6. Recital 25.10.24

Recap

Last week we discovered that for an asymptotically stable, causal, LTI system described by

$$y(t) = C e^{At} x_0 + C \int_0^t e^{A(t-\tau)} B u(\tau) d\tau + D u(t),$$

we can calculate the steady state output y_{ss} to the input e^{st} as follows:

$$y_{ss} = G(s) e^{st} \quad \text{with } G(s) = C(sI - A)^{-1} B + D$$

The complex function $G(s)$ is known as the transfer function and describes how a stable system G transforms an input e^{st} into the output $G(s) e^{st}$. You can thing of it like the Σ in the block diagrams. But why do we care about the response to e^{st}?

We discovered that most functions $f(t)$ can be written as a linear combination of terms e^{st}, where s is a complex number of the form: $s = \sigma + j\omega$.

In the context of control systems this means that any input $u(t)$ can be re-written as

$$u(t) = \sum_i U_i e^{s_i t}$$

with U_i representing the weights in the linear combination. Consequentley the output can be written as:

$$y(t) = \sum_i G(s_i) U_i e^{s_i t}$$

With the help of the Laplace transform we arrive at the expression:

$$Y(s) = G(s) U(s)$$

Where $Y(s)$ and $U(s)$ are the Laplace transform of $y(t)$ and $u(t)$ respectivley. From now on we will mainly work with TFs in the s-domain.

Transfer Functions
contd.

Ways to write TFs

Mostly we can write TFs as rational functions. We already saw an example last week:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$$

We can do partial fraction expansion and get:

$$G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \dots \frac{r_n}{s-p_n}$$

We can also factorize the numerator and denominator in different ways to obtain:

- Root-locus form:

$$G(s) = \frac{k_{rl}}{s^q} \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_{n-q})}$$

- Bode form:

$$G(s) = \frac{k_{Bode}}{s^q} \frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\dots\left(\frac{s}{-z_m}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\dots\left(\frac{s}{-p_{n-q}}+1\right)}$$

in all cases $p_1, \dots, p_{n-q}$ are called poles and $z_1, \dots, z_m$ zeros. Poles are the roots of the denominator and the zeros the roots of the numerator. There are other ways to write the TF, depending on the application some are more usefull then others. What insides can we gain form the TFs ?

Steady state response to a unit step:

Given a TF of a stable system, we can obtain the stead state response by looking at how the system reacts to a unit step, i.e. $u(t) = h(t) = e^{0t} = 1, t \geq 0$

Example: $G(s) = 3\frac{s+8}{s^2+5s+4}$

we know that: $y_{ss} = G(s)\, e^{st}$

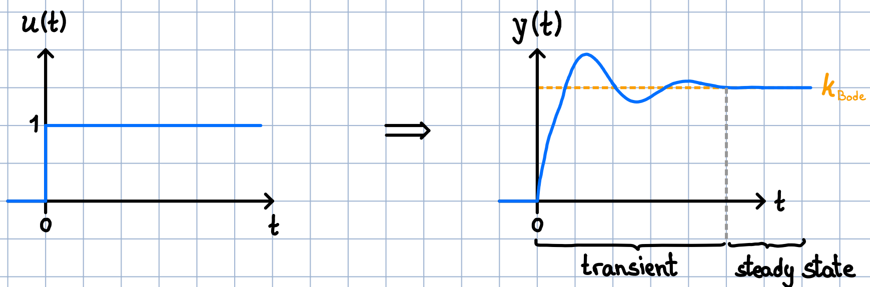
$x(t)$	$X(s)$
impulse: $\delta(t)$	1
step: $h(t)$	$\frac{1}{s}$

pluging in: $y_{ss} = G(0)\, e^{0t} = G(0) = 6$

You can also use the final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to0} sY(s) = \lim_{s\to0} s\, G(s) U(s) = \lim_{s\to0} s\, G(s)\underbrace{\frac{1}{s}}_{\mathcal{L}\{h(t)\}} = \lim_{s\to0} G(s) = 6$$

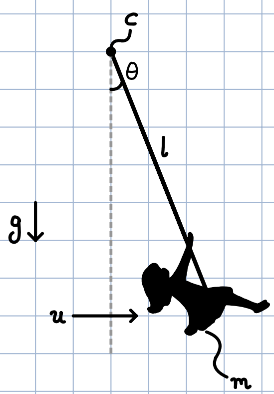
This is also called the Bode gain (k_{Bode})



Steady state response to a sinusoid:

Given some TF $G(s)$ of a LTI system, we can compute the steady state output to an input $u(t) = \sin(t)$, i.e. $s = \pm j$ with $y_{ss} = |G(j)| \sin(t + \angle G(j))$ (see example from last week). We now need to compute $|G(j)|$ and $\angle G(j)$.

