \le e_{max}$	$\left\lvert \frac{P(0)}{1 + P(0) \cdot C(0)} \right\rvert \le e_{max}$
Static error is not tolerable $e_\infty = 0$	$P(s)$ or $C(s)$ have to be of type $k \ge 1$	$C(s)$ has to be of type $k \ge 1$

$$r(t) = \frac{1}{m!} \cdot t^m \qquad t \ge 0$$

e_{ss}	$m = 0$	$m = 1$	$m = 2$
$type\ 0: 1$	$\frac{1}{1 + k_{Bode}}$	∞	∞
$type\ 1: \frac{1}{s}$	0	$\frac{1}{k_{Bode}}$	∞
$type\ 2: \frac{1}{s^2}$	0	0	$\frac{1}{k_{Bode}}$

Where : m = Ordung des Systems. Insert m in r(t)

Example Steady-state error (HS 2005)

Calculate the Steady-state error for the loop gain

$$P(s) = \frac{1}{s^2 + 5s + 25}; \quad C(s) = \frac{1}{s}$$

Solution

$$L(s) = C(s) \cdot P(s) = \frac{1}{s^3 + 5s^2 + 25s}$$
$$E(s) = \left\lvert \frac{1}{1 + L(0)} \right\rvert$$

Since

$$L(0) = \frac{1}{0} = \infty \to E(s) = \left\lvert \frac{1}{\infty} \right\rvert = \underline{\underline{0}}$$

Example Steady-state error (HS 2005)

Calculate the steady-state error of a unit step for the closed loop control system.

$$P(s) = \frac{2}{s^2 + s + 7}; \quad C(s) = 9 + 4.5s$$

Solution

$$L(s) = C(s) \cdot P(s) = \frac{9s + 18}{s^2 + s + 7}$$
$$T(s) = \frac{L(s)}{1 + L(s)} = \dots = \frac{9s + 18}{s^2 + 10s + 25}$$
$$Y(s) = R(s) \cdot T(s) = \frac{1}{s}\frac{9s + 18}{s^2 + 10s + 25}$$
$$\boxed{e_\infty = r - y_\infty}$$

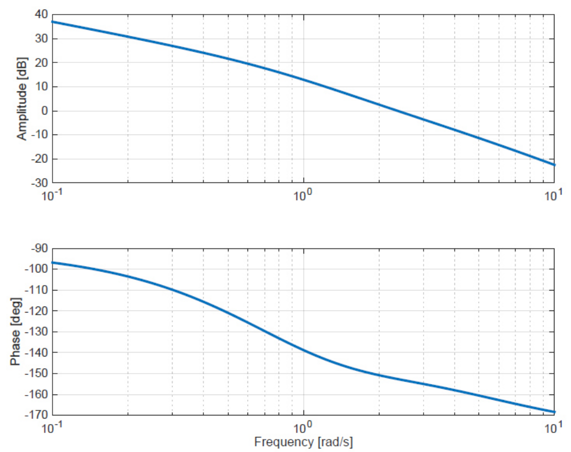
Where

$$r = h(t)$$

$$e_\infty = \lim_{t\to\infty} e(t) = \lim_{t\to\infty}\big(h(t)-y(t)\big) = \lim_{s\to 0} s\left(\frac{1}{s}-Y(s)\right)$$
$$= \lim_{s\to 0}\big(1-sY(s)\big) = 1-\frac{18}{25} = \underline{\frac{7}{25}}$$

Example Steady-state error (Set.10, Ex. 1)

Determine the static error e_∞ of the system below with an input of a unit step.



Solution

$$e_\infty = \lim_{t\to\infty} e(t) = \lim_{t\to 0} E(s)\cdot s = \lim_{t\to 0} S(s)\cdot s\cdot\frac{1}{s} = S(0)$$
$$S(0) = \frac{1}{1+L(0)}$$

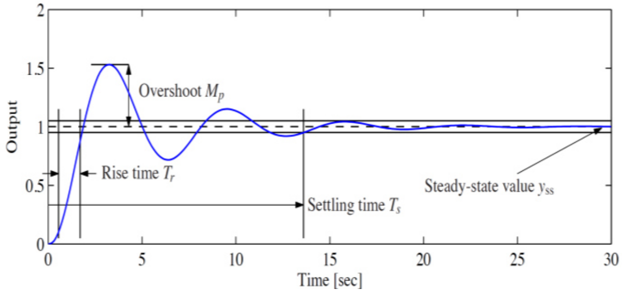
As in the Bode plot can be seen

$$L(0) = \infty$$
$$\underline{e_\infty = \frac{1}{1+\infty} = 0}$$

Time domain specifications

Time-domain specifications impose constraints on the locations of the dominant closed-loop poles (e.g. peak overshoot, rise time, dominant poles)→ use root locus

- ➔ Usually expressed in open-loop frequency responses



- **Rise Time :** depends primarily on ω_n:
$$T_{100} = \frac{\pi}{2\omega}$$
$$T_{90} = \frac{2\pi\cdot(0.14+0.4\delta)}{\omega_0} = T_0(0.140.4\delta) = \frac{1.7}{\omega_c}$$
$$T_{90} \approx 0.4\cdot T_0 \approx \frac{2.4}{\omega_0}$$
$$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$$
$$\omega_c = \sqrt{\sqrt{4\delta^4+1}-2\delta^2\omega_0}$$
- **Peak time:** depends on the frequency ω:
$$T_p \approx \frac{\pi}{\omega}$$
- **% Overshoot:** depends on the damping δ:
$$M_p = \frac{max.\ overshoot}{steady\ state\ value} - steady\ state\ value$$
$$\ln(M_P) \approx -\frac{\sigma\cdot\pi}{\omega} = -\frac{\delta\pi}{\sqrt{1-\delta^2}}$$
$$M_P = \frac{(71^\circ-\varphi)}{117^\circ}$$
- **Settling time** (e.g. to 2%): depends on the real part of the poles σ:
$$T_S = \frac{-\ln(2\%)}{\sigma}$$
- **Gain**: $k = y_{ss}$
- **Further helpful equations:**
$$\omega_c = \sqrt{\sqrt{4\delta^4+1}-2\delta^2}\cdot\omega_0$$
$$\varphi = \frac{\pi}{2} - \arctan\left(\frac{\sqrt{\sqrt{4\delta^4+1}-2\delta^2}}{2\delta}\right)$$

Dominant Pole Approximation

If a closed-loop system is higher order, often one can approximate it with a second-order (or even first-order) system.

- Dominant poles are those with the larges real part (and the slowest decay rate)
- Exception: If the pole with the larges real part also have very small residues (zero-pole cancellation)
- $G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots \leftrightarrow g(t) = r_1 e^{p_1 t} + r_2 p^{p_2 t} + \cdots$

Example 1

$$G(s) = \frac{130}{(s+5)(s+1+5j)(s+1-5j)}$$

The contribution to the response of the pole at $s = -5$ will decay as e^{-5t} wile that of the poles at $s = -1\pm 5j$ will decay at e^{-t}

- $g_{dom}(s) = \frac{26}{(s+1+5j)(s+1-5j)}$

Example 2

$$G(s) = \frac{13}{(s+0.5)(s+1+5j)(s+1-5j)}$$

The contribution to the response of the pole at $s = -0.5$ will decay as $e^{-0.5t}$, while that of the ples at $s = -1\pm 5j$ will decay as e^{-t}

$$G_{dom}(s) = \frac{0.5}{s+0.5}$$

Example 3

$$G(s) = \frac{21.667(s+0.6)}{(s+0.5)(s+1+5j)(s+1-5j)}$$

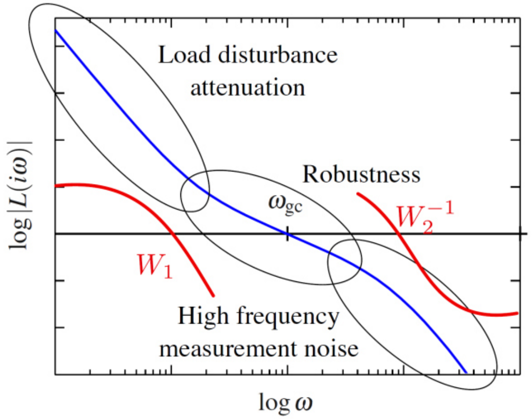
The zero at $s = -1$ makes the magnitude of the residue ot the pole at $s = -0.5$ small w.t.z. to the magnitudes of the residues of the other poles → zero-pole cancellation

$$G_{dom}(s) = \frac{26}{(s+1+5j)(s+1-5j)}$$

Frequency domain specifications

- Usually expressed in closed-loop frequency responses
- Make $|s(j\omega)| \ll 1$ (hence $|T(j\omega)| \approx 1$ at low frequencies) → ensure that commands are tracked with max 10% error up to a frequency of 10 Hz
- Make $|T(j\omega)| \ll 1$ at high frequencies. → ensure that noise is reduced by a factor of 10 at the output at frequencies higher than 100 Hz
-

Bode plot "obstacle course":

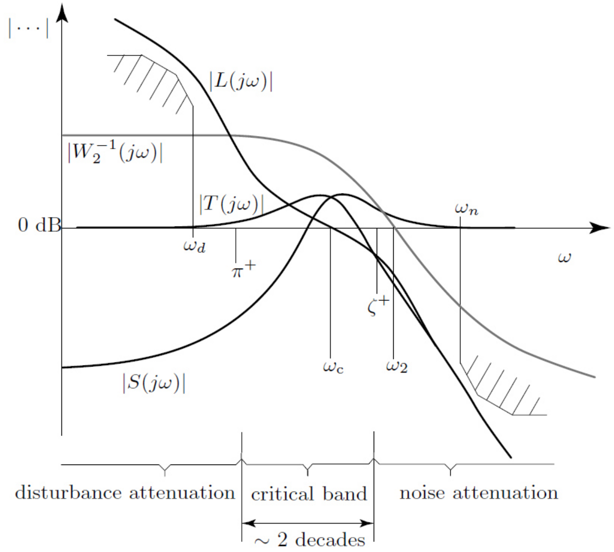


The following requirement must be fulfilled for the crossover frequency ω_c:

$$\max\{10\omega_d, 2\pi^+\} \le \omega_c \le \min\{0.5\omega_2, 0.5\zeta^+, 0.1\omega_n, 0.5\omega_{delay}\}$$

Where:

- ω_d : Highest Frequency of the disturbance
- π^+ : highest/fastest unstable pole
- ω_2 : uncertainty reaches 100%: $|W_2(j\omega)| = 1$
- ζ^+: Non-minimum phase zero
- ω_n: lowest frequency of the noise
- $\omega_{delay} = \frac{1}{T_{tot}}$: frequency where the delay starts affecting the system



Loop shaping

Goal: steer through the bode obstacle course → Improve the robustness of your controller

- **Proportional (static) compensation**: Choose a proportional controller with transfer function C(s)=k
- **Dynamic compensation**: By choosing a controller (compensator) with transfer function C(s) so that L(s) = P(s)C(s) satisfies the requirements

Procedure:

1. How many Integrators are needed?
2. Add Zeros/Poles
3. Add Lead/Lag elements
4. Fix the gain

Proportional control

Change the value of the gain k

Effects: shift of the magnitude plot of the transfer function up and down. The phase plot is not affected

Advantage: If the system is open-loop stable, we know that small enough gains (k → 0) yield stable closed-loops.

Disadvantage: we are not able to meet the other constraints (crossover/bandwidth, or command tracking/disturbance rejection), without compromising stability.

