4. Recital 11.10.24

Recap

System Classification

Generally we classify system in these categories:

→ Linear vs. Nonlinear

→ Causal vs. Non-causal

→ Static (memoryless) vs. Dynamic

→ Time invariant vs. Time-varying

Linearity:

For a system to be linear two conditions have to be fullfilled.

→ Additivity: $\Sigma(u_1+u_2)=\Sigma u_1+\Sigma u_2$

→ Homogeneity: $\Sigma k u = k\Sigma u \quad , \quad k\in\mathbb{R}$

Differentiation and integration are linear operations!

We can summarize both to:

$$\Sigma(\alpha u_1+\beta u_2) = \alpha\Sigma u_1+\beta\Sigma u_2 = \alpha y_1+\beta y_2 \qquad \alpha,\beta\in\mathbb{R}$$

This implies the idea of superposition.

Causality:

A system is said to be causal, iff the future input does not affect the present output. All practically realizable systems are causal. Otherwise you could predict the future.

Static vs. Dynamic

An input-output system Σ is static or memoryless if for all t, $y(t^*)$ is only a function of $u(t^*)$.

In other words: the present output depends only on the present input and not on past or future inputs.

Time invariant vs. Time-varying:

A time invariant system will always have the same output to a certain input, independent of when the input is applied.

But what kind of systems do we care about ?

- → Linear
- → Time invariant
- → Causal
- → Single input, Single output

$\Big\}$ LTI SISO Systems

very restrictive class of systems
many systems can be well approximated by LTI SISO systems

One characteristic of LTI systems, is that we can write the state space model in the form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

where $A, B, C,$ and D are constant matrices or vectors. The vector $x = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ is the state of the system. It contains the information needed, together with the current input, to uniquely predict future outputs. The dimension n of the state vector $x \in \mathbb{R}^n$ is equal to the dimension of the system. Every different choice of states is called realization (there are infinitley many realizations). If we choose a realization with the smallest possible state vector, it is called minimal realization.

Linearization:

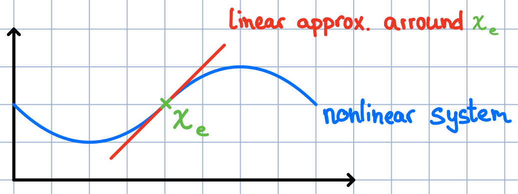
When Linearizing, we take some finite-dimensional, time-invariant, causal nonlinear system and approximate it as a LTI system. This approximation works very well and let's us use control systems designed for LTI systems even on nonlinear systems.

The main idea is to pick an equilibrium point of the system and then make a linear approximation arround that equilibrium point. For a system modeled with ODEs $\dot{x}(t) = f(x(t), u(t))$, we can define an equilibrium point (x_e, u_e) to be at

$$f(x_e, u_e) = 0$$

Next we use the Jacobian Linearization procedure. Where we do a Taylor Series expansion arround (x_e, u_e) of the nonlinear system's dynamic.

Graphically you can think of this approach as follows:



Important: we always linearize a system arround some x_e, u_e! If we linearize near another equilibrium point x_e, u_e we will get a different state space representation.

This way of approximating nonlinear systems as LTI systems works well if we stay close to the equilibrium point.

Time Response

We know now how to represent physical systems with ODEs and how to uniformally represent them in the LTI state space form:

$$\dot{x}(t) = A x(t) + B u(t)$$

$$y(t) = C x(t) + D u(t)$$

But how do we compute the solution of these ODEs?

First, we can take advantage of linearity of our equations. Since the system is linear, we can consider two seperate cases:

→ Initial-conditions response: $\begin{cases} x_{IC}(0) = x_0 \\ u_{IC}(t) = 0 \quad , t \geq 0 \end{cases} \rightarrow y_{IC}$

→ Forced response: $\begin{cases} x_F(0) = 0 \\ u_F(t) = u(t) \quad , t \geq 0 \end{cases} \rightarrow y_F$

After solving each case separatley we just add y_{IC} and y_F to get the complete output. This separation allows us to analyze the effects of non-zero initial conditions and non-zero inputs separatley.

