4. Recital 11.10.24

Recap

Time response:

We know now how to represent physical systems with ODEs and how to uniformally represent them in the LTI state space form:

$$\dot{x}(t) = A x(t) + B u(t)$$
$$y(t) = C x(t) + D u(t)$$

The solution for the state space LTI system is given by:

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

$$y(t) = C e^{At} x_0 + C \int_0^t e^{A(t-\tau)} B u(\tau) d\tau + D u(t)$$

Initial condition (homogeneous) responses:

Let us now take a closer look at systems where A is diagonal. More specific we will look at the initial condition response, i.e. $u(t) = 0$.

→ For a diagonal, real matrix:

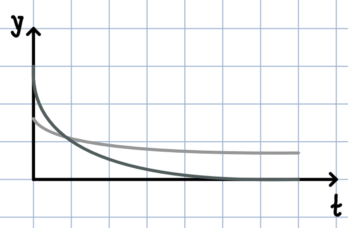
$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \lambda_i \in \mathbb{R}$$

$$y(t) = C e^{At} x_0$$

where we can write out all terms and simplify for A being diagonal.

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$y(t) = c_1 e^{\lambda_1 t} x_1(0) + c_2 e^{\lambda_2 t} x_2(0)$$



So for diagonal, real matrices the initial condition response is the linear combination of two exponentials.

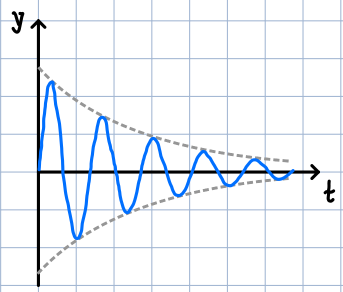
→ For a diagonal, complex matrix:

$$A = \begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix}$$

$$y(t) = Ce^{At}x_0$$

where we can write out all terms and simplify for A being diagonal.

$$\begin{aligned} y(t) &= c_1 e^{\sigma t} e^{j\omega t} x_1(0) + c_2 e^{\sigma t} e^{-j\omega t} x_2(0) \\ &= e^{\sigma t}\left(c_1 e^{j\omega t} x_1(0) + c_2 e^{-j\omega t} x_2(0)\right) \\ &= e^{\sigma t}\left(\alpha_1 \sin(\omega t) + \alpha_2 \cos(\omega t)\right) \\ &= \alpha e^{\sigma t} \sin(\omega t + \phi) \end{aligned}$$



Stability:

In the two cases above we can see that the output is somehow linked to exponential terms.

$$y(t) = c_1 e^{\lambda_1 t} x_1(0) + c_2 e^{\lambda_2 t} x_2(0)$$

$$y(t) = \alpha e^{\sigma t} \sin(\omega t + \phi)$$

The growth of these terms is dictated by the real part of the eigenvalues of A. We can see that if the eigenvalues λ have a positive real part, the output will grow exponentially over time, i.e. become unstable. ($y \to \infty$)

Given the time-response we can asses the stability of our system. We generally distinguish between three types of stability:

→ Lyapunov Stability: $\|x_0\| < \epsilon$, and $u = 0 \Rightarrow \|x(t)\| < \delta \ \forall t \geq 0$

→ Asymptotic Stability: $\|x_0\| < \epsilon$, and $u = 0 \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0$

→ BIBO Stability: $\|u(t)\| < \epsilon \ \forall t \geq 0$, and $x_0 = 0 \Rightarrow \|y(t)\| < \delta \ \forall t \geq 0$

We can check stability by looking at the eigenvalues of A!

→ Lyapunov stable if $Re(\lambda_i) \leq 0 \ \forall i$

→ Asymptotically stable if $Re(\lambda_i) < 0 \ \forall i$

Time Response
contd.

Forced response:

We have seen how systems react to initial conditions, i.e. how the homogeneous solution $Ce^{At}x_0$ behaves. It gives us a feeling about the natural dynamics of the system. But what about the forced response, given by the convolution integral:

$$C\int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau + D\,u(t)$$

This is harder to interpret. To gain some intuition, we will look at how elementary inputs affect the system. Since we are dealing with linear(ized) systems, we can later construct more complex inputs by superposition.