Example:



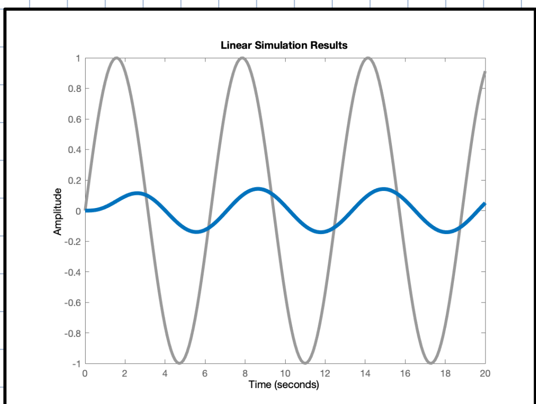
$$\Longrightarrow \quad G(s) = \frac{0.2}{s^2 + s + 2}$$

Lets compute both terms with $|G(j)|$ and $\angle G(j)$

$|G(j\omega)| =$

$\angle G(j) =$

now $y_{ss} =$



Poles and Zeros

We know that we can write TFs as rational functions in the form:

$$G(s) = \frac{b_m s^m + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

the roots of the nominator (zeros) and denominator (poles) are very important and tell us a lot about the behavior of a system.

Poles:

To understand poles, lets look at the impulse response.

Impulse response:

Let us consider the response to a unit impuls, i.e. $u(t) = \delta(t)$

$\int_0^{\varepsilon} \delta(t)\,dt = 1 \;\forall\; \varepsilon > 0$ and $\int_0^{t} f(\tau)\delta(\tau)\,d\tau = f(0) \;\forall\; t > 0$

we can solve the general equation for $y(t)$ by pluging in: $D = 0$, $x_0 = 0$, and $u(t) = \delta(t)$

$$y(t) = Ce^{At}\cancel{x_0}^{\,0} + C\int_0^t e^{A(t-\tau)} B u(\tau)\,d\tau + \cancel{D}^{\,0} u(t)$$

$\int_0^t f(\tau)\delta(\tau)\,d\tau = f(0)$

$$y_{imp}(t) = C\int_0^t e^{A(t-\tau)} B\,\delta(\tau)\,d\tau = Ce^{At}\int_0^t e^{-A\tau} B\,\delta(\tau)\,d\tau$$

$$y_{imp}(t) = Ce^{At}B$$

now we can apply our knowledge from the Laplace transform. Since $Y(s) = G(s)U(s)$, we can apply the Laplace transform to y_{imp} and u to obtain $G(s)$. Lets consider a first-order system.

$$y_{imp}(t) = ce^{at}b \quad \overset{\mathcal{L}}{\Longrightarrow} \quad Y(s) = \quad \overset{\tau = cb}{=}$$

$$u(t) = \delta(t) \quad \overset{\mathcal{L}}{\Longrightarrow} \quad U(s) =$$

$x(t)$	$X(s)$
$impulse: \delta(t)$	1
$step: h(t)$	$\frac{1}{s}$
$h(t) \cdot t^n$	$\frac{n!}{s^{n+1}}$
$h(t) \cdot e^{at}$	$\frac{1}{s-a}$

now $G(s) = Y(s) = \text{———}$ and $y(t) =$

So the impulse response totally defines the response of a system!