Example Stability

Question

Stabilize the Plant $P(s) = \frac{s-5}{s-1}$ with a P-Controller $C(s) = k_p$.

a) Determine k_p.
b) The absolute value of the amplification of the high-frequency sensor noise should be equal to 1, determine k_p
c) Calculate the steady state error

Solution

a) To assess the stability, we have to look at the poles of the closed loop transfer function:

$$T(s) = \frac{k_p(s-5)}{(s-1)+k_p(s-5)}$$

The pole of T(s) must be negative:

$$\pi_{T(s)} = \frac{1+5k_p}{1+k_p} < 0$$

Which leads to:

$$-1 < k_p < -\frac{1}{5}$$

b)

$$|T(s\to\infty)| = \left|\frac{k_p}{1+k_p}\right| = 1$$

Solve for k_p:

$$k_p = -0.5$$

c)

$$e_\infty = S(0) = \frac{1}{1+L(0)} = \frac{1}{1+5k_p} = -0.667$$

Dynamic control

Lead compensator

$$C_{lead} = k\frac{\frac{s}{a}+1}{\frac{s}{b}+1} = \frac{b}{a}\cdot\frac{s+a}{s+b}, \qquad 0 < a < b$$

Where

$\omega_0 = \sqrt{ab}$, $(location\ phase\ maximum)$

$\varphi_{Max} = 90° - 2\cdot \arctan\left(\sqrt{\frac{a}{b}}\right), maximum\ phase\ shift$

Effects:

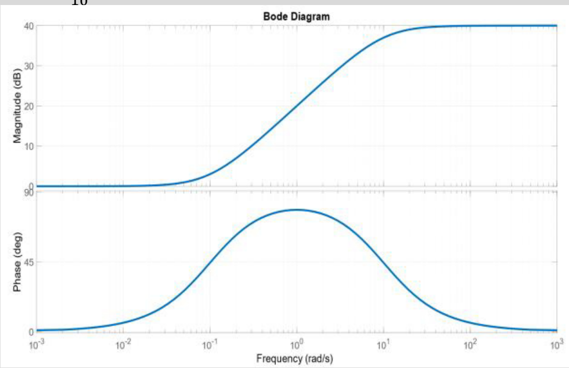
- Increase the magnitude at high frequencies, by b/a; magnitude at low frequencies is not affected.
- Increases the noise n(s)
- Increase the slope of the magnitude at frequencies between a and b by 20 dB/decade.
- Increase the phase around $\sqrt{ab}$ (i.e., the midpoint between a and b on the Bode plot, by up to 90 degrees.
- Increases the gain margin γ
- The larger b/a the larger the phase increase (max. 90 deg.)

Use:

- Pick $\sqrt{ab}$ at the desired ω_c
- Pick b/a depending on the desired phase gain
- Adjust k to put ω_c at the desired frequency

Example 1:

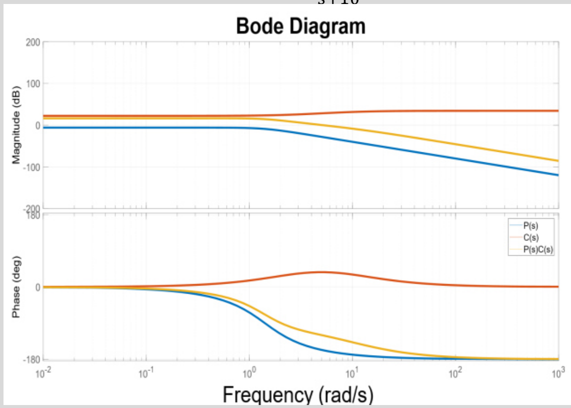
$$C(s) = \frac{\frac{s}{0.1}+1}{\frac{s}{10}+1}$$



Example 2:

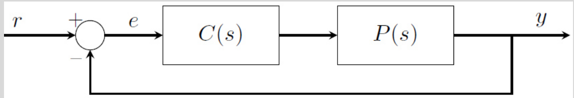
- Desired phase margin: > 45°
- Desired bandwidth: $5\frac{rad}{s}$
- Plant: $P(s) = \frac{1}{s^2+2s+2}$

➔ Controller $C(s) = 52\cdot\frac{s+2.5}{s+10}$



Example 3

Design a lead-lag compensator $C(s)$ for a system modeled by the figure below and the given plant $P(s)$ and the following requirements:



$$P(s) = 5\frac{\left(1-\frac{s}{200}\right)}{\left(1+\frac{s}{20}\right)\left(1+\frac{s}{4}\right)}$$

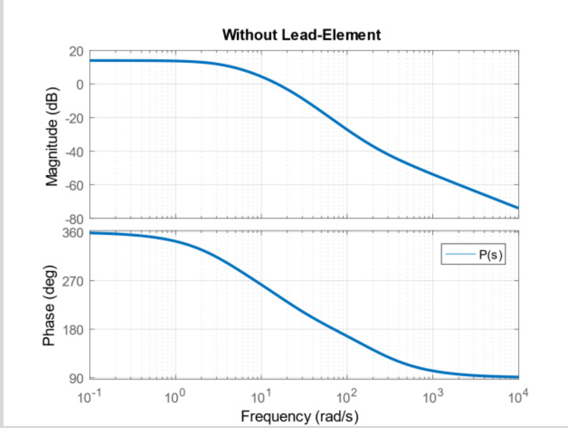
1. Small steady state error to a step response ($e_\infty = 0.005$)
2. Rise time as fast as possible
3. Phase margin around 40° to 45°

Solution

The Compensator is given as

$$C(s) = k\frac{\left(1+\frac{s}{a}\right)}{\left(1+\frac{s}{b}\right)}$$

The bode-plot without Lead



We start with the fast rise time:

$$T_{90} = \frac{1.7}{\omega_c} \to Small\ T_{90} \to Big\ \omega_c$$

The crossover frequency should be as big as possible and is limited by

$$\max\{10\omega_d, 2\pi^+\} \le \omega_c \le \min\{0.5\omega_2, 0.5\zeta^+, 0.1\omega_n, 0.5\omega_{delay}\}$$

With a zero at 200 rad/sec

$$\omega_c \le 100\, rad/s$$

We calculate the existing phase margin at 100 rad/s

$$\angle(P(j\omega))_{\omega=100} = \cdots - 192.96° \to \varphi = -12.96°$$

As φ should be ≈ 45° we can increase the phase margin by ≈ 57.96°

$$\omega_{c,Lead} = \sqrt{ab} = 100 \to a = \frac{10000}{b}$$

$$\angle(C(j\omega))_{\omega=100} = \arctan\left(\frac{100}{a}\right) - \arctan\left(\frac{100}{b}\right)$$

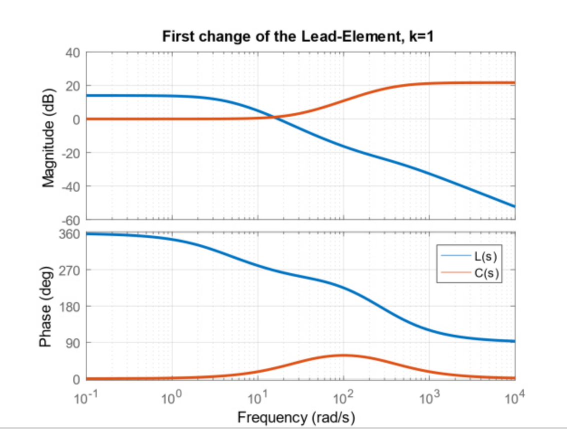
$$= \arctan\left(\frac{b}{100}\right) - \arctan\left(\frac{100}{b}\right) \to^* \begin{cases} a = 28.74 \\ b = 348\end{cases}$$

$$* \to We\ can't\ solve\ this\ by\ hand$$

So the lead element is

$$C_{lead}(s) = k\frac{\left(1+\frac{s}{28.74}\right)}{\left(1+\frac{s}{348}\right)}$$

With the bode-plot:



Now, we have a phase margin of 45° at $\omega = 100\ rad/s$, but because of the Lead-element, we also increased the phase, so ω_c is no more at $100\ rad/s$. We move it to the right place with the right k.

$$|L(j\omega)|_{\omega=100} = 0dB = 1$$

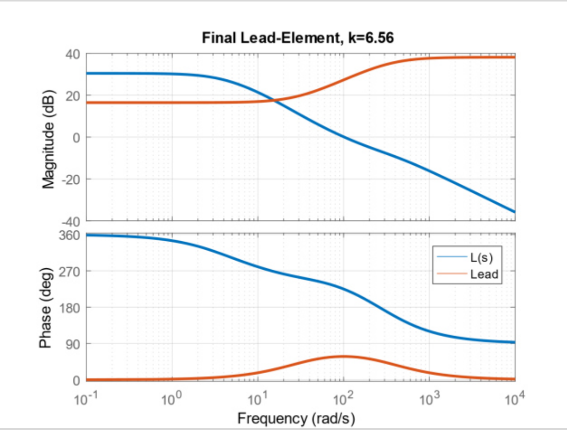
$$|L(j\omega)|_{\omega=100} = \left|k\frac{\left(1+\frac{s}{28.74}\right)}{\left(1+\frac{s}{348}\right)}\right| \cdot 5\left|\frac{\left(1-\frac{s}{200}\right)}{\left(1+\frac{s}{20}\right)\left(1+\frac{s}{4}\right)}\right| = \cdots$$

$$= k \cdot 0.152$$

$$\underline{k = 6.557}$$

So the final lead element is

$$C_{Lead}(s) = 6.56 \cdot \frac{\left(1+\frac{s}{28.74}\right)}{\left(1+\frac{s}{348}\right)}$$



Now we want to make the steady-state error smaller ≈ 0.005

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \cdot E(s) \cdot R(s) = E(0) = \frac{1}{1+L(0)}$$

We see that

$$\frac{1}{1+L(0)} = 0.005 \to L(0) \approx \frac{1}{0.005} = 200$$

At the instant we have

$$L(0) = 6.65 \cdot 5 = 33.25$$

So we need to multiply by 6.01 to get $L(0) = 200$

For this, we take an easy Lag compensator $C_{Lag} = \frac{s+a}{s+b}$

Since we don't want to disturb ω_c and φ, a rule of thumb says, that the zero ζ should be one decade below ω_c.

$$a \to 10$$

We want a static gain of 6.01, so $C_{Lag}(0)$ must be 6.01

$$C_{Lag}(0) = \frac{0+a}{0+b} = \frac{a}{b} = \frac{10}{b} = 6.01 \to b = \frac{10}{6.01} = 1.66$$

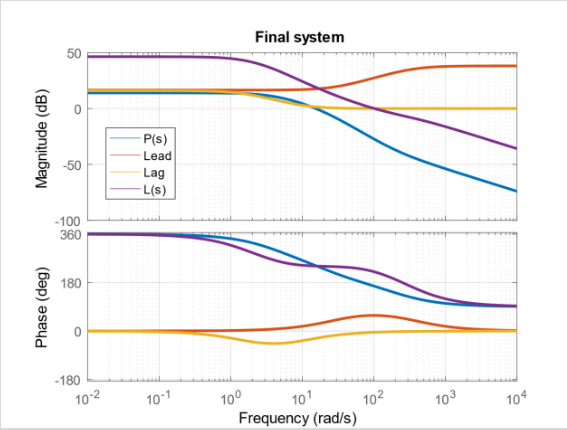
So the lag compensator is

$$C_{Lag} = \frac{s+10}{s+1.66}$$

And the whole system

$$\underline{L(s) = P(s) = 5\frac{\left(1-\frac{s}{200}\right)}{\left(1+\frac{s}{20}\right)\left(1+\frac{s}{4}\right)} \cdot 6.56 \cdot \frac{\left(1+\frac{s}{28.74}\right)}{\left(1+\frac{s}{348}\right)} \cdot \frac{s+10}{s+1.66}}$$