Step-response of 1st order system:

Let's look at a simple example where we apply a step input, given by the Heaviside function

$$u(t) = h(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$$ as input to the first order system: $$\dot{x}(t) = -\frac{1}{\tau}x(t) + \frac{k}{\tau}u(t)$$ $$y(t) = 1\,x(t)$$

$$u(t)=h(t) \longrightarrow \boxed{\begin{aligned} \dot{x}(t) &= -\tfrac{1}{\tau}x(t) + \tfrac{k}{\tau}u(t) \\ y(t) &= 1\;x(t) \end{aligned}} \longrightarrow y(t)$$

we can compute the output $y(t)$ as follows:

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-p)}B\,\cancel{u(p)}^{\,1}\,dp + \cancel{D}^{\,0}u(t)$$

$$e^{-\frac{1}{\tau}t}x_0 + \int_0^t e^{-\frac{1}{\tau}(t-p)}\frac{k}{\tau}\,dp$$

$$= e^{-\frac{1}{\tau}t}x_0 + \frac{k}{\tau}e^{-\frac{1}{\tau}t}\int_0^t e^{\frac{1}{\tau}p}\,dp$$

$$= e^{-\frac{1}{\tau}t}x_0 + \frac{k}{\tau}e^{-\frac{1}{\tau}t}\left[\tau e^{\frac{1}{\tau}p}\right]_0^t$$

$$= e^{-\frac{1}{\tau}t}x_0 + k\,e^{-\frac{1}{\tau}t}\left(e^{\frac{1}{\tau}t} - 1\right)$$

$$= e^{-\frac{t}{\tau}}x_0 + k\left(1 - e^{-\frac{t}{\tau}}\right)$$

General response:

Since we are working with linear systems, we can decompose any arbitrary input to simpler bits. So any u can be rewritten as $u = u_1 + \ldots + u_n$. We can then apply $u_1, \ldots, u_n$ seperatley to the system and sum all outputs $y = y_1 + \ldots + y_n$. In this case it would be convinient if we could find some input v that when linearly combined with itself could represent all, or most other signals, i.e. $u = a_1 v + \ldots + a_n v$.

Luckily we can use a mathematical tool from Analysis III to help us with that:

The Laplace Transform:

The Laplace Transform $\mathcal{L}$ and it's inverse are given by:

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi j} \lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s)\, e^{-st}\, ds$$

where s is a complex number of the form: $s = \sigma + j\omega$.

In Analysis III we mainly use the Laplace transform to solve ODE's. This characteristic is very important, but we will now look a bit closer at the inverse Laplace transform.

The inverse Laplace transform tells us, how to write a function $f(t)$ as a linear combination of terms e^{st} weighted by $F(s)$, the Laplace transform of $f(t)$.

So now we know that if we compute the output to some general e^{st} we can later easily compute the output to any input, since it will be a linear combination of e^{st} terms.

Let's see how the response to e^{st} will generally look like:

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau + D u(t)$$

with $u(t) = e^{st}$:

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)} B e^{s\tau}\, d\tau + D e^{st}$$

rearrange:

$$y(t) = Ce^{At}x_0 + Ce^{At}\underbrace{\int_0^t e^{(sI-A)\tau} B\, d\tau}_{} + D e^{st}$$

if $(sI-A)$ is invertible:

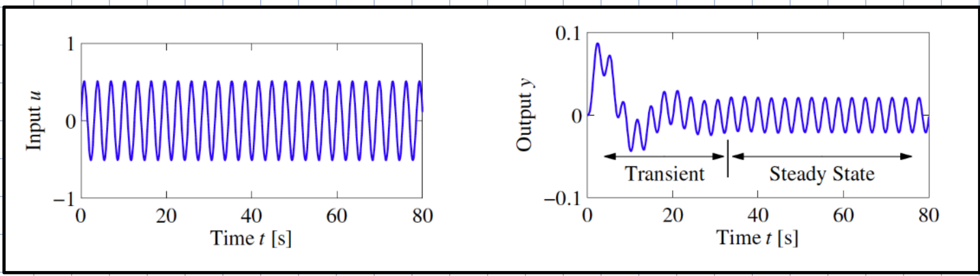
$$y(t) = Ce^{At}x_0 + Ce^{At}\overbrace{\left((sI-A)^{-1} e^{(sI-A)\tau} B\right)\Big|_0^t} + D e^{st}$$