We can extend this to higher order systems, and our TF will take the form

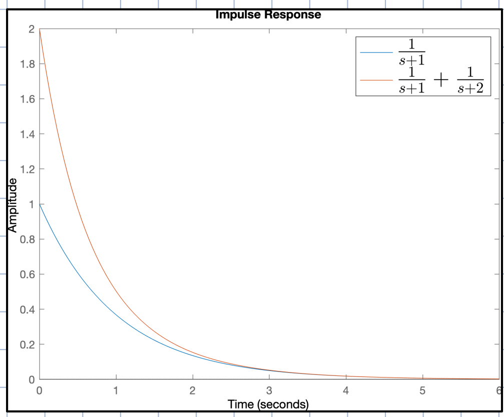
$$G(s) = \frac{\tau_1}{s - p_1} + \frac{\tau_2}{s - p_2} + \ldots + \frac{\tau_n}{s - p_n}$$

which translates to

$$y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \ldots + r_n e^{p_n t}$$

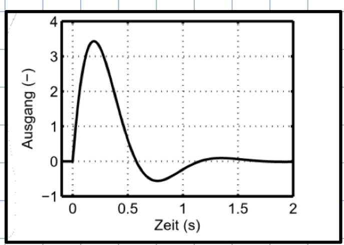
So each pole in our TF generates an exponential in the time domain. For real poles these are exponentials and for complex-conjugate pole pairs, sinusoids.

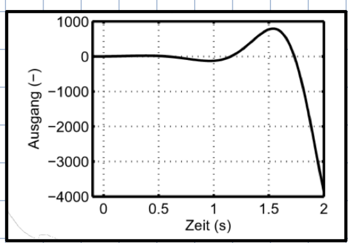
We are essentially "decomposing" a higher order system into multiple first order systems.

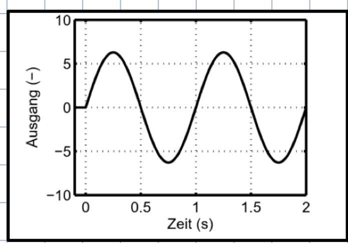


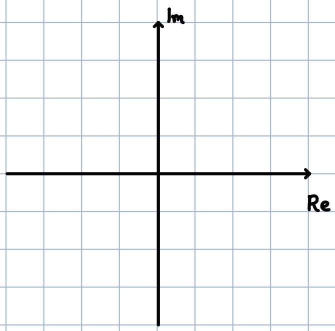
Exam Question: (poles and impulse response)

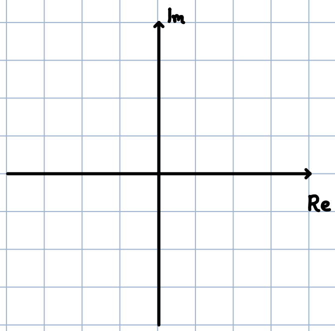
Qualitativley draw the poles of the systems below described by their impulse response:

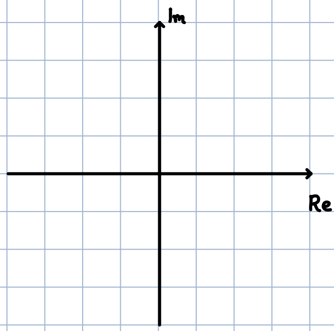












Zeros:

To understand the effects of zeros consider the following system:

$$u(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow y(t) = \frac{d}{dt}u(t)$$

This is a differentiator. Let's see what happens to some general $u(t) = e^{st}$. Since $y(t) = \frac{d}{dt}u(t)$, in this case $y(t) = s e^{st}$. We can conclude that the TF of a differentiator is given by:

$$G(s) = s$$

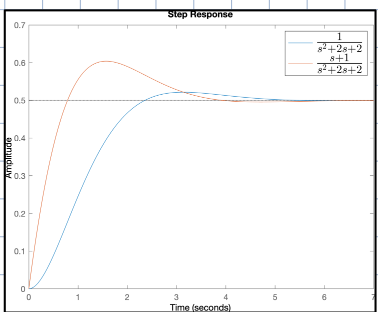
Side note: Integrators are given by $G(s) = s^{-1}$

So by multiplying $G(s)$ with s, essentially adding a zero to the original TF, introduces some derivative action. This usually has an "anticipatory effect".