With the bode-plot:



Example 4

Design a compensator $C(s)$ for the plant $P(s)$ which meets the following performance specifications:

1. The steady-state error following ramp inputs must not exceed 2%.
2. The error in response to sinusoidal inputs up to 5 rad/sec should not exceed about 5%.
3. The crossover frequency should be about 50 rad/sec.
4. The phase margin should be at least 50°.

$$P(s) = \frac{1}{s(0.1s+1)}$$

Solution

Part 1:

$$e_\infty = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \cdot R(s) \cdot S(s) = \lim_{s\to 0} s \cdot \frac{1}{s^2} \cdot \frac{1}{1+L(s)}$$

$$\approx \frac{1}{s \cdot L(0)} \le 0.02$$

$$\frac{1}{s \cdot C(0)L(0)} \le 0.02$$

We can take the controller $C(s) = k$

$$\frac{1}{k \cdot \frac{1}{1}} = \frac{1}{k} \to k = 50 \to \underline{C(s) = 50}$$

Part 2:

The error in response to sinusoidal input can be evaluated using $S(s)$

$$|S(j\omega)| = \frac{1}{|1+L(j\omega)|} \le 0.05 \to |1+L(j\omega)| \approx |L(j\omega)| \ge \frac{1}{0.05}$$

$$|L(j\omega)| = 8.944 \ge 20 \to multiply\ by \approx \frac{20}{9}$$

$$\underline{C_{new}(s) = 50 \cdot \frac{20}{9} \approx 115}$$

Part 3:

Currently, ω_c is at

$$|L(j\omega)| = 0 \to \omega_c \approx 33.2°$$

The magnitude at $\omega = 5\,rad/s$ is ≈ 0.4, so we need to lower the magnitude by 0.4

$$\underline{C_{new}(s) = \frac{1}{0.4}C(s) \approx 288}$$

Part 4:

Currently, the phase margin φ is $\approx 12°$, so it needs to be lifted up by $\approx 38°$, we take a lead compensator:

$$C_{Lead}(s) = k \cdot \frac{1+\frac{s}{a}}{1+\frac{s}{b}}$$

We don't want to change the low frequencies, so $k = 1$

We take the formula

$$\omega_c = \sqrt{ab} = 50 \to a = \frac{2500}{b}$$

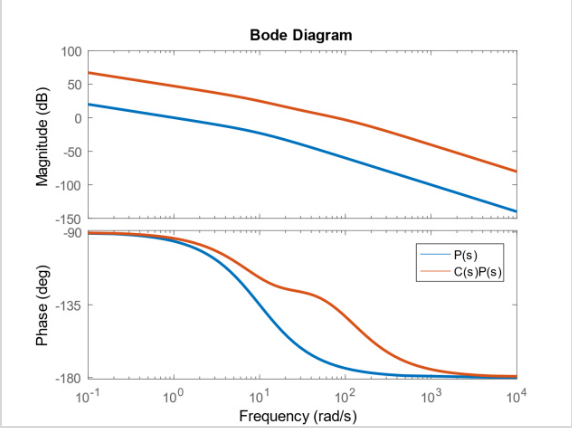
$$90° - 2 \cdot \arctan\left(\sqrt{\frac{a}{b}}\right) = \varphi_{change} = 38°$$

$$\frac{a}{b} = (\tan(26°))^2 \to \begin{cases} a = 24.38 \\ b = 102.52 \end{cases}$$

And with the superposition-principle, we get

$$\underline{C_{final}(s) = C(s)C_{Lead}(s) = 288\frac{1+\frac{s}{24.38}}{1+\frac{s}{102.52}}}$$

Bode plot on the next page!



Lag compensator

$$C_{lag} = k\frac{\frac{s}{a}+1}{\frac{s}{b}+1} = \frac{b}{a}\cdot\frac{s+a}{s+b}; \qquad 0 < b < a$$

Effects:

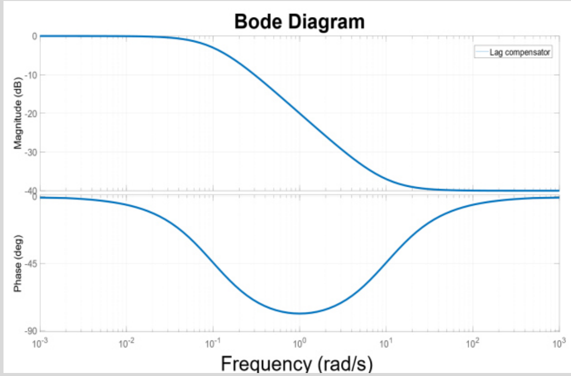
- Decreases the magnitude at high frequencies by b/a
- Magnitude at low frequencies is not affected
- Decreases the slope of the magnitude at frequencies between a and b, by 20 dB/decade
- Decreases the phase around $\sqrt{ab}$, i.e. the midpoint between a and b on the Bode plot by up to 90 degrees
- Improves command tracking/disturbance rejection

Use:

- Pick a/b as the desired increase in magnitude at low frequencies
- Pick a so that it is sufficiently smaller than the crossover frequency, not to affect ω_c and γ
- Increase the gain k by a/b

Example 1:

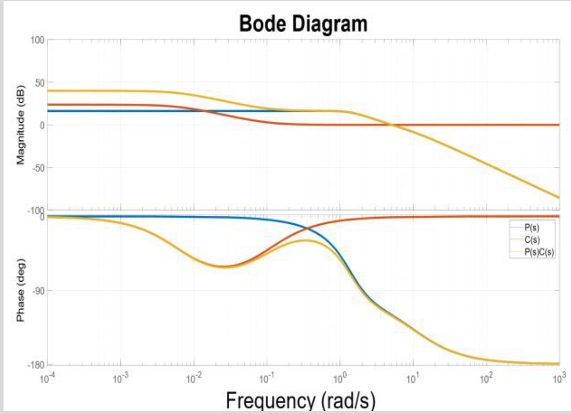
$$C(s) = \frac{\frac{s}{10}+1}{\frac{s}{0.1}+1}$$



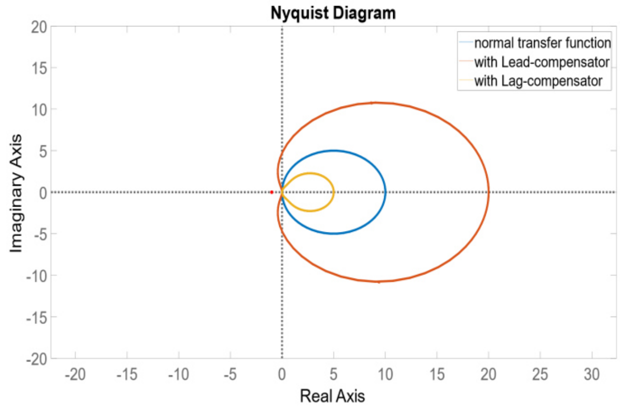
Example 2: (vgl. Ex 2 lead)

- Desired phase margin: > 45°
- Desired bandwidth: $5\frac{rad}{s}$
- Desired steady-state error to a unit step: 1%
- Plant: $P(s) = 52\cdot\frac{s+2.5}{s+10}\cdot\frac{1}{s^2+2s+2}$

➔ Controller: $C(s) = \frac{s+0.1}{s+\frac{0.1}{15.3}}$



Influences of Lead/Lag compensators in the Nyquist diagram:



Nonminimum phase /unstable systems

- Non-minimum-phase zeros limit the crossover frequency (closed-loop bandwidth)
- Open loop unstable poles require the crossover frequency to be higher

$$P(s) = P_{mp}(s)D(s)$$

Where:

- $P_{mp}(s)$: mirror image of the poles/zeros of P(s)
- $D(s) = P(s)\cdot P_{mp}{}^{-1}(s)$ ➔ $|D(j\omega)| = 1$

Example

$$P(s) = \frac{s-z}{s-p}$$

$$P_{mp}(s) = \frac{s+z}{s+p}$$

➔ $D(s) = \frac{z-s}{s+z}\cdot\frac{s+p}{s-p}$

Feedback Control Design

PID Controller

Tune a PID Controller by choosing the parameters k_p, k_i, k_d

P: Proportional

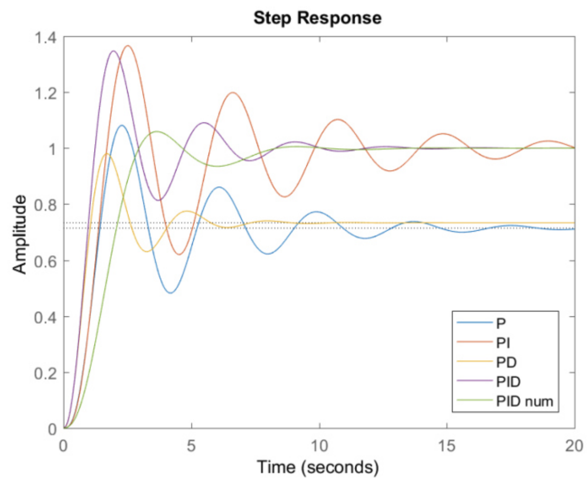
- Control action is proportional to control error
- Fast reduction of control error, but static error may result
- Shouldn't be used for reaching far away set points, because a large gain k_p leads to:
  - Large overshoot
  - Maybe unstable response

I: Integral

- Control action is proportional to integral of previous control error
- Slow but complete reduction of control error
- Faster decay of magnitude in the bode plot
- Lower crossover frequency ω_c → Prevents problems with noise
- To prevent a static error e_∞ an Integral is a better alternative than using a large gain k_p

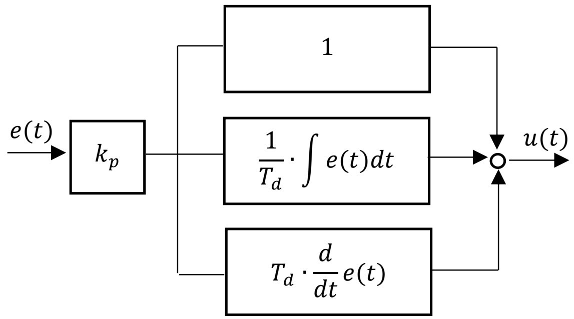
D: Derivative

- Control action is proportional to change of control error
- Attenuation of control action to a change in the plant output signal → damping
- Amplification of control action to a change of the reference signal
- D-Part leads to:
  - Less overshoot
  - Faster rise time (?)
- A too large gain in the D-Part leads to a slow response
- Don't use it with high frequency, because it may be corrupted by noise:
  - $\lim_{s\to\infty} k_p T_d s = \infty$
- To prevent this, use a low pass filter:
  - $D = \frac{k_p T_d s}{1+T_f s}$
  - $\lim_{s\to\infty} \frac{k_p T_d s}{1+T_f s} = \frac{k_p T_d}{T_f}$
  - Bounded output → Noise has no influence



Time Domain

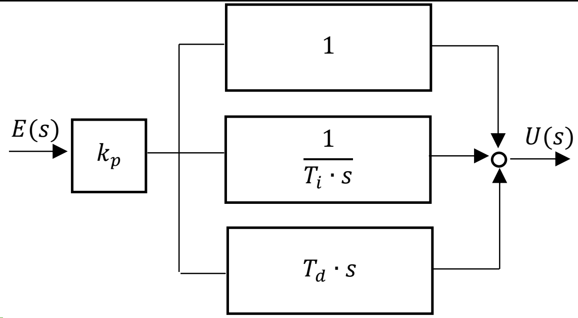
$$u(t) = k_p \cdot \left(e(t) + \frac{1}{T_i} \int_0^t e(\tau)d\tau + T_d \frac{d}{dt} e(t) \right)$$



Frequency Domain

$$U(s) = \underbrace{k_p \cdot \left(1 + \frac{1}{T_i \cdot s} + T_d \cdot s\right)}_{C(s)} \cdot E(s)$$

Where: $k_i = \frac{k_p}{T_i}$, $\quad k_d = k_p \cdot T_d$

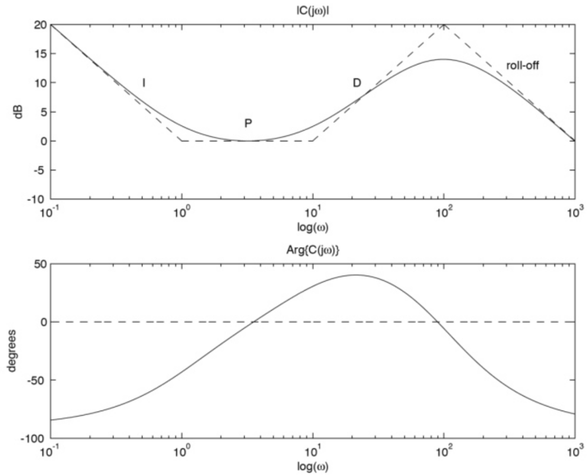


Roll-off

A system is non-causal, thus not implementable in real-time conditions. Therefore, in practice a PID controller is augmented with a roll-off term with a small-time constant τ

Goal: Turn of the controller at high frequencies.

