## 9. Recital 15.11.24

# Recap

### Time-Domain Specifications:

When looking at the step response of a system, we can specify certain characteristics our system should have. By looking at the plot we can define a few of them. Consider a second order system.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \iff \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} x + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u \\ y = x \end{cases}$$

for an underdamped system ($\zeta < 1$) with zero initial condition we get:

$$y(t) = 1 - \frac{1}{\cos\varphi} e^{\sigma t} \cos(\omega t + \varphi)$$

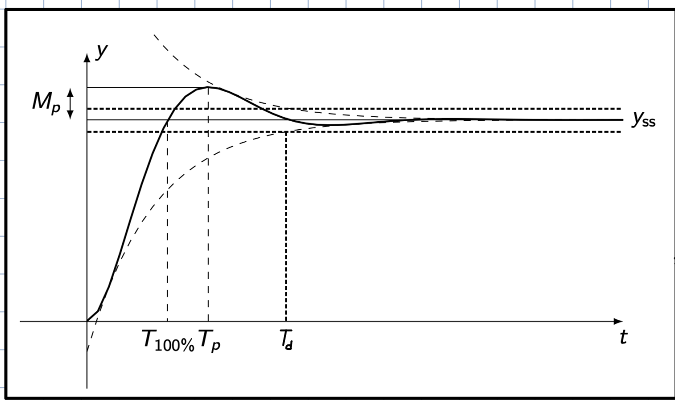
We can define important characteristics of the step response:

→ Time to peak: $T_p = \frac{\pi}{\omega}$

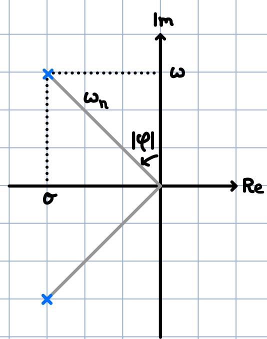
→ Peak overshoot: $M_p = e^{\frac{\sigma\pi}{\omega}}$

→ Rise time: $T_{100\%} = \frac{\frac{\pi}{2} - \varphi}{\omega} \approx \frac{\pi}{2\omega_n}$

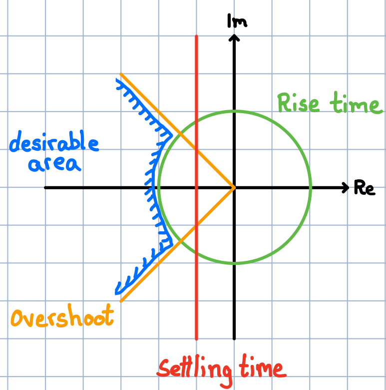
→ Settling time: $T_d = \frac{1}{|\sigma|} \ln(100/d)$