rearrange:

$$y(t) = Ce^{At}x_0 + Ce^{At}(sI-A)^{-1}(e^{(sI-A)t}-I)B + De^{st}$$

and finally:

$$y(t) = \underbrace{Ce^{At}\left[x_0 - (sI-A)^{-1}B\right]}_{} + \underbrace{\left[C(sI-A)^{-1}B + D\right]e^{st}}_{}$$

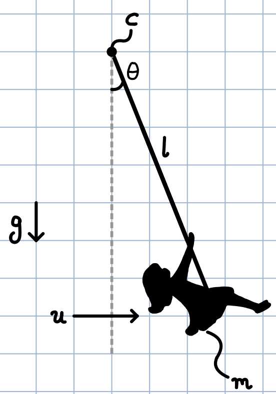


We can now generally say that the steady state response to the input e^{st} is given by

$$y_{ss} = G(s)\,e^{st} \qquad \text{with } G(s) = C(sI-A)^{-1}B + D \;\in \mathbb{C}$$

The complex function $G(s)$ is known as the transfer function and describes how a stable system G transforms an input e^{st} into the output $G(s)\,e^{st}$. You can thing of it like the Σ in the block diagrams.

<u>Example:</u>



arround equilibrium point $x_e = (0,0)$, $u_e = 0$

$g = 10$, $l = 5$, $c = 25$, $m = 1$.

$$\Rightarrow \quad A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Find the complex transfer function $G(s) = C(sI-A)^{-1}B + D$

But how exactly do we decompose some arbitrary input? Let's see how this looks if our input is a sinusoid, e.g.:

$$u(t) = \cos(\omega t) =$$

We notice that we can decompose $u(t)$ as follows:

$$u(t) = \sum_i U_i e^{s_i t} \qquad \text{with } U_{1,2} = \qquad \text{and } s_{1,2} =$$

The output is then given by

$$y(t) = G(j\omega)\tfrac{1}{2}e^{j\omega t} + G(-j\omega)\tfrac{1}{2}e^{-j\omega t}$$

or

$$y(t) = \sum_i G(s_i) U_i e^{s_i t}$$

we can rewrite $G(j\omega)$ as $Me^{j\phi}$ with $M = |G(j\omega)|$ Magnitude of $G(j\omega)$
$\phi = \angle G(j\omega)$ Phase of $G(j\omega)$

Magnitude

$A, B, C \in \mathbb{C}$

$$\left|A \cdot \frac{B}{C}\right| = |A| \cdot \frac{|B|}{|C|}$$

$$\left|\frac{(a+jb)^x}{(c+jd)^y}\right| = \frac{\left(\sqrt{a^2+b^2}\right)^x}{\left(\sqrt{c^2+d^2}\right)^y}$$

$$\left|e^{-j\omega \cdot T}\right| = |\cos(\omega \cdot T) - j \cdot \sin(\omega \cdot T)| = 1$$

Phase

$$\angle\left(A \cdot \frac{B}{C}\right) = \angle A + \angle B - \angle C$$

$$\angle\left(\frac{(a+jb)^q}{(c+jd)^k}\right) = q \cdot \angle(a + j \cdot b) - k \cdot \angle(c + j \cdot d)$$

$$= q \cdot \arctan\left(\frac{b}{a}\right) - \arctan\left(\frac{d}{c}\right)$$

$$\angle(a + j \cdot b) = \begin{cases} \arctan\left(\frac{b}{a}\right) & , a > 0, b \text{ beliebig} \\ \arctan\left(\frac{b}{a}\right) + \pi & , a < 0, b \geq 0 \\ \arctan\left(\frac{b}{a}\right) - \pi & , a < 0, b < 0 \end{cases}$$