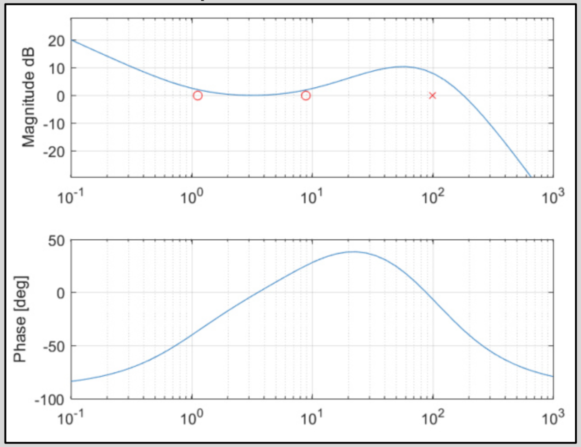
$$C_{pracical}(s) = C(s) \cdot \frac{1}{(s\tau + 1)^2}$$
$$= k_p \cdot \frac{1 + T_i s + T_d T_i s^2}{T_i s} \cdot \frac{1}{(s\tau + 1)^2}$$



Example on the next page!

Example:

PID controller with $k_p = 1;\ \ T_i = 1;\ \ T_d = 0.1;\ \ \tau = 0.01$



Example PID

Question:

The Bode Plot of the Plant P(s) is given (The y axis is not in dB!!). Design a PD controller $C(s) = k_p(1 + T_d \cdot s)$ with the following specifications:

- Crossover frequency $\omega_c = 1\frac{rad}{s}$
- Phase margin $\varphi = 60°$

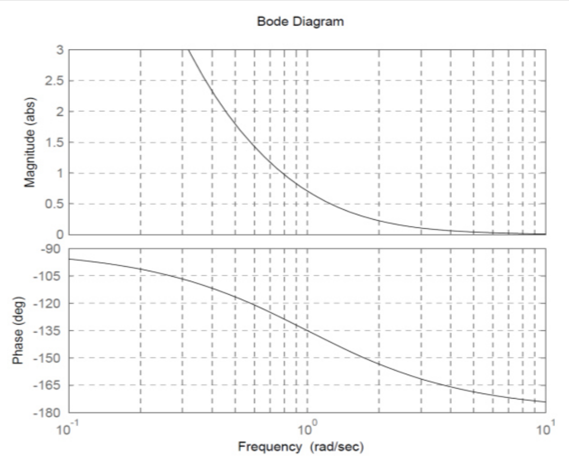


Abbildung 2: Frequenzgang der Strecke $P(s)$

Solution:

We can find k_p and T_d with the following equations:

- $|L(j\omega_c)| = |P(j\omega_c)| \cdot |C(j\omega_c)| = 1$
- $\angle(L(j\omega_c)) - \varphi = \angle(P(j\omega_c)) + \angle(C(j\omega_c)) = -180°$

Magnitude and Phase at the crossover frequency can be found in the bode plot:

- $|P(j\omega_c)| \approx 0.75$
- $\angle(P(j\omega_c)) \approx -135°$

Therefore the controller C(s) is:

- $|C(j\omega_c)| \approx \frac{4}{3}$
- $\angle(C(j\omega_c)) \approx 15°$

Add the magnitude and phase from the Controller to the formula of the PD controller:

- $|C(j\omega_c)| = k_p\sqrt{1^2 + (T_d \cdot \omega_c)^2}$

→ $k_p \cdot \sqrt{1 + T_d^2} = \frac{4}{3}$

- $\angle(C(j\omega_c)) = \arctan\left(\frac{T_d \cdot \omega_c}{1}\right)$

→ $\arctan(T_d) = 15°$

Now we can find the Parameters:

- $T_d = \tan(15°) \approx 0.27$
- $k_p = \frac{4}{3 \cdot \sqrt{1 + T_d^2}} \approx 1.29$

Follow up question:

Would it also be possible to design the Controller C(s) wit a PI Controller?

Solution:

No! A PI controller always leads to a phase loss.

Ziegler Nichols

- Useful when no model of the plant is available
- Not precise

Assumption: The Plant can be approximated by the transfer function

$$P(s) = \frac{k}{\tau s + 1} \cdot e^{-Ts}$$

Procedure Nyquist:

1. Set $T_i = \infty,\ \ T_d = 0,\ \ \tau = 0,$ →
   P-Controller: $L(j\omega) = C(j\omega) \cdot P(j\omega) = k_p P(j\omega)$
2. Increase k_p until it goes through $(-1{,}0)$ → it is in a steady-state oscillation
3. Note critical k_p^* and the corresponding critical oscillation period T^*
4. Use k_p^* and T^* to calculate the control gains:

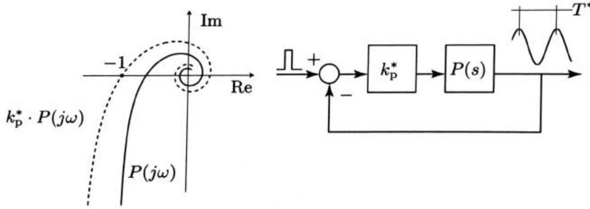
Procedure Bode:

1. Frequency where the Phase Plot crosses the 180° line → ω^*
2. $T^* = \frac{2\pi}{\omega^*}$
3. Where does the Magnitude Plot cross ω^* → …dB → convert this number to "no units" → k_p^*

Equations:

1. $\angle\left(k_p^* \cdot P(j\omega^*)\right) = -\pi$
   a. $\angle(P(j\omega^*)) = -\pi$ → solve for ω^*
   b. $T^* = \frac{2\pi}{\omega^*}$
2. $|k_p^* \cdot P(j\omega^*)| = 1$ →solve for k_p^*

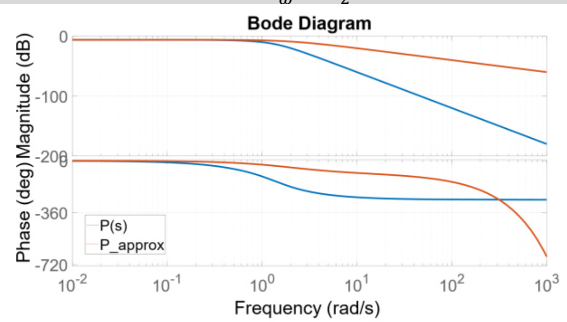
Type	$k_p\ [-]$	$T_i\ [sec]$	$T_d\ [sec]$
P	$0.5 \cdot k_p^*$	$\infty \cdot T^*$	$0 \cdot T^*$
PI	$0.45 \cdot k_p^*$	$0.85 \cdot T^*$	$0 \cdot T^*$
PD	$0.55 \cdot k_p^*$	$\infty \cdot T^*$	$0.15 \cdot T^*$
PID	$0.6 \cdot k_p^*$	$0.5 \cdot T^*$	$0.125 \cdot T^*$



Example on the next page!

Example:

- Plant: $P(s) = \frac{1}{(s+1)\cdot(s^2+2s+2)}$
- Approximation: $P_{approx} = \frac{0.5}{0.5\cdot s+1} \cdot e^{-0.01s}$
- Set $T_i = \infty,\ T_d = 0,\ \tau = 0,$ and increase gain k_p
- $k_p^* = 10 \ \rightarrow\ T^* = \frac{2\pi}{\omega^*} = \frac{2\pi}{2} = \pi$



Nuisances

Time Delay

Definition: The evaluation of sensory information aimed at deciding the best course of action, will require a finite computation time.

Input: $u(t) = e^{sT}$

Output: $y(t) = e^{s(t-T)} = e^{-sT}u(t)$

Magnitude : $|e^{j\omega T}| = 1,$

Phase : $\angle(e^{-j\omega T}) = -\omega T$

➔ $L'(s) = e^{-sT}L(s)$

➔ $|L'(j\omega)| = |L(j\omega)|, \quad \angle L'(j\omega) = \angle(L(j\omega) - \omega T)$

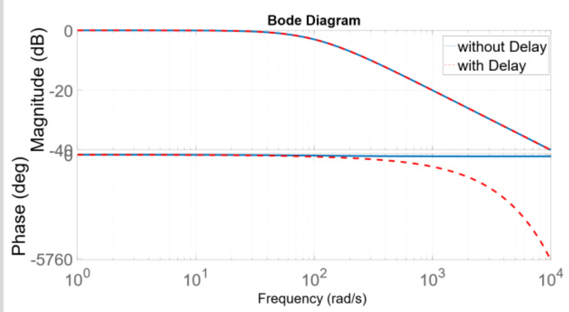
How to find T: at which frequency $f = \frac{1}{T}$ is the phase $-57°$

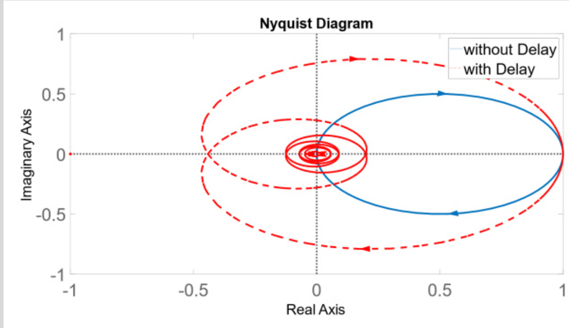
Effects:

- Reduction of phase margin: $\varphi_{delay} = \varphi - \omega_c T$
- Phase margin reduction → crossover frequency increase

Example

- Without time delay: $L(s) = \frac{1}{0.01s+1}$
- With time delay: $L'(s) = \frac{1}{0.01s+1} \cdot e^{-0.01s}$





Approximation

The root locus method cannot be used for continuous time models with delays → transfer function must be rational Therefor we use approximations:

Taylor series expansion

$$e^{-sT} \approx 1 - sT + \frac{1}{2}(sT)^2$$

This is a rather naïve approximation. It only holds for $|sT| \ll 1$.