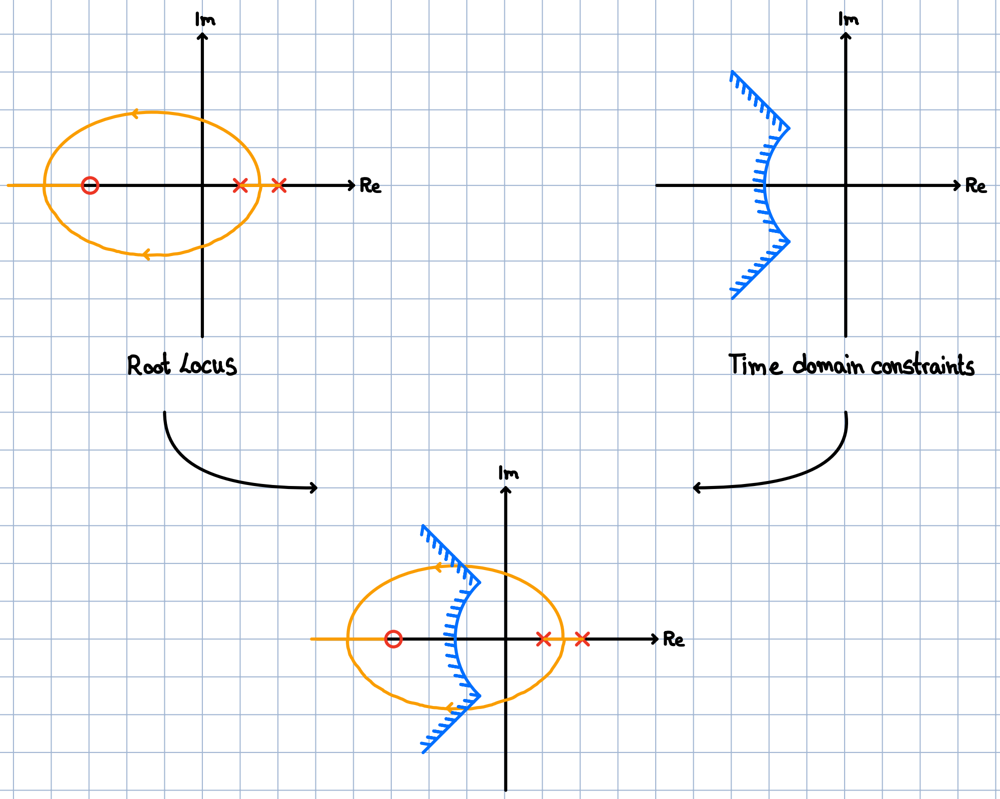

We can map out the poles of $G(s)$ as a function of $\sigma, \omega, \omega_n$, and $\varphi$ in the complex plane. We observe that each component of the time response affects the poles differently.




That means that we can take requirements in the time domain and transform them into constraints on the location of the poles of the system. When designing our closed loop system we can take these restrictions into account.



A way to relate these restrictions to the root locus is overlaying both




we can now choose k s.t. the poles fall in the restricted area.

## Dominant Pole Approximation:

What if our system has more than two poles?

Often we can approximate the higher order system. Remember that every pole corresponds to an exponential. The real part of the pole indicates how fast the exponential grows or decays. For poles in the LHP we can say that poles further away from the imaginary axis are "faster" since they decay at a higher rate.

We observe that the be combined behavior can be well approximated by the slower pole, i.e. the pole closer to the imaginary axis.

## Steady-state error:

To compute the steady-state error of a standard feedback system to a ramp input we can use the table below:

| $e_{ss}$ | $q=0$ | $q=1$ | $q=2$ |
|---|---|---|---|
| Type 0 | $\frac{1}{1+k_{\text{Bode}}}$ | $\infty$ | $\infty$ |
| Type 1 | 0 | $\frac{1}{k_{\text{Bode}}}$ | $\infty$ |
| Type 2 | 0 | 0 | $\frac{1}{k_{\text{Bode}}}$ |

Where the type refers to the number of poles at the origin, and $q$ to the order of the ramp input.

# Frequency Response

Back in week 5 we introduced the TF and how, for LTI systems, it can modell the steady-state output to some general input $e^{st}$. The TF is given by:

$$y_{ss} = G(s)\, e^{st} \qquad \text{with } G(s) = C(sI-A)^{-1}B + D \quad , \quad s \in \mathbb{C}$$

We chose the general input $e^{st}$ since virtually ervery input can be generated as a linear combination of terms $e^{st}$. Since our system is LTI, we can thus calculate the response to complex inputs by breaking it down, and summing up the individual outputs. If the input is a cosine we saw that:

$$u(t) = \cos(\omega t) = \tfrac{1}{2} e^{j\omega t} + \tfrac{1}{2} e^{-j\omega t}$$

We notice that we can decompose $u(t)$ as follows:

$$u(t) = \sum_i U_i e^{s_i t} \qquad \text{with } U_{1,2} = \tfrac{1}{2} \text{ and } s_{1,2} = \pm j\omega$$

The output is then given by

$$y(t) = G(j\omega)\tfrac{1}{2} e^{j\omega t} + G(-j\omega)\tfrac{1}{2} e^{-j\omega t}$$

we can rewrite $G(j\omega)$ as $Me^{j\phi}$ with $M = |G(j\omega)|$ Magnitude of $G(j\omega)$
$\phi = \angle G(j\omega)$ Phase of $G(j\omega)$

and then

$$y(t) = Me^{j\phi}\tfrac{1}{2} e^{j\omega t} + Me^{-j\phi}\tfrac{1}{2} e^{-j\omega t}$$