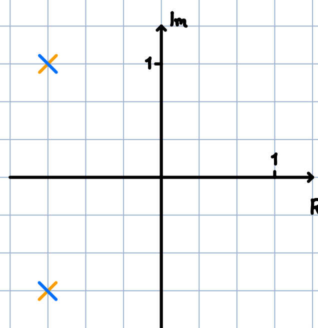
Step response:

The step response is the systems output given a step input i.e.: $u(t) = h(t)$. The resulting graphs give good insights. In the last problem set you derived a general solution for a first order system. Now we will look more at the plots.

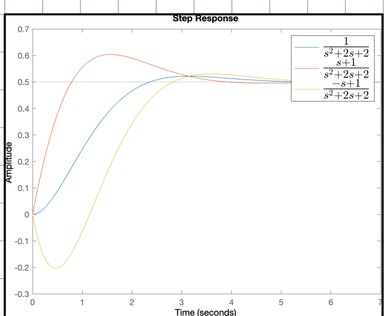
Consider the plot below, showing the step response of two systems. Both have the same poles, but the orange system has an additional zero at $s = -1$

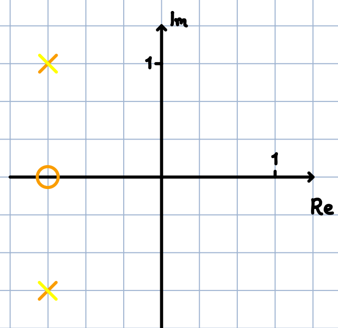


minimum phase zero



Zeros in the left half plane are called minimum phase zeros. They add some derivative action to the response. What happens when the zero is in the right half plane, i.e. positive real part?



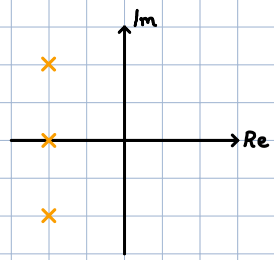


non-minimum phase zero

we observe that the stability is conserved, but the non-minimum phase zero causes the output to initially move in the wrong direction, i.e. we have some sort of "negative" derivative action.

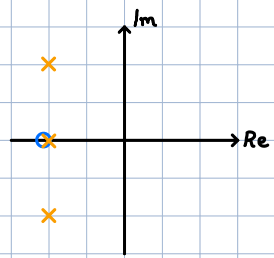
What happens if we add a zero near a pole? Consider the TF given by:

$$G_1(s) = \frac{1}{(s+1)(s+1+j)(s+1-j)} \Longrightarrow$$



now lets add a zero near the pole s+1:

$$G_2(s) = \frac{s+1+\varepsilon}{(s+1)(s+1+j)(s+1-j)} \Longrightarrow$$



Remember from earlier that in the partial fraction expansion form we can gauge how strong the influence of each pole in the output is:

$$G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \dots + \frac{r_n}{s-p_n} \longrightarrow y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \dots + r_n e^{p_n t}$$

So let's compute the partial fraction expansion of both TFs and compare. We can use the cover-up method to compute r_i:

$$r_i = \lim_{s \to p_i} (s-p_i)\, G(s)$$

Lets compute the r_i

$r_1 =$

$r_2 =$

$r_3 =$

$$G(s) = \frac{}{s+1} + \frac{}{s+1+j} + \frac{}{s+1-j}$$

similarly: $$G(s) = \frac{\varepsilon}{s+1} + \frac{-1/2\,j}{s+1+j} + \frac{1/2\,j}{s+1-j}$$

So the smaller ε, the smaller the influence of the pole s+1. If the zero coincides with the pole they cancel out!

$$G_3(s) = \frac{\cancel{s+1}}{\cancel{(s+1)}(s+1+j)(s+1-j)}$$

Since the TF describes input-output behavior, we can cancel out poles, such that we can't observe the asociated behavior, or we cannot influence it through the input. If the pole that is being cancelled is stable this is of no concern, but if the pole is unstable this becomes a big problem.