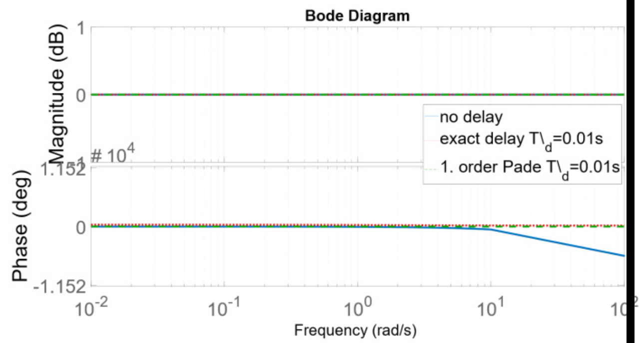
The magnitude of the frequency response diverges for $\omega \rightarrow \infty$, while the magnitude of $e^{-sT} = 1$

Padé Approximation

1. Order: $e^{-sT} \approx k \cdot \frac{\frac{2}{T}-s}{\frac{2}{T}+s}$

Advantage: magnitude is equal to 1

➔ Useful to plot root locus



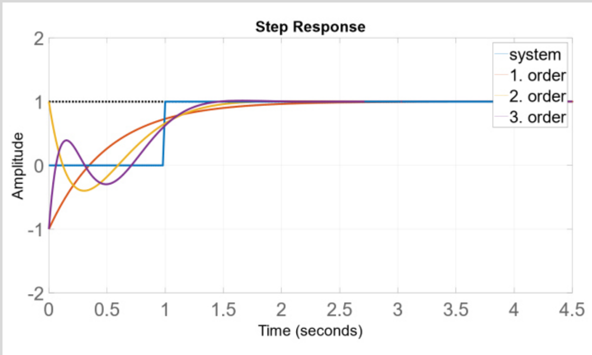
Example:

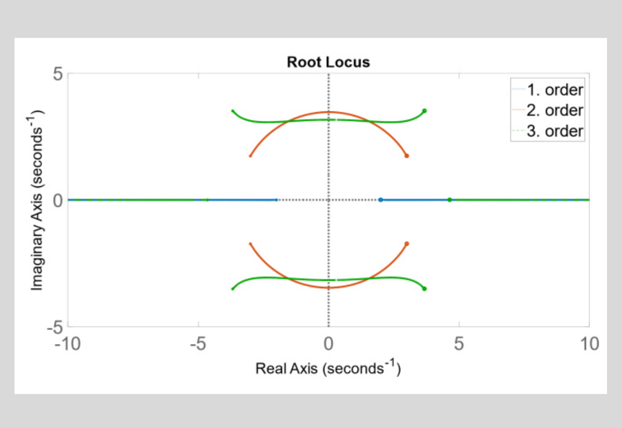
Transfer function ($T_d = 1$): $G_d(s) = e^{-s}$

1. Order: $2^{-s} = \frac{2-s}{2+s}$

2. Order: $e^{-s} \approx \frac{s^2-6s+12}{s^2+6s+12}$

3. Order: $e^{-s} \approx \frac{-s^3+12s^2-60s+120}{s^3+12s^2+60s+120}$





Nonlinearities

Most real-world systems are not linear:

- Linear Input ⇏linear Output
- Principle of Superposition doesn't hold

Nonlinear System:

$$\frac{d}{dt}x(t) = f\big(t, x(t), u(t)\big)$$
$$y(t) = h\big(t, x(t), u(t)\big)$$

Jacobian Linearization

- Approximation only holds for very small ξ, v

Hartman-Grossman: "If the linearized system is closed-loop BIBO stable, then the nonlinear system is also stable, for (ξ, v) in a neighborhood of (0,0)"

Procedure:

- Find the desired equilibrium condition
- Linearize the non-linear model around the equilibrium
- View p.3: Linearization

Anti-reset windup (ARW)

Problem:

- Once the input saturates, the integral of the error keeps increasing

Idea:

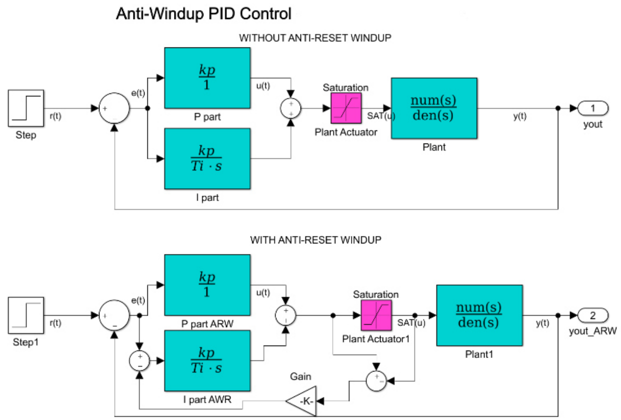
- Once the input saturates, stop integrating the error

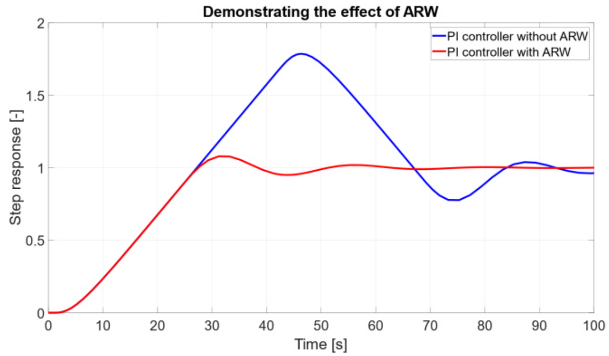
Implementation:

Integral gain: $K_1' = \begin{cases} K_1 \rightarrow if\ the\ input\ doesn't\ saturate \\ 0 \ \rightarrow If\ the\ input\ saturates \end{cases}$

Effects of anti-windup schemes:

- guarantee the stability of the compensator when the (original) feedback loop is effectively opened by the saturation.
- Prevent divergence of the integral error when the control cannot keep up with the reference.
- Maintain the integral errors "small".





How to implement a compensator

Pseudo Code(Euler approximation):

- Make state space realization
- Initialize the state to, e.g. $x \leftarrow 0$
- Initialize the error to, e.g. $\boldsymbol{e_{old}} \leftarrow 0$
- Let dt be some small-time interval
- **Loop**
  - → Read the reference value r
  - → Read the measured output value y
  - → Compute the error e
  - → Update the state as $x \leftarrow x + (Ax + Be) \cdot dt$
  - → Compute the output as $u = Cx + De + K_D \cdot \frac{e - e_{old}}{dt}$
  - → Compute the output as $u = Cx + De$
  - → Send the command u to the actuators
  - → Store the error: $e_{old} \leftarrow e$

→ Additional steps for Non-proper functions are red

(In)Proper

$$\frac{s^a + s^{a-1} + \cdots + s^1}{s^b + s^{b-1} + \cdots + s^1}$$

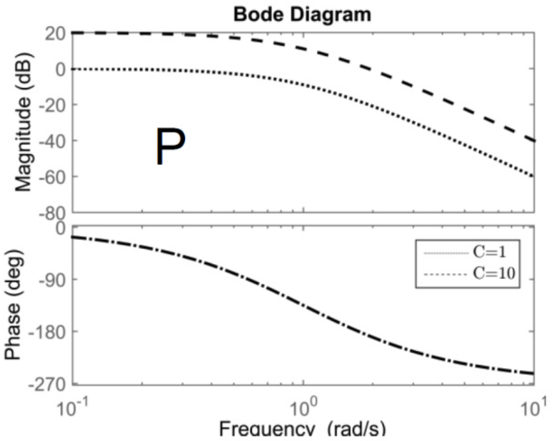
A system is:

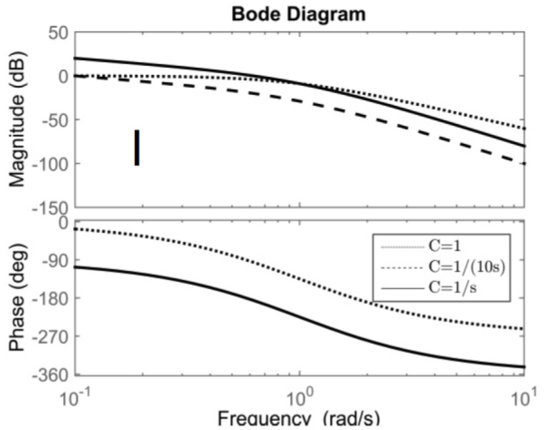
- Proper: $a = b$
- Improper: $a > b$
- Strictly proper: $a < b$

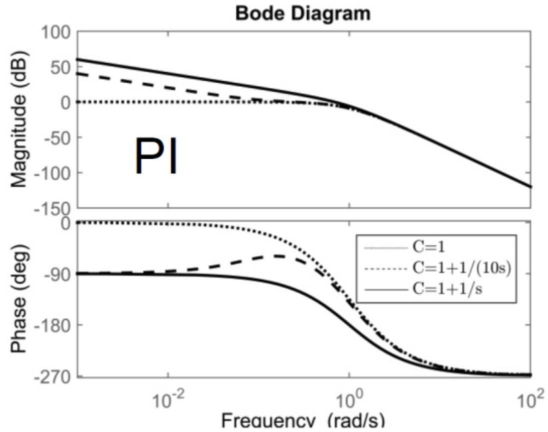
Influence of PID Controller

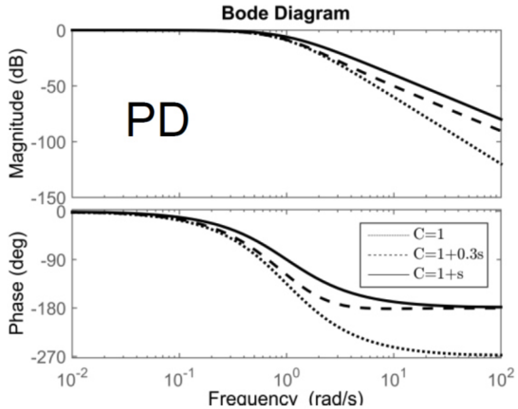
Influence of PID without roll-off on the open loop gain L(s):

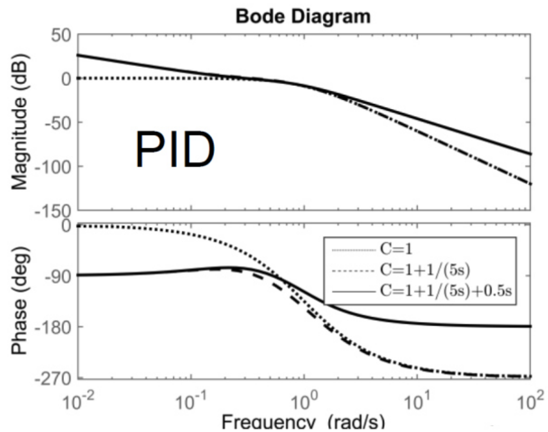
$$C(s) = k_p \cdot \frac{1 + T_i s + T_d T_i s^2}{T_i s}$$