In the following we will solve both cases for a first-order system i.e. $A, B, C,$ and D are scalars.

Initial condition (homogeneous) response:

In this case the initial condition is $x(0) = x_0$ and the input $u(t) = 0$.

$$\begin{cases} \dot{x}(t) = a\,x(t) & ,\ x(0) = x_0 \\ y(t) = c\,x(t) \end{cases}$$

The solution is given by:

$$x_{IC}(t) = e^{at} x_0 = \phi(t) x_0$$

$$y_{IC}(t) = c\,e^{at} x_0 = c\,\phi(t) x_0$$

Where $\phi(t) = e^{at}$ is the state-transition function.

Forced response:

In this case the initial condition is $x(0) = 0$ and the input $u(t) \neq 0$.

$$\begin{cases} \dot{x}(t) = a\,x(t) + b\,u(t) & ,\ x(0) = 0 \\ y(t) = c\,x(t) + d\,u(t) \end{cases}$$

The solution is given by: (derivation in lecture)

$$x_F(t) = \int_0^t e^{a(t-\tau)}\, b\,u(\tau)\,d\tau = \int_0^t \phi(t-\tau)\, b\,u(\tau)\,d\tau$$

$$y_F(t) = c\int_0^t e^{a(t-\tau)}\, b\,u(\tau)\,d\tau + d\,u(t) = c\int_0^t \phi(t-\tau)\, b\,u(\tau)\,d\tau + d\,u(t)$$

Complete response: (First-order system)

Due to linearity, we can just sum up both cases. Therefore $x = x_{IC} + x_F$ and $y = y_{IC} + y_F$.

$$x(t) = e^{at} x_0 + \int_0^t e^{a(t-\tau)}\, b\,u(\tau)\,d\tau$$

$$y(t) = \underbrace{c\,e^{at} x_0}_{\text{natural response}} + \underbrace{c\int_0^t e^{a(t-\tau)}\, b\,u(\tau)\,d\tau}_{\text{forced response}} + \underbrace{d\,u(t)}_{\text{Feedthrough}}$$

Remember that a, b, c, and d are scalars here.

Complete response: (Higher-order system)

In general $A, B, C,$ and D are not scalars like above. Fortunatley the solution looks very similar.

Keep in mind that whenever $A, B, C,$ and D are matrices you have to maintain the order of multiplication!

The solution for the state space LTI system is given by:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

If we take a closer look, we see that some terms contain the matrix exponential e^{At}.

But how do we compute e^{At}?

Throwback: Linear Algebra II

The matrix exponential can be defined through a Taylor-Series: (also valid for scalars)

$$e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n = I + At + \frac{1}{2}(At)^2 + \dots + \frac{1}{n}(At)^n$$

Therefore we would have to calculate infinetley many terms. But for some matrices we can drastically simplify the calculations:

→ Diagonal: $\exp\left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} t\right) = \begin{bmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{bmatrix}$

→ Jordan Form: $\exp\left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t\right) = \begin{bmatrix} \exp(\lambda t) & t\exp(\lambda t) \\ 0 & \exp(\lambda t) \end{bmatrix}$

Where λ_i are the eigenvalues of the respective matrix.

To facilitate calculations we can therefore do a coordinate transformation, $x = T\tilde{x}$ such that e^{At} is easier to compute:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \rightarrow \begin{cases} T\dot{\tilde{x}}(t) = AT\tilde{x}(t) + Bu(t) \\ y(t) = CT\tilde{x}(t) + Du(t) \end{cases}$$

$$\begin{cases} \dot{\tilde{x}}(t) = (T^{-1}AT)\tilde{x}(t) + (T^{-1}B)u(t) \\ y(t) = CT\tilde{x}(t) + Du(t) \end{cases}$$