$$\phi = \begin{cases} \frac{\pi}{2} & , a > 0, b \text{ beliebig} \\ -\frac{\pi}{2} & , a = 0, b < 0 \\ \text{unbestimmt} & , a = 0, b = 0 \end{cases}$$

$$\begin{cases} \arctan(\infty) = \frac{\pi}{2} \\ \arctan(-\infty) = -\frac{\pi}{2} \end{cases}$$

$$\begin{cases} \arctan(1) = \frac{\pi}{4} \\ \arctan(-1) = -\frac{\pi}{4} \\ \arctan(0) = 0 \end{cases}$$

$$\angle(e^{-j\omega \cdot T}) = \angle\left(\cos(\omega \cdot T) - j \cdot \sin(\omega \cdot T)\right) = -\omega \cdot T$$

$$\angle(a + jb)^c = c \cdot \arctan\left(\frac{b}{a}\right)$$

$$\lim(\angle(a + j \cdot b)^c) = c \cdot \lim(\angle(a + j \cdot b))$$

and then

$$y(t) = Me^{j\phi}\tfrac{1}{2}e^{j\omega t} + Me^{-j\phi}\tfrac{1}{2}e^{-j\omega t}$$

$$y(t) = M\cos(\omega t + \phi)$$

The output is another sinusoid with a different amplitude and phase, but <u>same frequency.</u>

In general we can say that:

$$u(t) = \sum_i U_i e^{s_i t} \implies y(t) = \sum_i G(s_i) U_i e^{s_i t}$$

We can now make use of the inverse Laplace transform that generalizes this sum, such that we can represent all inputs:

$$u(t) = \frac{1}{2\pi j} \lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} U(s) e^{-st} ds = \mathcal{L}^{-1}\{U(s)\}$$

the same logic can be applied to the outputs:

$$y(t) = \frac{1}{2\pi j} \lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} G(s) U(s) e^{-st} ds = \mathcal{L}^{-1}\{G(s)U(s)\}$$

This means that

$$\boxed{Y(s) = G(s)U(s)}$$

By using the Laplace transform the output can be computed by muliplying the input with the transfer function!

Another way to think about TFs is that they are the result of solving the LTI state space ODE by using the Laplace transform. For this we make use of the property that a time derivative is transformed to a multiplication by s. , i.e.:

$$\mathcal{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - \sum_{i=1}^{n} s^{n-i} \cdot \left.\frac{d^{(i-1)} x(t)}{dt^{(i-1)}}\right|_{t=0}$$

we can use this now with:
assume $d=0$

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(0) = x_0$$

$$y(t) = Cx(t)$$

Lets transform the first equation:

$$sX(s) - x_0 = AX(s) + BU(s)$$

$$X(s) = (sI-A)^{-1}BU(s) + (sI-A)^{-1}x_0$$

from this the output can be obtained:
for $x_0 = 0$

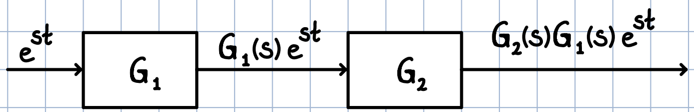
$$Y(s) = \underbrace{C(sI-A)^{-1}B}\,U(s) \iff Y(s) = G(s)\,U(s)$$

We still have to compute the Laplace transform but for that we have tables:

$x(t)$	$X(s)$
impulse: $\delta(t)$	1
step: $h(t)$	$\frac{1}{s}$
$h(t)\cdot t^n$	$\frac{n!}{s^{n+1}}$
$h(t)\cdot e^{at}$	$\frac{1}{s-a}$
$h(t)\cdot t^n \cdot e^{a\cdot t}$	$\frac{n!}{(s-a)^{n+1}}$
$h(t)\cdot \sin(\omega\cdot t)$	$\frac{\omega}{s^2+\omega^2}$
$h(t)\cdot \cos(\omega\cdot t)$	$\frac{s}{s^2+\omega^2}$
$h(t)\cdot \sinh(\omega\cdot t)$	$\frac{\omega}{s^2-\omega^2}$
$h(t)\cdot \cosh(\omega\cdot t)$	$\frac{s}{s^2-\omega^2}$
$k\cdot u(t-a)$	$k\cdot\frac{e^{-as}}{s}, a>0$
$\delta(t-a)$	$e^{-as}, a>0$

A very nice property of transfer functions is that system interconnections are easy to calculate.