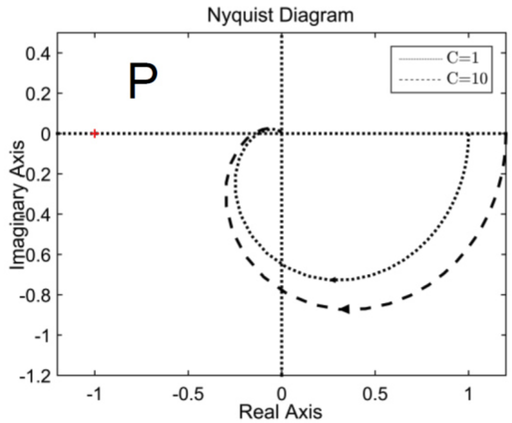


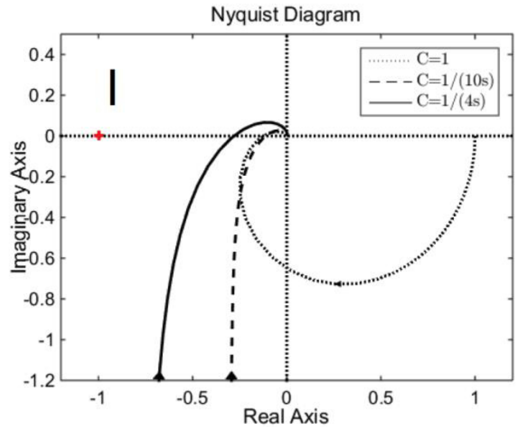


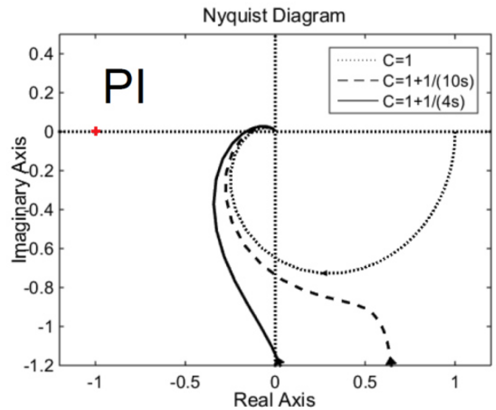


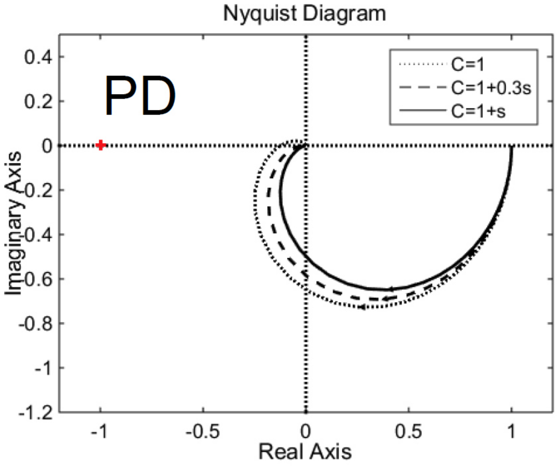


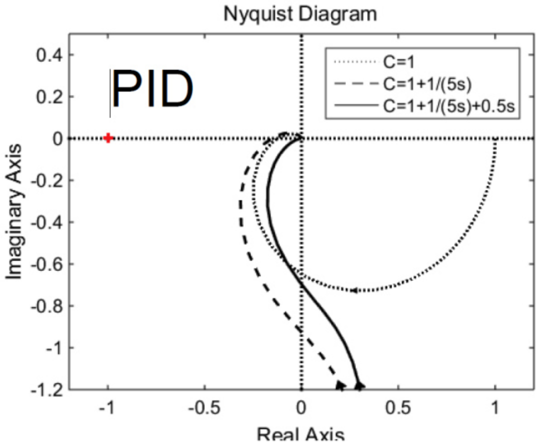












Relevant standard elements Guzzella

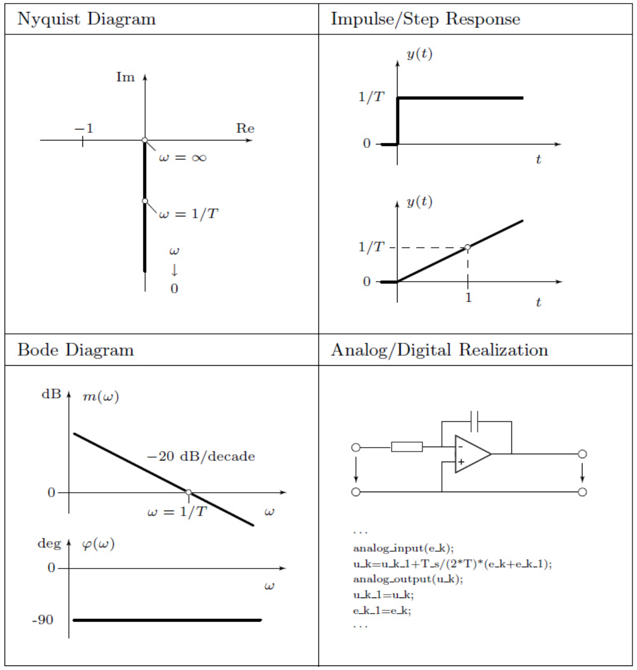
A.1 Integrator Element

Element Acronym: I

Transfer Function: $\Sigma(s) = \frac{1}{T \cdot s}$

Poles/Zeros: $\pi_1 = 0,\ \zeta_1 = \infty$

Internal Description: $\frac{d}{dt}x(t) = \frac{1}{T} \cdot u(t)$
$y(t) = x(t)$

Nyquist Diagram	Impulse/Step Response
	
Bode Diagram	**Analog/Digital Realization**

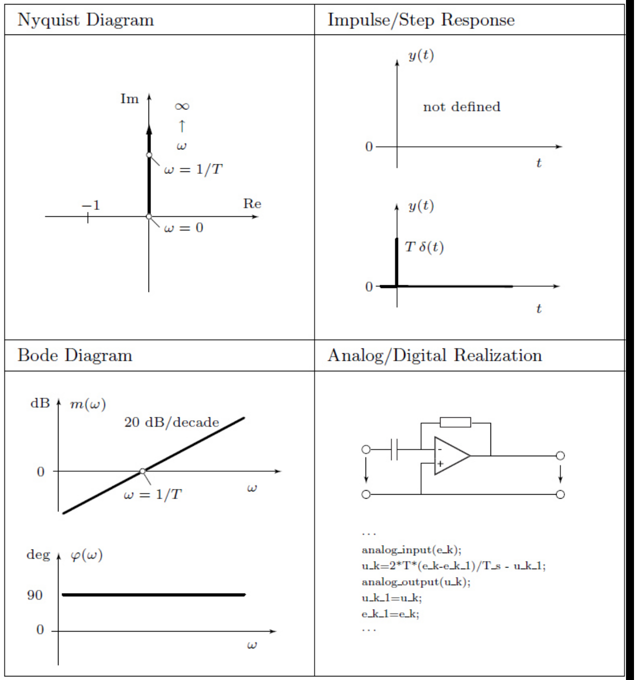
A.2 Differentiator Element

Element Acronym: D

Transfer Function: $\Sigma(s) = T \cdot s$

Poles/Zeros: $\pi_1 = \infty,\ \zeta_1 = 0$

Internal Description: $y(t) = T \cdot \frac{d}{dt}u(t)$



A.3 First-Order Element

Element Acronym: LP-1

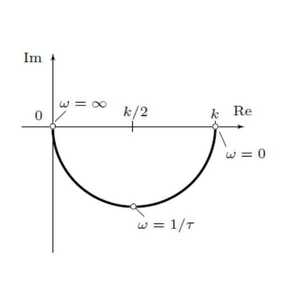
Transfer Function: $\Sigma(s) = \frac{k}{\tau \cdot s+1}$

Poles/Zeros: $\pi_1 = -\frac{1}{\tau}$, $\zeta_1 = \infty$

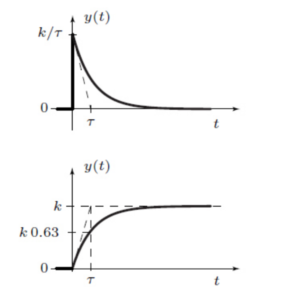
Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\frac{1}{\tau}\cdot x(t) + \frac{1}{\tau}\cdot u(t)$

$y(t) = k \cdot x(t)$

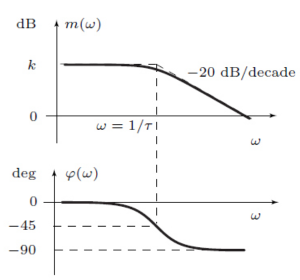
Nyquist Diagram



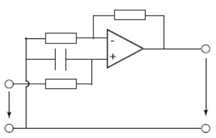
Impulse/Step Response



Bode Diagram



Analog/Digital Realization



```
...
analog_input(e_k);
u_k=u_k_1*(2*tau-T_s)/(2*tau+T_s)+...
    (e_k_1+e_k)*(k*T_s)/(2*tau+T_s);
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

A.4 Realizable Derivative Element

Element Acronym: HP-1

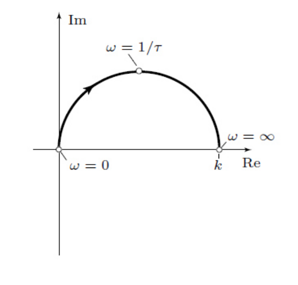
Transfer Function: $\Sigma(s) = k \cdot \frac{\tau \cdot s}{\tau \cdot s+1} = k \cdot \left(1 - \frac{1}{\tau \cdot s+1}\right)$

Poles/Zeros: $\pi_1 = -\frac{1}{\tau}$, $\zeta_1 = 0$

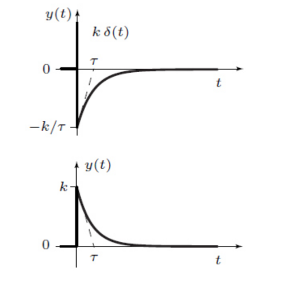
Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\frac{1}{\tau}\cdot x(t) + \frac{1}{\tau}\cdot u(t)$

$y(t) = -k \cdot x(t) + k \cdot u(t)$

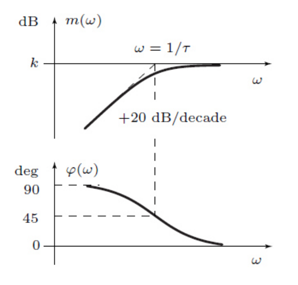
Nyquist Diagram



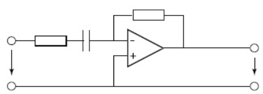
Impulse/Step Response



Bode Diagram



Analog/Digital Realization



```
...
analog_input(e_k);
u_k=1/(T_s+2*tau)*(u_k_1*(2*tau-T_s)+...
    (e_k-e_k_1)*2*k*tau);
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

A.5 Second-Order Element

Element Acronym: LP-2

Transfer Function: $\Sigma(s) = k \cdot \frac{\omega_0^2}{s^2+2\cdot\delta\cdot\omega_0\cdot s+\omega_0^2}$

Poles/Zeros: $\pi_{1,2} = -w_0 \cdot \delta \pm w_0\sqrt{\delta^2-1}$, $\zeta_{1,2} = \infty$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = x_2(t)$,

$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = -\omega_0^2 \cdot x_1(t) - 2\cdot\delta\cdot\omega_0\cdot x_2(t) + \omega_0^2 \cdot u(t)$

$y(t) = k \cdot x_1(t)$

$$|\Sigma(j\omega_{max})| = \frac{1}{2\delta\sqrt{1-\delta^2}}$$

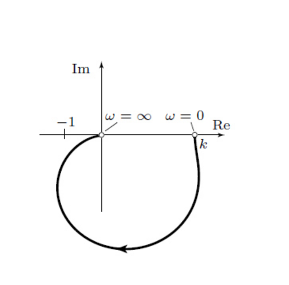
$$\omega_{max} = \omega_0\sqrt{1-2\delta^2}\,;\delta < \frac{1}{\sqrt{2}}$$

<u>*if* $\delta \ll 0$</u>

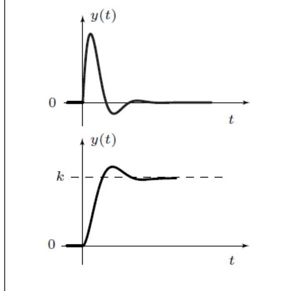
$$|\Sigma(j\omega_{max})| \approx \frac{1}{2\delta}$$

$$\omega_{max} \approx \omega_0$$

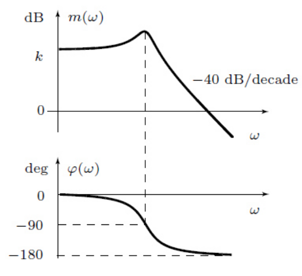
Nyquist Diagram



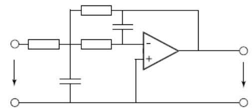
Impulse/Step Response



Bode Diagram



Analog/Digital Realization



```
...
analog_input(e_k);
...
use Matlab's c2dm
...
analog_output(u_k);
u_k_2=u_k_1;
e_k_2=e_k_1;
u_k_1=u_k;
e_k_1=e_k;
...
```