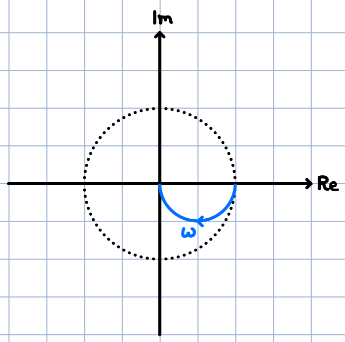
$$y(t) = M\cos(\omega t + \phi)$$

The output is another sinusoid with a different amplitude and phase, but same frequency. This means that, in order to analyze how a sinusoid affects our system we only have to know how the magnitude and phase change. These changes are given by $M = |G(j\omega)|$ and $\phi = \angle G(j\omega)$. So by pluging in $s = j\omega$ we can completley define the steady-state response to a sinusoidal input, this is also called frequency response.

The frequency response is very important in control systems and allows us to asses the stability, performance, behavior and many other things of LTI systems.

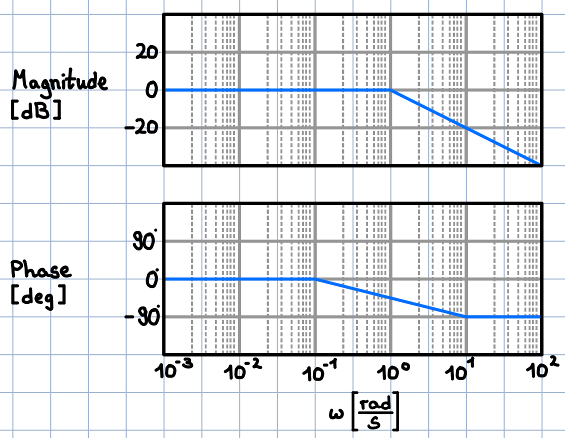
When representing the frequency response we are essentially plotting a complex function $G(j\omega) \in \mathbb{C}$ with a real argument $\omega \in \mathbb{R}$. There are two ways to represent this:

→ Polar Plot ( Nyquist Plot)




as parametric curve, $\omega$ is implicit

→ Bode Plot




two separet plots as a function of $\omega$

We will start with the Bode plot.

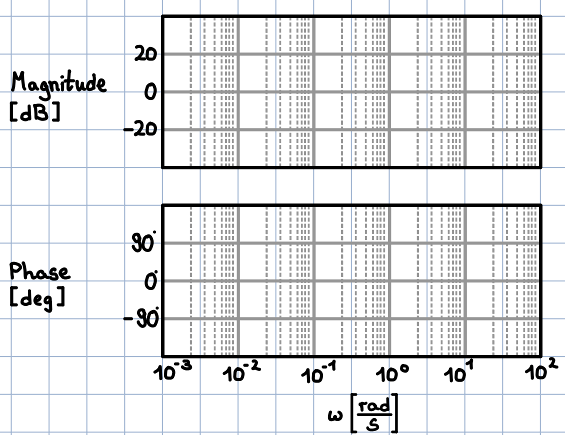
## Bode Plot:

We want to plot $M = |G(j\omega)|$ and $\phi = \angle G(j\omega)$ as a function of $\omega$. We will use two plots with a shared frequency axis. The frequency axis is shown on a $\log_{10}$ scale in $\frac{rad}{s}$. The magnitude is plotted in deciBells, i.e.

$$|G(j\omega)|\,[dB] = 20 \log_{10} |G(j\omega)|,$$

| $x$ | $\frac{1}{1000}$ | $\frac{1}{100}$ | $\frac{1}{10}$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $1$ | $\sqrt{2}$ | $2$ | $10$ | $100$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_{dB}$ | $-60$ | $-40$ | $-20$ | $\approx -6$ | $\approx -3$ | $0$ | $\approx 3$ | $\approx 6$ | $20$ | $40$ |

and the phase usually in degrees. This results in a plot of this form.




One reason for choosing this scaling is that when we have multiple inputs the magnitude gets multiplied and the phases added. Since the magnitude plot is in log scale we can then add two lines in the plot instead of multiplying.

