7. Recital 01.11.24

Recap

We can write TFs as rational functions.

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$$

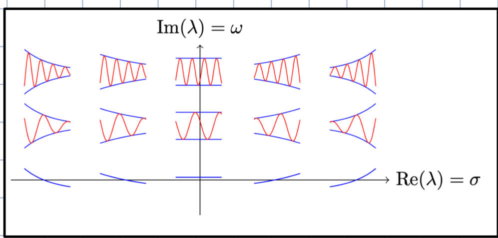
We can also factorize the numerator and denominator in different ways to obtain:

→ Root-locus form:

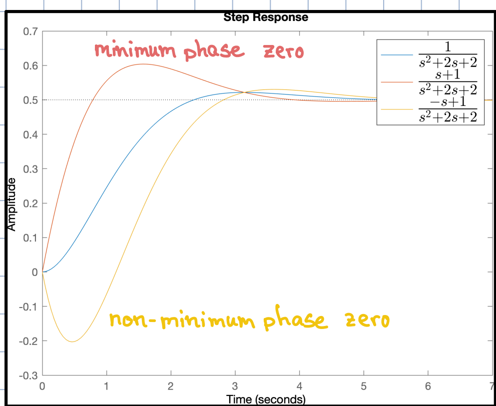
$$G(s) = k_{rl}\,\frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

$p_1, \dots, p_{n-q}$ are called poles and $z_1, \dots, z_m$ zeros. Poles are the roots of the denominator and the zeros the roots of the numerator.

→ Each pole in our TF generates an exponential in the time domain. For real poles these are pure exponentials and for complex-conjugate pole pairs, sinusoids.



→ Zeros in the left half plane are called minimum phase zeros. They add some derivative action to the response. A non-minimum phase zero causes the output to initially move in the wrong direction, i.e. we have some sort of "negative" derivative action. Stability is conserved.



→ If a zero coincides with the pole they cancel out! If the pole that is being cancelled is stable this is of no concern, but if the pole is unstable this becomes a big problem.

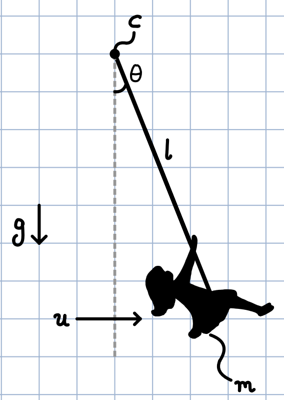
Remember course objectives:

Modeling: represent real world systems with mathematical equations.

Analysis: understand how a given system behaves; how the input affects the output, how feedback influences the system.

Synthesis: Change the system, so that it behaves in a desirable way.

What we did so far



Physical system

Modeling ⇘

$$\dot{x}_1(t) = x_2(t)$$

$$m l^2 \dot{x}_2(t) = -lmg \sin x_1(t) - c\, x_2(t) + l \cos x_1(t)\, u(t)$$

$$y(t) = x_1(t)$$

Differential equations

Linearization ⇙

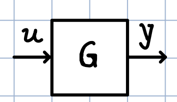
$$x_e = (\pi, 0) \quad , u_e = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{c}{ml^2} \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ \frac{1}{ml} \end{bmatrix}$$

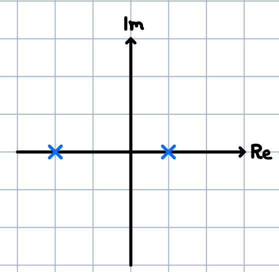
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} , \quad D = 0$$

LTI State space form

S-Domain ⇘



$$G(s) = \frac{0.2}{s^2 + s - 2}$$

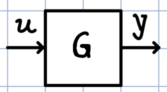


Transfer Functions

Root Locus

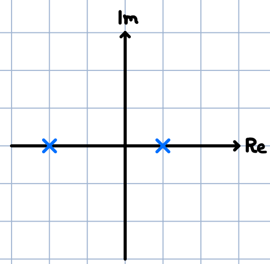
We arrived at a point where we can start to look at how different controler influence our system.