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ y(t) = \tilde{C}\tilde{x}(t) + \tilde{D}u(t) \end{cases}$$

For a matrix $A \in \mathbb{R}^{n \times n}$ with n eigenvalues $\lambda_1, \dots, \lambda_n$ and n linearly independent eigenvectors $v_1, \dots, v_n$ one can do a coordinate transformation such that

$$\tilde{A} = T^{-1}AT = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

where $\tilde{A}$ is a diagonal matrix with the eigenvalues $\lambda_1, \dots, \lambda_n$ on the diagonal and T a transformation matrix containing the eigenvectors $v_1, \dots, v_n$ as columns.

Note that the time response remains unchanged. Through the coordinate transformation, we simply use a different realization of the system i.e. a different state vector.

Initial condition (homogeneous) responses:

Let us now take a closer look at systems where A is diagonal. More specific we will look at the initial condition response, i.e. $u(t) = 0$.

→ For a diagonal, real matrix:

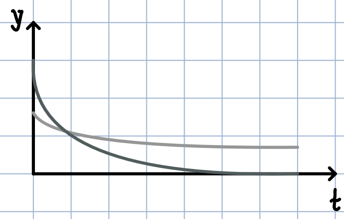
$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \lambda_i \in \mathbb{R}$$

$$y(t) = Ce^{At}x_0$$

where we can write out all terms and simplify for A being diagonal.

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$y(t) = c_1 e^{\lambda_1 t} x_1(0) + c_2 e^{\lambda_2 t} x_2(0)$$



So for diagonal, real matrices the initial condition response is the linear combination of two exponentials.

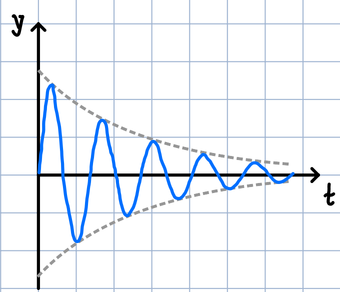
→ For a diagonal, complex matrix:

$$A = \begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix}$$

$$y(t) = Ce^{At}x_0$$

where we can write out all terms and simplify for A being diagonal.

$$\begin{aligned} y(t) &= c_1 e^{\sigma t} e^{j\omega t} x_1(0) + c_2 e^{\sigma t} e^{-j\omega t} x_2(0) \\ &= e^{\sigma t}(c_1 e^{j\omega t} x_1(0) + c_2 e^{-j\omega t} x_2(0)) \\ &= e^{\sigma t}(\alpha_1 \sin(\omega t) + \alpha_2 \cos(\omega t)) \\ &= \alpha e^{\sigma t} \sin(\omega t + \phi) \end{aligned}$$



Stability

In the two cases above we can see that the output is somehow linked to exponential terms.

$$y(t) = c_1 e^{\lambda_1 t} x_1(0) + c_2 e^{\lambda_2 t} x_2(0)$$

$$y(t) = \alpha e^{\sigma t} \sin(\omega t + \phi)$$

The growth of these terms is dictated by the real part of the eigenvalues of A. We can see that if the eigenvalues λ have a positive real part, the output will grow exponentially over time, i.e. become unstable. ($y \to \infty$)

But what does stability really mean? There are few ways to classify stability:

→ Lyapunov Stability: a system is Lyapunov stable if, for any bounded initial condition, and zero input, the state remains bounded, i.e.:

$$\|x_0\| < \epsilon, \text{ and } u = 0 \Rightarrow \|x(t)\| < \delta \quad \forall t \geq 0$$

→ Asymptotic Stability: a system is asymptotically stable if, for any bounded initial condition, and zero input, the state converges to zero, i.e.:

$$\|x_0\| < \epsilon, \text{ and } u = 0 \Rightarrow \lim_{t \to \infty} \|x(t)\| = 0$$