To sketch the Bode Plot we follow these basic rules:

| Magnitude / Phase | $-20$ dB/dec | $+20$ dB/dec |
|---|---|---|
| $-90°$ | stable pole | non-minimum phase zero |
| $+90°$ | unstable pole | minimum phase zero |

Let's try drawing some plots. To draw these plots by hand we can do a straight line approximation with the table above. Also recall the Bode Form of a TF:

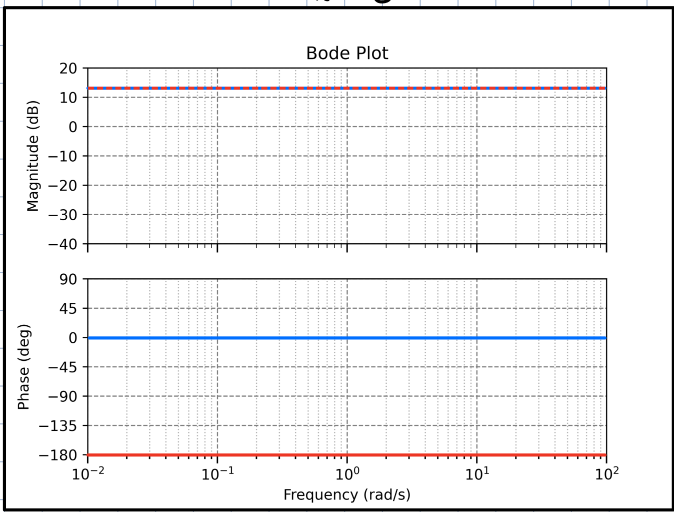
$$G(s) = \frac{k_{Bode}}{s^q} \frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\dots\left(\frac{s}{-z_m}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\dots\left(\frac{s}{-p_{n-q}}+1\right)}$$