But how do we know how to chose our controler? Let's take our inverted swing example. First of all, we want our system to be stable, i.e. no poles in the right half plane. Our original system has two poles, which can be represented as follows:

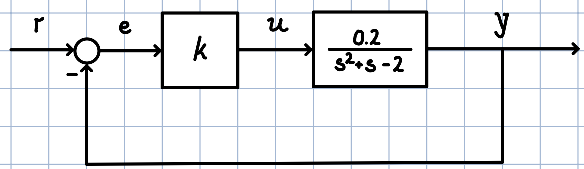


$$G(s) = \frac{0.2}{s^2+s-2} = \frac{0.2}{(s+2)(s-1)} = L(s)$$

open Loop TF



In order to control this system, let's add some controller $C(s)$ and introduce some feedback. The block diagram would look as follows:



For now, the controller C will just be a constant that multiplies the error e. This also called a gain and we will denote it with the letter k.

The TF mapping r to y, in this closed loop system, is now given by:

$$T(s) = \frac{k\frac{0.2}{s^2+s-2}}{1+k\frac{0.2}{s^2+s-2}} = \frac{0.2k}{s^2+s-2+0.2k}$$

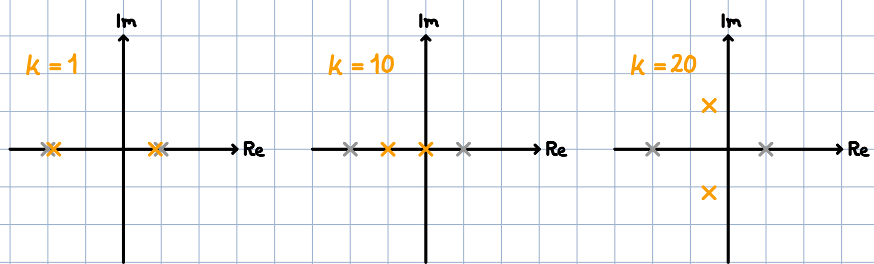
$T(s)$ is also referred to as the complementary sensitivity.

So ideally, by changing k, we can bring the unstable pole to the left half plane, stabalizing the system.

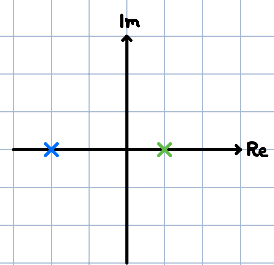
Now instead of looking at $L(s)$ we have to consider $T(s)$. Luckily the poles of $T(s)$ are dependend on k, i.e. by adjusting k we influence the position of the closed loop poles. Let's see how they behave.

Solve: $s^2+s-2+0.2k=0$

for different k



We can trace out the paths that the poles take and obtain:



We can now see that for some value k both poles are in the LHP and the system is stable.

The plot we just created is called root locus, and it shows how the location of the closed loop poles, based on the open loop poles. It allows us to quickly see if some controler is feasible or not.

Ideally we don't want to calculate the position of the poles for all values of k to be able to sketch a bode plot.

Luckily there are some rules we can follow:

1. The closed loop poles are symmetric about the real axis.
2. The number of closed loop poles is equal to the number of open loop poles.
3. The closed loop poles approach the open loop poles as $k \to 0$
4. If there are as many open loop poles as open loop zeros, then the closed loop poles approach the open loop zeros as $k \to \infty$. If theres more poles then they go "to infinity".