Exam Problems:

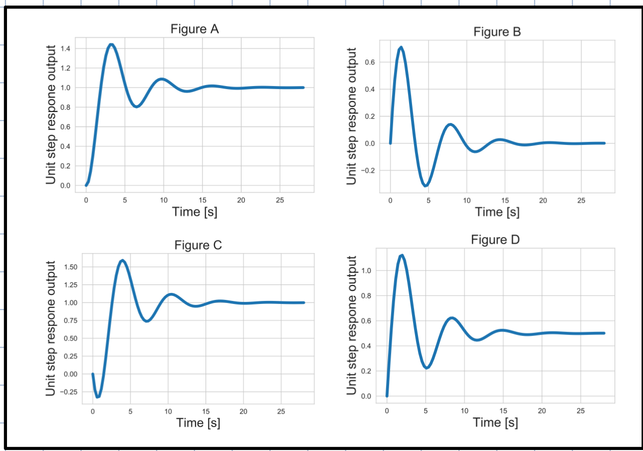
You are given the following set of input to output transfer functions:

1. $$G(s) = \frac{s+0.5}{s^2+0.5s+1}$$

2. $$G(s) = \frac{1-s}{s^2+0.5s+1}$$

3. $$G(s) = \frac{s}{s^2+0.5s+1}$$

4. $$G(s) = \frac{1}{s^2+0.5s+1}$$



Assign each TF to a step response:

1 →

2 →

3 →

4 →

Question 12 *Choose the correct answer. (1 Point)*

What are the initial y_0 and final values y_∞ of an *impulse response* for the following input to output transfer function?

$$G(s) = \frac{s+1}{2s^2+0.5s+1}$$

A $y_0 = 0.5, \quad y_\infty = 0$

B $y_0 = 1, \quad y_\infty = 0$

C $y_0 = 0, \quad y_\infty = 1$

D $y_0 = 0, \quad y_\infty = 0.5$

Hint:

Initial value theorem:

$$\boxed{\lim_{t\to 0^+} x(t) = \lim_{s\to\infty} s \cdot X(s)}$$

Final value theorem:

$$\boxed{\lim_{x\to\infty} x(t) = \lim_{s\to 0^+} s \cdot X(s)}$$

These two theorems only hold, if $X(s)$ is a stable function.

Todays Summary

→ Mostly we can write TFs as rational functions. We already saw an example last week:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$$

We can do partial fraction expansion and get:

$$G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \dots \frac{r_n}{s-p_n}$$

We can also factorize the numerator and denominator in different ways to obtain:

→ Root-locus form:

$$G(s) = \frac{k_{rl}}{s^q} \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_{n-q})}$$

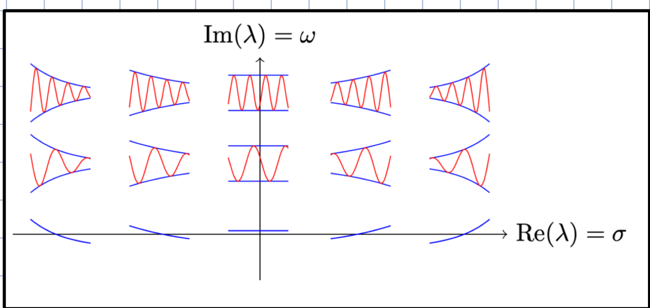
→ Bode form:

$$G(s) = \frac{k_{Bode}}{s^q} \frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\dots\left(\frac{s}{-z_m}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\dots\left(\frac{s}{-p_{n-q}}+1\right)}$$

→ Given a TF of a <u>stable system</u>, we can obtain the stead state response by looking at how the system reacts to a unit step, i.e. $u(t) = h(t) = e^{0t} = 1,\ t \geq 0$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,Y(s)$$

→ Each pole in our TF generates an exponential in the time domain. For real poles these are exponentials and for complex-conjugate pole pairs, sinusoids.



→ Zeros in the left half plane are called minimum phase zeros. They add some derivative action to the response. For non-minimum phase zeros the output to initially moves in the wrong direction, i.e. we have some sort of "negative" derivative action.

→ If a zero coincides with the pole they cancel out! If the pole that is being cancelled is stable this is of no concern, but if the pole is unstable this becomes a big problem.

Why are we doing this again?

Remember course objectives:

Modeling: represent real world systems with mathematical equations.

We're here → **Analysis**: understand how a given system behaves; how the input affects the output, how feedback influences the system.

Synthesis: Change the system, so that it behaves in a desirable way.