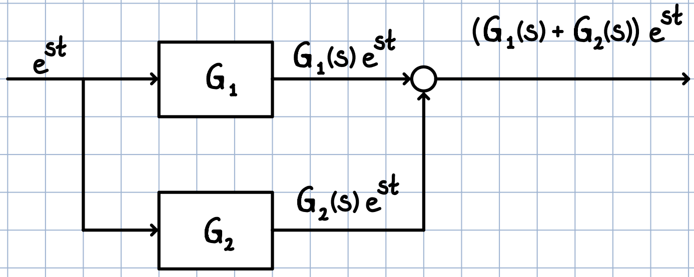
→ Serial interconnection:



$G(s) =$

→ Parallel interconnection:

$G(s) =$



Relationsphip between state-space representation and TF:

→ If A is diagonal:

$$G(s) = \frac{P_1}{s-\lambda_1} + \frac{P_2}{s-\lambda_2} + \dots + \frac{P_n}{s-\lambda_n} + d \iff A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} \sqrt{P_1} \\ \vdots \\ \sqrt{P_n} \end{bmatrix}$$

$$C = \begin{bmatrix} \sqrt{P_1} & \dots & \sqrt{P_n} \end{bmatrix}, \quad D = d$$

→ If A is in the controllable canonical form:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_0}{\underbrace{s^n + a_{n-1}s^{n-1} + \dots + a_0}_{\text{characteristic polynomial of } A}} \iff A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \ddots & \\ & & & & & 1 \\ -a_0 & -a_1 & \dots & & & a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

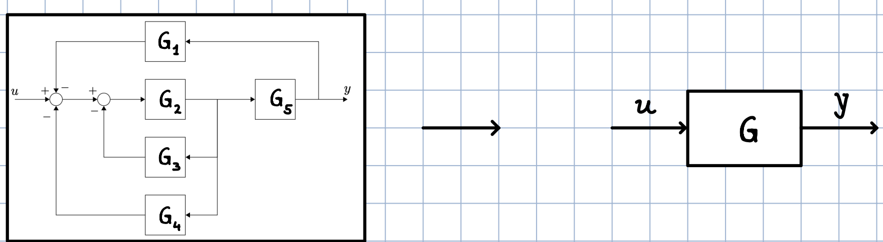
$$C = \begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \end{bmatrix}, \quad D = d$$

The roots of the denominator of $G(s)$ are called poles of the system. They are also the eigenvalues of A. So if we want do analyze the stability of a system, we have to look at it's poles. (We will talk a lot about poles!)

Now we have two ways to represent a system, but which representation is usefull for what?

1.) State-space to TF

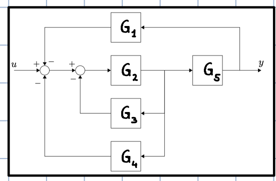
Say you are given a system with some architecture. And you want to know the input output relation. With the State-space representation you would have to solve a lot of ODE's, but with TFs you can just multiply everything together.



2.) TF to State-space

If we know a certain output we want to achieve for a given input we can do :

$$\frac{Y(s)}{U(s)} = G(s) \longrightarrow$$



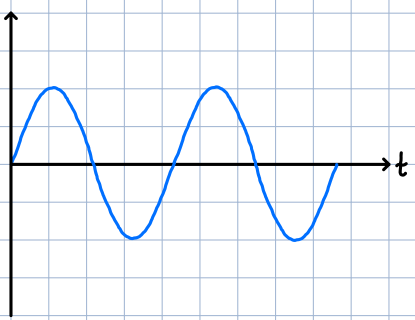
and from there derive the right system architecture.