Asymptotes

Contact point / Centroid of asymptotes

$$\boxed{S_{com} = \frac{\sum x_{Poles} - \sum x_{Zeros}}{\#Poles - \#Zeros}}$$

$$x_i \rightarrow Coordinates\ on\ the\ Real\ axis$$

Angle of asymptotes

$$\boxed{\alpha_n = \frac{2n+1}{\#Poles - \#Zeros} \cdot 180°}$$

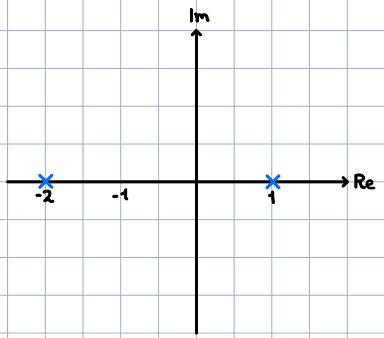
$$n = \{0; 1; \ldots; (\#Poles - \#Zeros - 1)\}$$

Sketching Rules

1. Root loci start at poles → go to zeros
2. There are n lines (loci) where n is the degree of Poles or Zeros (whichever is greater).
3. As k increases from 0 to ∞, the roots move from the poles of $G(s)$ to the zeros of $G(s)$.
4. When roots are complex, they occur in conjugate pairs.
5. At no time will the same root cross over its path.
6. The portion (Anteil) of the real axis to the left of an odd number of open loop poles and zeros are part of the loci. → **Roots are always sketched from the right to the left.**
7. Lines leave and enter the real axis at 90°.
8. If there are not enough poles or zeros to make a pair, then the extra lines go to / come from infinity.
9. Lines go to infinity along asymptotes.
10. If there are at least two lines to infinity, then the sum of all roots is constant.
11. K going from 0 to $-\infty$ can be drawn by reversing rule 5 and adding 180° to the asymptote angles.

Examples:

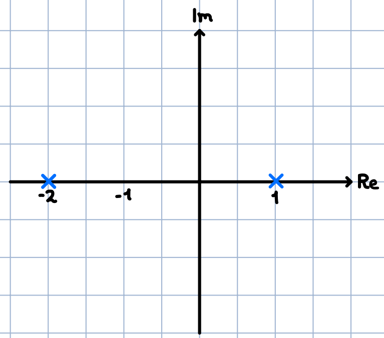
Let's take the example from above, and follow the rules.



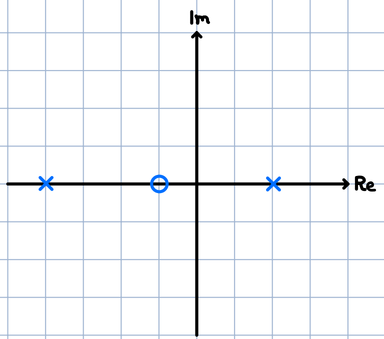
The portion (Anteil) of the real axis to the left of an odd number of open loop poles and zeros are part of the loci. → **Roots are always sketched from the right to the left.**

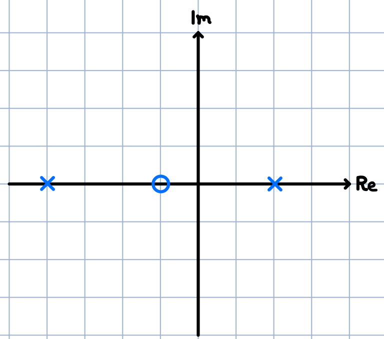
Contact point of asymp. : $\frac{-2+1}{2} =$

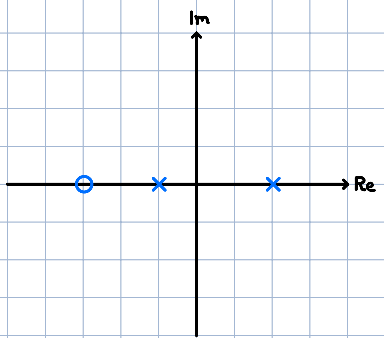
Angle of asymp.: $\frac{3}{2} \cdot 180° =$