A.6 Lag Element

Element Acronym: LG-1

Transfer Function: $\Sigma(s) = k \cdot \frac{T \cdot s+1}{\alpha \cdot T \cdot s+1} = \frac{k}{\alpha} + k \cdot \frac{1-1/\alpha}{\alpha \cdot T \cdot s+1} \quad 1 < \alpha$

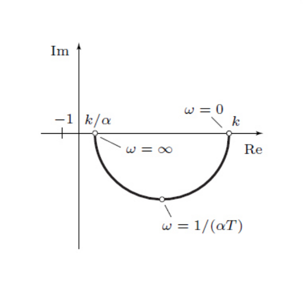
Poles/Zeros: $\pi_1 = -\frac{1}{\alpha \cdot T}$, $\zeta_1 = -\frac{1}{T}$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\frac{1}{\alpha \cdot T} \cdot x(t) + \frac{1}{\alpha \cdot T} \cdot u(t)$

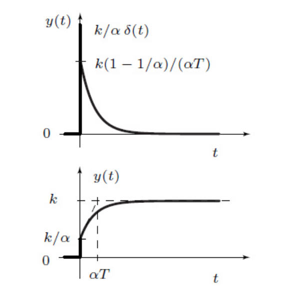
$y(t) = \frac{k \cdot (\alpha - 1)}{\alpha} \cdot x(t) + \frac{k}{\alpha} \cdot u(t)$

Phase minimum: $\hat{\varphi} = \arctan(1/\sqrt{\alpha}) - \arctan(\sqrt{\alpha})$ at $\hat{\omega} = (T \cdot \sqrt{\alpha})^{-1}$

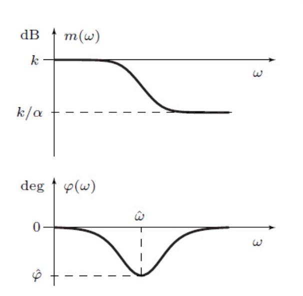
Nyquist Diagram



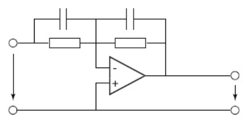
Impulse/Step Response



Bode Diagram



Analog/Digital Realization



```
...
analog_input(e_k);
u_k=u_k_1*(2*T*T_s*alpha)/ ...
    (T_s+2*T*alpha)+...
    e_k*k(T_s+2*T*alpha)+...
    e_k_1*k(T_s-2*T)/(T_s+2*T*alpha);
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

A.7 Lead Element

Element Acronym: LD-1

Transfer Function: $\Sigma(s) = k \cdot \frac{T \cdot s+1}{\alpha \cdot T \cdot s+1} = \frac{k}{\alpha} + k \cdot \frac{1-1/\alpha}{\alpha \cdot T \cdot s+1} \quad 0 < \alpha < 1$

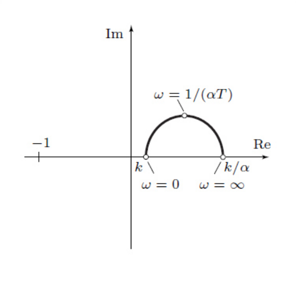
Poles/Zeros: $\pi_1 = -\frac{1}{\alpha \cdot T}$, $\zeta_1 = -\frac{1}{T}$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\frac{1}{\alpha \cdot T} \cdot x(t) + \frac{1}{\alpha \cdot T} \cdot u(t)$

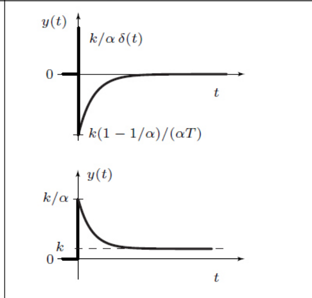
$y(t) = \frac{k \cdot (\alpha - 1)}{\alpha} \cdot x(t) + \frac{k}{\alpha} \cdot u(t)$

Phase maximum: $\hat{\varphi} = \arctan(1/\sqrt{\alpha}) - \arctan(\sqrt{\alpha})$ at $\hat{\omega} = (T \cdot \sqrt{\alpha})^{-1}$

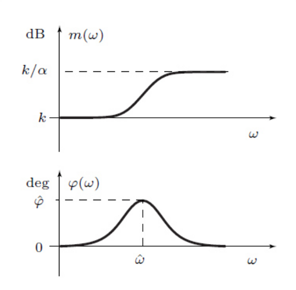
Nyquist Diagram



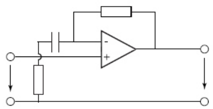
Impulse/Step Response



Bode Diagram



Analog/Digital Realization



```
...
analog_input(e_k);
u_k=u_k_1*(2*T*T_s*alpha)/ ...
    (T_s+2*T*alpha)+...
    e_k*k(T_s+2*T)/(T_s+2*T*alpha)+...
    e_k_1*k*(T_s-2*T)/(T_s+2*T*alpha);
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

A.8 PID Element

Element Acronym: PID

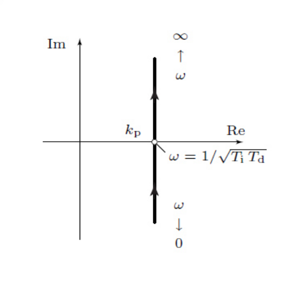
Transfer Function: $\Sigma(s) = k_\mathrm{p} \cdot \frac{T_\mathrm{d} \cdot T_\mathrm{i} \cdot s^2 + T_\mathrm{i} \cdot s + 1}{T_\mathrm{i} \cdot s} = k_\mathrm{p} \cdot (1 + \frac{1}{T_\mathrm{i} \cdot s} + T_\mathrm{d} \cdot s$

Poles/Zeros: $\pi_1 = 0$, $\pi_2 = \infty$, $\zeta_{1,2} = -\frac{1}{2 \cdot T_\mathrm{d}} \pm \sqrt{\frac{1}{4 \cdot T_\mathrm{d}^2} - \frac{1}{T_\mathrm{i} \cdot T_\mathrm{d}}}$

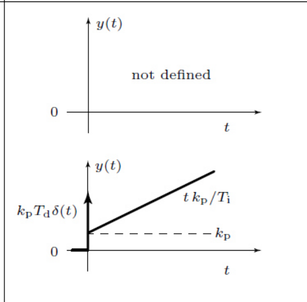
Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = \frac{1}{T_i} \cdot u(t)$

$y(t) = k_\mathrm{p} \cdot \left(u(t) + x_1(t) + T_\mathrm{d} \cdot \frac{\mathrm{d}}{\mathrm{d}t}u(t)\right)$

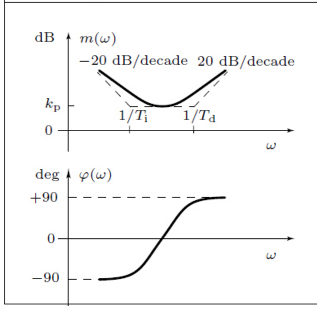
Nyquist Diagram



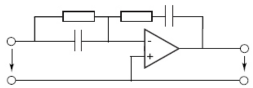
Impulse/Step Response



Bode Diagram



Analog/Digital Realization



```
...
analog_input(e_k);
u_k=e_k*kp*(1+T_s/(2*T_i)+2*T_d/T_s)+ ...
    e_k_1*kp*(T_s/T_i-4*T_d/T_s)+...
    e_k_2*kp*(-1+T_s/(2*T_i)+2*T_d/T_s)+...
    u_k_2;
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

A.9 First-Order All-Pass Element

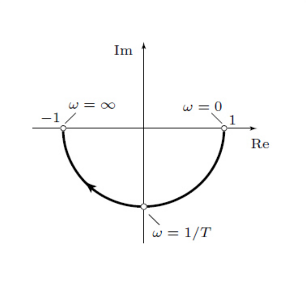
Element Acronym: AP-1

Transfer Function: $\Sigma(s) = \frac{-T\cdot s+1}{T\cdot s+1} = -1 + \frac{2}{T\cdot s+1}$

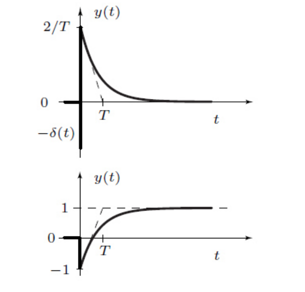
Poles/Zeros: $\pi_1 = -\frac{1}{T}$, $\zeta_1 = \frac{1}{T}$

Internal Description: $\frac{d}{dt}x(t) = -\frac{1}{T}\cdot x(t) + \frac{1}{T}\cdot u(t)$
$y(t) = 2\cdot x(t) - u(t)$

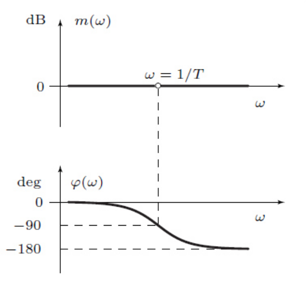
Nyquist Diagram



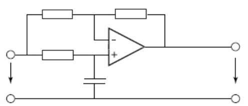
Impulse/Step Response



Bode Diagram



Analog/Digital Realization



```
...
analog_input(e_k);
u_k=(e_k-u_k_1)*(T_s-2*T)/(T_s+2*T)+e_k_1;
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

A.10 Delay Element

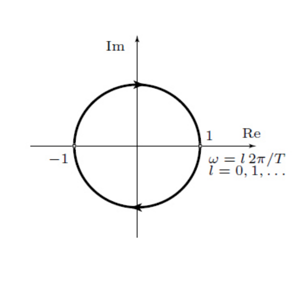
Element Acronym: -

Transfer Function: $\Sigma(s) = e^{-s\cdot T}$

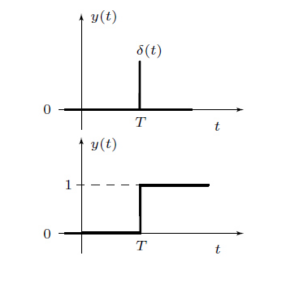
Poles/Zeros: not a real-rational element

Internal Description: $y(t) = u(t-T)$

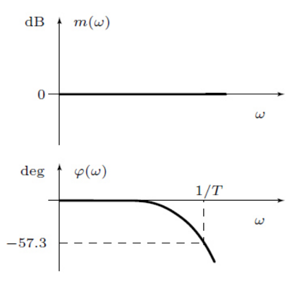
Nyquist Diagram



Impulse/Step Response



Bode Diagram



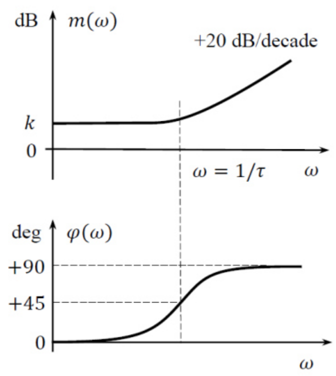
Analog/Digital Realization

Analog: use Padé elements (allpass elements) as approximation

```
KTZ=integer(T/T_s);
...
analog_input(e_k);
u_k=e_alt(KTZ);
analog_output(u_k);
for i=1:KTZ-1
        e_alt(i+1)=e_alt(i);
end;
e_alt(1)=e_k;
...
```