S-Domain:

With the TFs we start to represent systems in a new domain. So instead of characterizing how systems behave in the time domain, i.e. $y(t)=\dots$, we start to look at how it behaves in the s-Domain, i.e. $Y(s)=G(s)\,U(s)$. But what exactly is the s-Domain? $(s=\sigma+j\omega)$

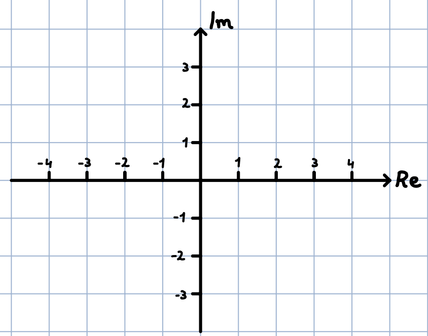
Lets look at examples: How can we represent time domain functions in the s-Domain?

time domain



$\sin(3t)$

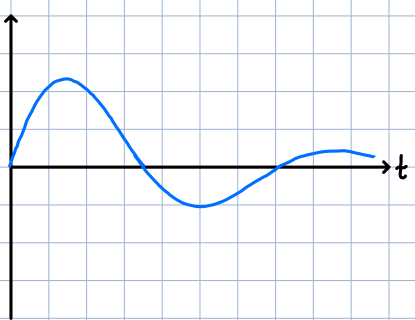
S-Domain



$\mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+9}$ with poles at

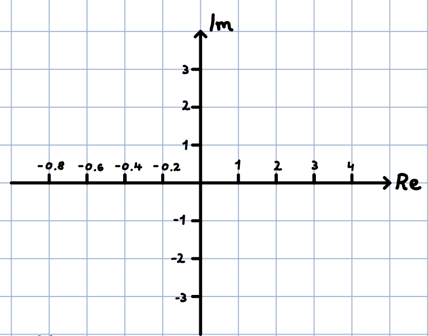
s =

time domain



$e^{-0,8t}\sin(3t)$

S-Domain

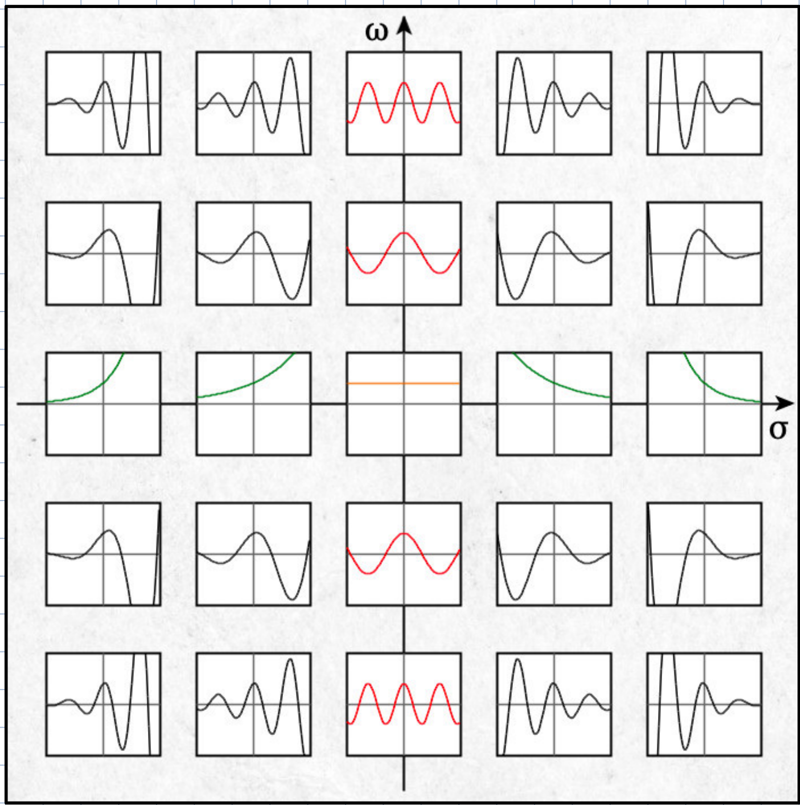


$\mathcal{L}\{e^{-0,8t}\sin(3t)\} = \frac{3}{(s+0,8)^2+9}$ with poles at

s =

For now you can think that in the s-Domain functions or systems are represented by their poles (and zeroes) plotted in the complex plane.

https://www.youtube.com/watch?v=n2y7n6jw5d0&t=711s ← video recomendation.