Let's look at simple examples

→ $G(s) = k$ , $k>0$ and $k<0$

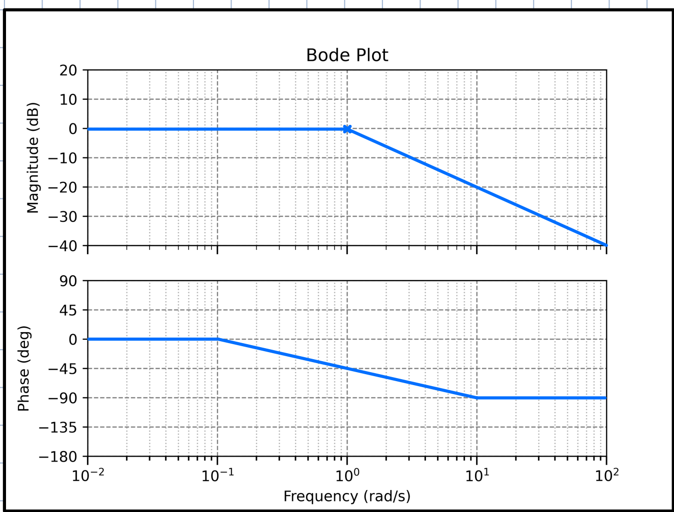
$k = \pm 5$

Magnitude : $20 \log|k| \approx 14\,dB$



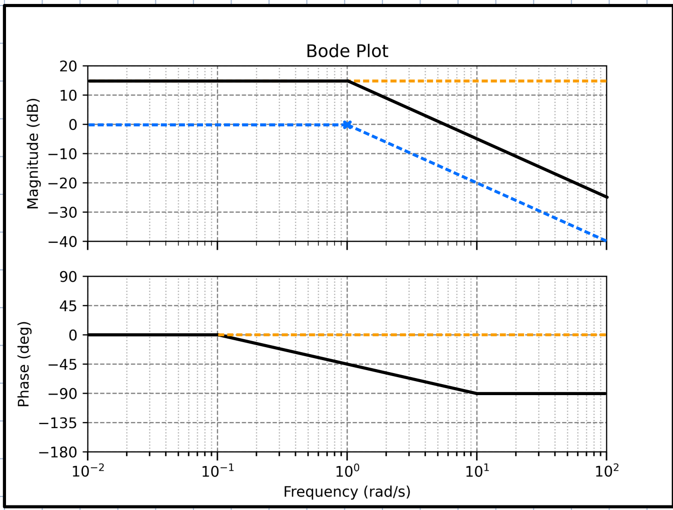

→ $G(s) = \frac{1}{s+1}$ in Bode Form

to draw this plot we take the absolute position of the pole and mark it on the plot. Draw the low freq. asymptote at 0 dB until the location of the pole. Then the high freq. asymptote from the pole until $\omega \to \infty$.
The phase usually starts changing 1 dec befor the pole location and continues until 1 dec after the pole.



we can put both together and draw the Bode plot for:

$\rightarrow G(s) = 5 \dfrac{1}{s+1}$ in Bode Form




Since we chose a log scale for the magnitude we can now add everything up.

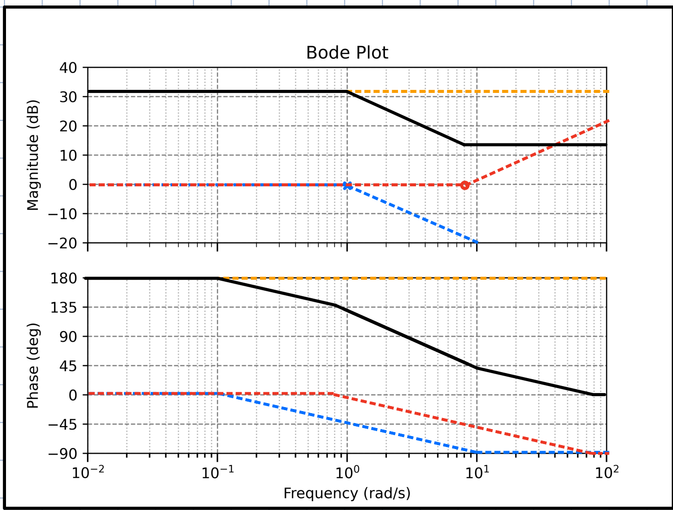
we can also add a zero:

$\rightarrow G(s) = 5 \dfrac{s-8}{s+1}$

bring to Bode Form!

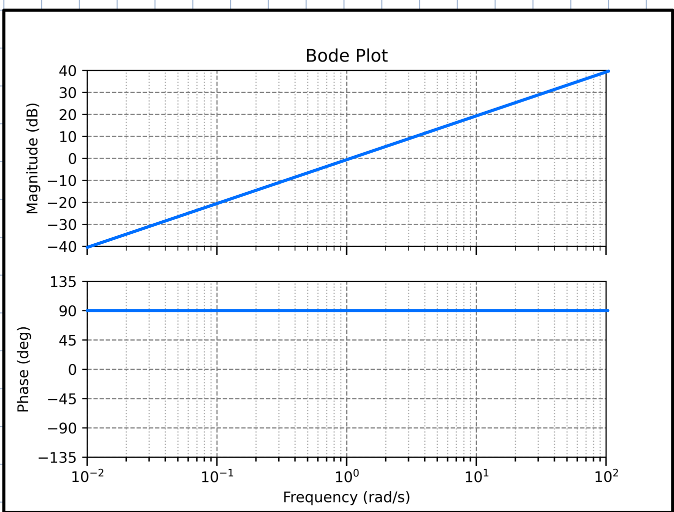
$$G(s) = -40 \frac{\frac{s}{-8}+1}{s+1}$$

Proceed as above, but look at the table for the behavior of the non-minimum phase zero.