Minimumphase Zero

$$\Sigma(s) = k\cdot(\tau\cdot s+1)$$

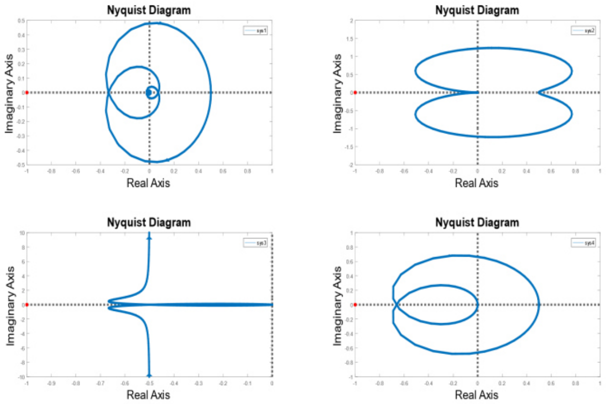


Examples

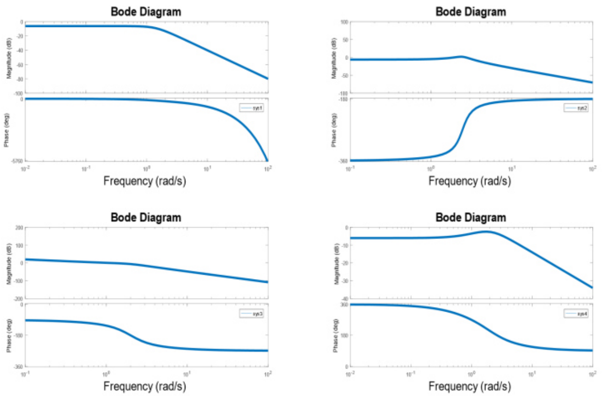
Legend:

1. $sys1 = \frac{1}{s^2+2s+2} \cdot e^{-s}$
2. $sys2 = \frac{3}{s^2-s+6}$
3. $sys3 = \frac{4}{s^3+2s^2+4s}$
4. $sys4 = \frac{-2s+2}{s^2+3s+4}$

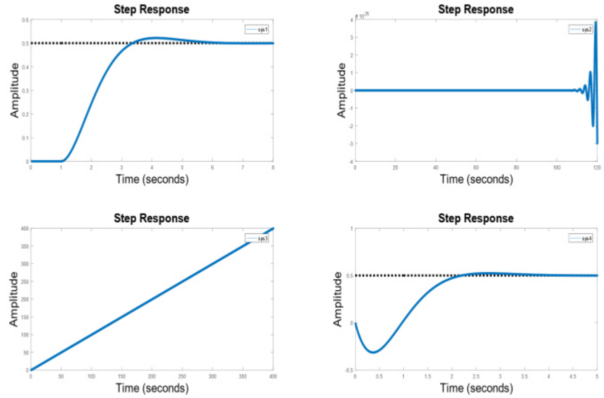
Nyquist



Bode

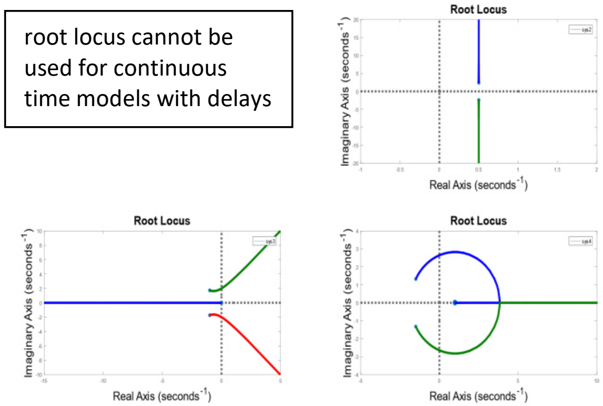


Step Responses



Root Locus

root locus cannot be used for continuous time models with delays



Appendix

Mechanics

Energy

Spring energy	$E_{Feder} = \frac{1}{2}k \cdot (x - x_0)^2$
Kinetic energy	$E_{Kin} = \frac{1}{2}m\dot{x}(t)^2$
Potential energy	$E_{Pot} = m \cdot g \cdot (x - x_0)$
Rotational energy	$E_{Rot} = \frac{1}{2}\theta\omega^2(t)$

Forces

Spring force	$F = k \cdot (x - x_0)$
Damping force	$F = d \cdot \dot{x}$

Power

Translational Power	$P_t = F \cdot \dot{x}(t)$
Rotational Power	$P_r = M \cdot \dot{\varphi}$

Partial fraction expansion

$$\frac{s}{(s-1)^2(s+1)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$\frac{s}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s^2+1} + \frac{Cs}{s^2+1}$$

$$\frac{1}{s^2(s-1)} = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}$$

$$\frac{1}{s^2(s+1)} = \frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}$$

Trigonometric functions

$\alpha[°]$ / $\alpha[rad]$	0 / 0	30 / $\frac{\pi}{6}$	45 / $\frac{\pi}{4}$	60 / $\frac{\pi}{3}$	90 / $\frac{\pi}{2}$
$sin(\alpha)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$cos(\alpha)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$tan(\alpha)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$
$cot(\alpha)$	$\pm\infty$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Euler equations

$$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$$

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$\tan(x) = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\sinh(0) = 0$$

$$\cosh(0) = 1$$

$$e^{ix} = \cos(x) + i \cdot \sin(x)$$

Conversion ODG n^{th} order to ODE 1st order

Consider the general ODE:

$$y^{(n)} = a_0 y + a_1 y' + a_2 y'' + \cdots + a_{n-1} y^{(n-1)}$$

Substitute

$$y_0 := y$$
$$y_1 := y'$$
$$y_2 := y''$$
$$\vdots$$
$$y_{n-1} := y^{(n-1)}$$

We form a matrix

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix} \cdot \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{pmatrix}$$

Eigenvalue problem

The eigenvalue of a Matrix $A \in \mathbb{R}^{n \times n}$ are the solution of the equation

$$\det(\lambda\mathbb{I} - A) = 0$$

The eigenspace v_λ of an eigenvalue λ solves this equation:

$$kern(\lambda\mathbb{I} - A)$$

Or equivalent

$$(\lambda\mathbb{I} - A) \cdot v = 0$$

Matrix Exponential

Consider a matrix $A \in \mathbb{R}^{n \times n}$. The matrix exponential is

$$e^{At} := \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Remark: $A^0 = \mathbb{I}$

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^{At} = \sum_{k=0}^{\infty} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^k \cdot \frac{t^k}{k!}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^0 \cdot \frac{t^0}{1} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \frac{t^1}{1} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 \cdot \frac{t^2}{2} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot t^0 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot t + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \frac{t^2}{2} + \cdots$$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Matlab

General Commands

Command	Description
`A(i,j)`	Matrix(Zeile, Spalte)
`abs(X)`	Betrag
`angle(X)`	Phase in Bogenmas
`X'`	Transponierte & complex konjugiert
`X.'`	Transponierte ohne complex konjugiert
`conj(X)`	Complex konjugiert
`real(X)`	Realteil
`imag(X)`	Imaginärteil
`eig(A)`	Eigenwerte
`[V,D]=eig(A)`	Eigenwerte D und Eigenvektoren V
`s=svd(A)`	Singulärwert
`[U,Sigma,V]=svd(A)`	Singular value decomposition
`rank(A)`	Rang
`det(A)`	Determinante
`inv(A)`	Inverse
`diag([a1,...,an])`	Diagonalmatrix
`zeros(x,y)`	Nullmatrix
`zeros(x)`	
`eye(x,y)`	Identitätsmatrix
`eye(x)`	
`ones(x,y)`	Matrix mit allen Einträgen =1
`ones(x)`	
`max(A)`	Grösstes Element in Vektor
`min(A)`	Kleinstes Element in Vektor
`sum(A)`	Summer aller Elemente
`dim=size(A)`	Dimension der Matrix
`dim=size(A,a)`	a=1→dim Zeilen; a=2→dim Spalten
`t=a:i:b`	Zeilenvektor(Anfangswert, schrittgrösse,Endwert)
`y=linspace(a,b)`	Zeilenvektor mit 100 "linear-spaced" Punkte im Intervall [a,b]
`y=linspace(a,b,n)`	n : Anzahl Punkte
`y=logspace(a,b)`	Zeilenvektor mit 50 "logarithmically-spaced" Punkte im Intervall [10^a,10^b]
`y=logspace(a,b,n)`	n : Anzahl Punkte
`I=find(A)`	Indizen von nichtnull Elemente von A
`disp(A)`	Auf Kommandozeile ausgeben

Control Systems Commands

Command	Description
`sys=ss(A,B,C,D)`	State Space M im Zeitbereich
`sys=ss(A,B,C,D,Ts)`	Ts= sampling Zeit
`sys=zpk(Z,P,K)`	State Space M. mit Nullstellen Z, Pole P und Gain K
`sys=zpk(Z,P,K,Ts)`	
`sys=tf([bm ...b0],[an ...a0])`	Übertragungsfkt., b:Zähler, a:Nenner
`P=tf(sys)`	Übertragungsfkt. Von sys
`P.iodelay=...`	Mit Todzeit
`pole(sys)`	Pole
`zero(sys)`	Nullstellen
`[z,p,k]=zpkdata(sys)`	z: Nullstellen, p: Pole, k: statische Verstärkung
`ctrb(sys)` oder `ctrb(A,b)`	Steuerbarkeitsmatrix
`obsv(sys)` oder `obsv(A,c)`	Beobachtbarkeitsmatrix
`series(sys1,sys2)`	Serieschaltung
`feedback(sys1,sys2)`	sys1 mit sys2 als (negative) Feedback
`[Gm,Pm,Wgm,Wpm]=margin(sys)`	Gm: Verstärkungsreserve, Pm: Phasenreserve, Wpm: Durchtrittsfrequenz
`[y,t]=step(sys,Tend)`	y: Sprungantwort von sys bis T, t: Zeit
`[y,t]=impulse(sys,Tend)`	Impulsantwort
`y=lsim(sys,u,t)`	Simulation von sys mit dem Input u für die Zeit t
`sim('Simulink model',Tend)`	Simultion von Simulink Model' bis Tend
`p0=dcgain(sys)`	Statische Verstärkung (P(0))
`K=lqr(A,B,Q,R)`	Verstärkungsmatrix K (Lösung des LQR-Problems)

Plotting Diagrams

Command	Description
`nyquist(sys)`	Nyquist Diagram
`nyquist(sys,fa,bg)`	Im Intervall [a,b]
`bode(sys)`	Bode Diagram
`bode(sys,fa,bg)`	Im Intervall [a,b]
`bodemag(sys)`	Nur Magnitude Plot
`bodemag(sys,fa,bg)`	
`rlocus(sys)`	Root locus
`impulse(sys)`	Impulsantwort
`step(sys)`	Sprungantwort
`pzmap(sys)`	Pole-Nullstelle Map
`svd(sys)`	Singularwertverlauf
`plot(X,Y)`	Plot von Y als Funktion von X
`plot(X,Y,...,Xn,Yn)`	
`stem(X,Y)`	Diskreter Plot von Y als Funktion von X
`stem(X,Y,...,Xn,Yn)`	
`xlabel('name')`	x-Achsen Name
`ylabel('name')`	y-Achsen Name
`title('name')`	Titel
`xlim([a b])`	Schranke für x-Achse
`ylim([a b])`	Schranke für y-Achse
`grid on`	Gitter ein
`legend('name1',..., 'namen')`	Legende
`subplot(m,n,p)`	Mehere Plots in Figur, m: Zeilen, n: Spalten, p: Position
`semilogx(X,Y)`	Logarithmischer Plot mit y-Achse linear

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