ω
σ

Old exam questions:

Problem: You are given a linear time-invariant system of the form:

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t),$$

$$y(t) = C \cdot x(t) + D \cdot u(t).$$

Q16 (1 Points)

Given:

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = -1.$$

Which of the following transfer functions $G(s)$ is equivalent to the given state space system. Mark the correct answer.

- ☐ $G(s) = -\frac{s^2-3\cdot s+2}{s^2-4\cdot s+1}$
- ☐ $G(s) = -\frac{(s+2)\cdot(s-1)}{s^2-4\cdot s+1}$
- ☐ $G(s) = \frac{s^2-3\cdot s+2}{s^2-4\cdot s+1}$
- ☐ $G(s) = -\frac{s^2+3\cdot s+2}{s^2-4\cdot s+1}$

Q17 **(1 Points)**

Given:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -4 & -1 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} \frac{1}{2} & -2 & 0 & -\frac{1}{3} & 0 \end{bmatrix}, D = 0.$$

Which of the following transfer functions $G(s)$ is equivalent to the given state space system. Mark the correct answer.

- ☐ $G(s) = \frac{\frac{1}{2}\cdot s^3 - 2\cdot s + \frac{1}{3}}{s^5 + 2\cdot s^4 + 4\cdot s^3 + 1\cdot s^2 + 6}$
- ☐ $G(s) = \frac{-\frac{1}{3}\cdot s^3 - 2\cdot s + \frac{1}{2}}{s^5 + 6\cdot s^4 + s^3 + 4\cdot s^2 + 2}$
- ☐ $G(s) = \frac{-\frac{1}{3}\cdot s^3 - 2\cdot s + \frac{1}{2}}{s\cdot(s^4 + 6\cdot s^3 + s^2 + 4\cdot s + 2)}$
- ☐ $G(s) = \frac{s\cdot(\frac{1}{2}\cdot s^3 - 2\cdot s + \frac{1}{3})}{s^5 + 2\cdot s^4 + 4\cdot s^3 + 1\cdot s^2 + 6}$

Todays Summary

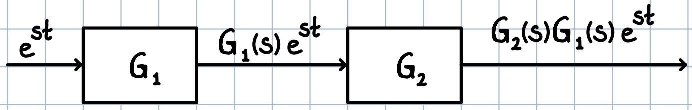
The complex function $G(s)$ is known as the transfer function and describes how a system reacts to some input.

$$Y(s) = \underbrace{C(sI-A)^{-1}B}_{G(s)}\,U(s) \iff Y(s) = G(s)\,U(s)$$

Where $Y(s)$ and $U(s)$ are Laplace transforms of $y(t)$ and $u(t)$ respectiveley.

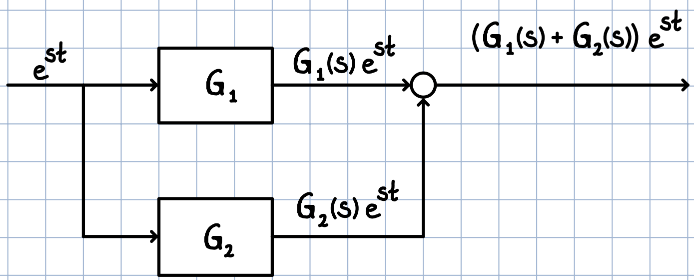
A very nice property of transfer functions is that system interconnections are easy to calculate.

→ Serial interconnection:



$$G(s) = G_2(s)G_1(s)$$

→ Parallel interconnection:



$$G(s) = G_2(s) + G_1(s)$$