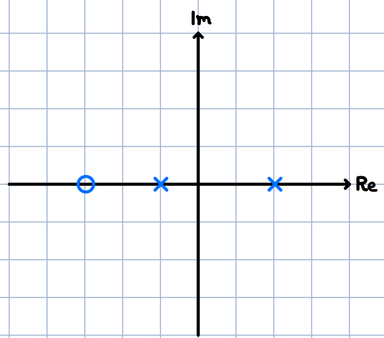


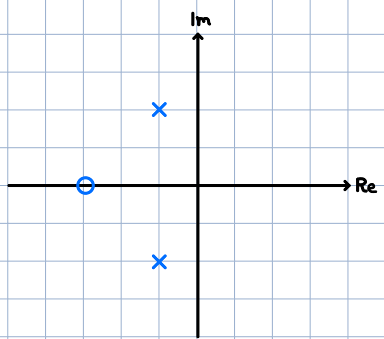
Let's add a zero, and look at some more examples.

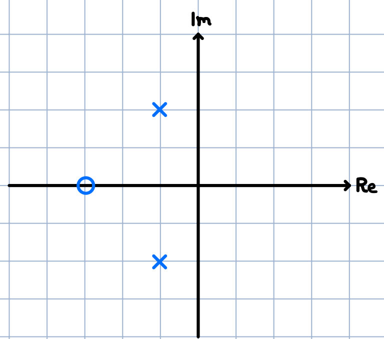


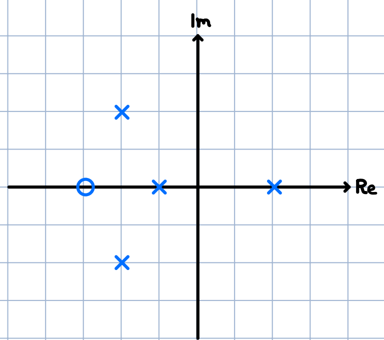


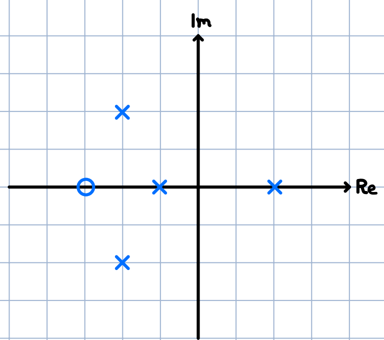






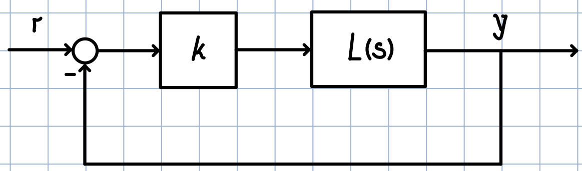






https://lpsa.swarthmore.edu/Root_Locus/RLDraw.html

Let's look at the TF again.



$L(s)$ will be some rational function of the form $L(s) = \frac{N(s)}{D(s)}$, now the closed loop TF will always have the form:

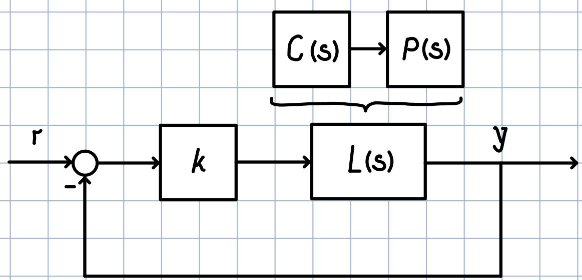
$$T(s) = \frac{k\,L(s)}{1 + k\,L(s)} = \frac{k\,N(s)}{D(s) + k\,N(s)}$$

That means that the closed loop poles are given by $D(s) + k\,N(s) = 0$. Here we can niceley see:

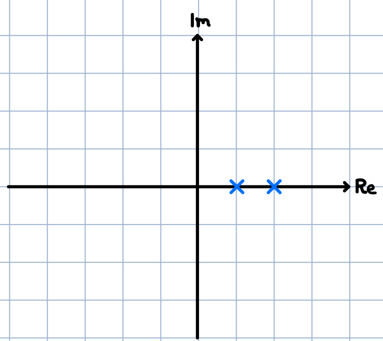
$$\lim_{k \to 0} D(s) + k\,N(s) = D(s) \quad \text{closed loop poles approach the open loop poles}$$

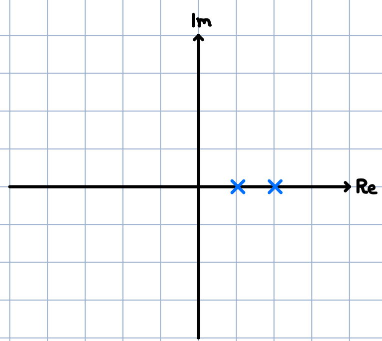
$$\lim_{k \to \infty} D(s) + k\,N(s) \cong N(s) \quad \text{closed loop poles approach the open loop zeros}$$

If we want to implement more complex cortroller, that is itself a dynamic system, the open loop TF is the product of the plant with controler, i.e. $L(s) = C(s)\,P(s)$

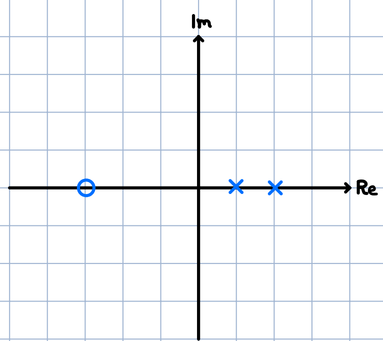


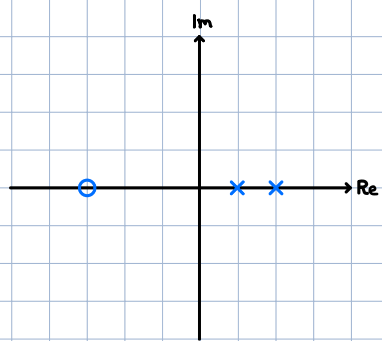
The controler $C(s)$ is often called dynamic compensator. We can use dynamic compensators to e.g. stabalize unstable systems. Lets look at a simple example:





now we can design the dynamic compensator to have a minimum phase zero.



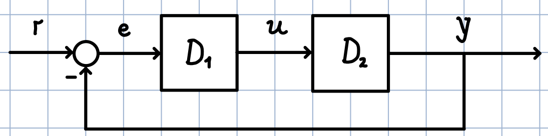


By adding a zero we are able to make the feedback connection stable. This is also called a PD controler. More on that later in the course.

Some additional considerations:

Well-posedness:

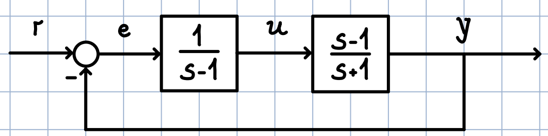
Consider the system below:



with D_1 and D_2 being scalar gains. Then the denominator of $T(s)$ will be $1 + D_2 D_1$. So if $D_2 D_1 = -1$ then $T(s) = \frac{1}{0}$. In this case the system is not well posed, this interconnection does not make sense.

Internal stability:

Up to now the stability criterea we defined only really looked at the stability of one specific TF. Now we will look at the stability of a whole system. Consider:



The closed loop TF $r \to y$ is given by the complementary sensitivity $T(s)$:

$$T(s) = \frac{1}{s+2}$$

this seems to be stable, but in such interconected system we have to check if all closed loop TFs between any two signals are stable. If we look at our example we can see that e.g. the TF from $r \to u$ is given by:

$$\frac{s+1}{(s-1)(s+2)}$$

we that it is unstable. So our u might be "blowing up", but we don't see it in the output. It is unobservable. This is one of the effects of pole-zero cancellation.