→ Bounded-Input, Bounded-Output Stability: a system is BIBO-stable if, for any bounded input, the output remains bounded, i.e.:

$$\|u(t)\| < \epsilon \quad \forall t \geq 0, \text{ and } x_0 = 0 \Rightarrow \|y(t)\| < \delta \quad \forall t \geq 0$$

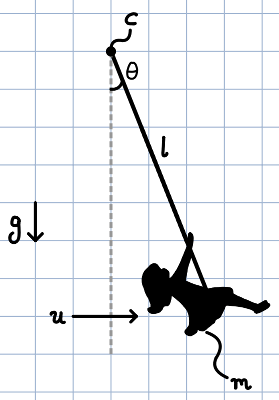
Every system that is not stable, is called unstable.

We can check stability by looking at the eigenvalues of A!

→ Lyapunov stable if $\text{Re}(\lambda_i) \leq 0 \quad \forall i$

→ Asymptotically stable if $\text{Re}(\lambda_i) < 0 \quad \forall i$

Example:



Modeling $\Longrightarrow$

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{1}{ml^2}\left[-lmg \sin x_1(t) - c\, x_2(t) + l \cos x_1(t)\, u(t)\right] \\ y(t) = x_1(t) \end{cases}$$

$\Downarrow$ Linearization

arround equilibrium point $x_e = (\pi, 0)$, $u_e = 0$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{c}{ml^2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{ml} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Lets solve this for the inital condition response, i.e. $x(0) = x_0$ and the input $u(t) = 0$.

Assume the following values for the internal parameters: $g = 10$, $l = 5$, $c = 25$, $m = 1$.

The initial condition is now given by:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\cancelto{0}{u(t)} \\ y(t) = Cx(t) + D\cancelto{0}{u(t)} \end{cases} \Longrightarrow \begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

We have to solve

$$\begin{bmatrix} e^t & -e^{-2t} \\ e^t & 2e^{-2t} \end{bmatrix} x_0$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x(t)$$

For that we need eigenvalues and eigenvectors: $\lambda_1 = 1$, $\lambda_2 = -2$, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

From LinAlg II we now know that: $x(t) = e^{At} x_0 = T \operatorname{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})\, T^{-1} x_0$

$$x(t) = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-2t} \end{pmatrix} \underbrace{\frac{1}{3} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} x_0}$$

$$x(t) = \begin{pmatrix} e^t & -e^{-2t} \\ e^t & 2e^{-2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

contains informations about I.C.

And now the output is given by:

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e^t & -e^{-2t} \\ e^t & 2e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 e^t - c_2 e^{-2t}$$

This system is unstable since $Re(\lambda_1) = 1 > 0$, also $\lim_{t\to\infty} c_1 e^{t} - c_2 e^{-2t} = \infty$

Let's quickly look at the other equilibrium point: $x_e = (0,0)$, $u_e = 0$

We now have to look at

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} x(t)$$

The eigenvalues are given by: $\lambda_1 = \frac{1}{2}(-1 + i\sqrt{7})$, $\lambda_2 = \frac{1}{2}(-1 - i\sqrt{7})$

Here $Re(\lambda_i) < 0$ $\therefore$ Asymptotically stable

Summary

Given a LTI SISO system in the state-space representation we got an expression to calculate the output $y(t)$. The time dependent output is also call time-response

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \Longrightarrow \quad \begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \\ y(t) &= Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t) \end{aligned}$$

Given the time-response we can asses the stability of our system. We generally distinguish between three types of stability:

→ Lyapunov Stability: $\|x_0\| < \epsilon$, and $u = 0$ $\Rightarrow$ $\|x(t)\| < \delta \;\; \forall\, t \geq 0$

→ Asymptotic Stability: $\|x_0\| < \epsilon$, and $u = 0$ $\Rightarrow$ $\lim_{t\to\infty} \|x(t)\| = 0$

→ BIBO Stability: $\|u(t)\| < \epsilon \;\; \forall\, t \geq 0$, and $x_0 = 0$ $\Rightarrow$ $\|y(t)\| < \delta \;\; \forall\, t \geq 0$