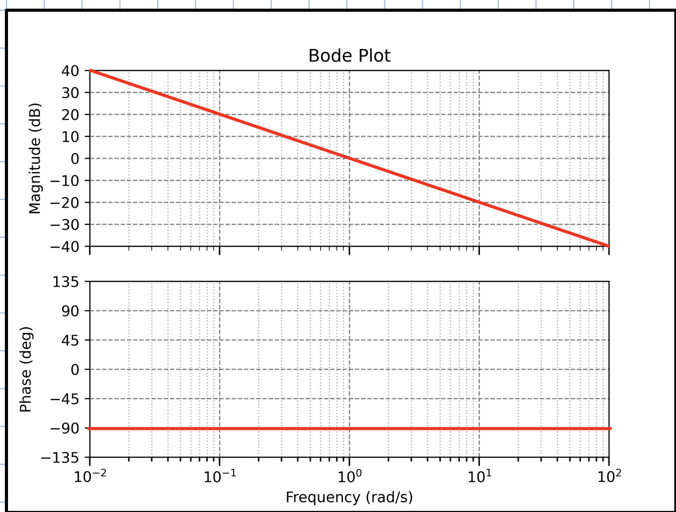



This can be done for an arbitrary numbers of poles and zeros. Two important cases are poles and zeros at the origin. The respective plots look like this:

$G(s) = s$




$G(s) = \dfrac{1}{s}$



A more complete table with sketching rules is given by:

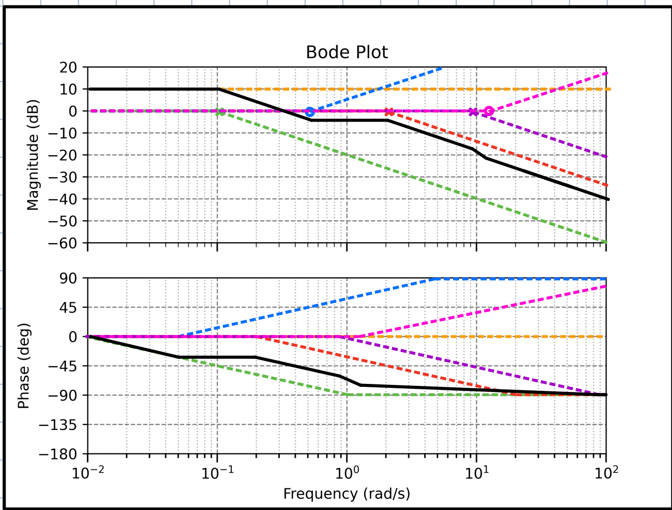
## Rules for Making Bode Plots

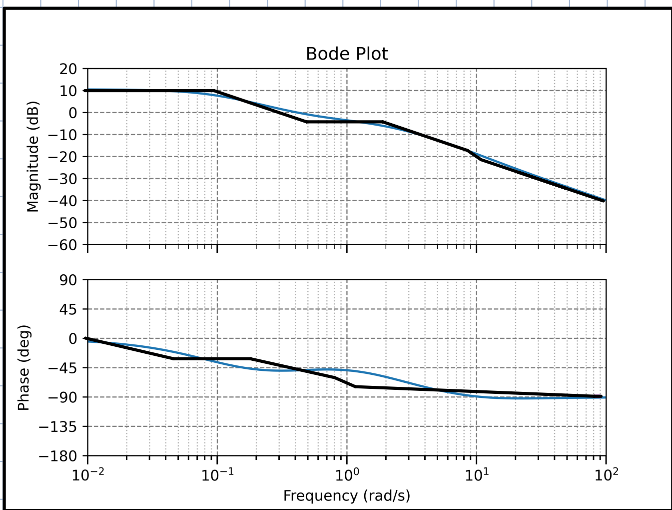
| Term | Magnitude | Phase |
|---|---|---|
| **Constant:** K | $20 \cdot \log_{10}(\lvert K \rvert)$ | $K>0$: $0°$<br>$K<0$: $\pm 180°$ |
| **Real Pole:** $\dfrac{1}{\frac{s}{\omega_0}+1}$ | • Low freq. asymptote at 0 dB<br>• High freq. asymptote at -20 dB/dec<br>• Connect asymptotic lines at $\omega_0$, | • Low freq. asymptote at 0°.<br>• High freq. asymptote at -90°.<br>• Connect with straight line from $0.1 \cdot \omega_0$ to $10 \cdot \omega_0$. |
| **Real Zero*:** $\frac{s}{\omega_0}+1$ | • Low freq. asymptote at 0 dB<br>• High freq. asymptote at +20 dB/dec.<br>• Connect asymptotic lines at $\omega_0$. | • Low freq. asymptote at 0°.<br>• High freq. asymptote at +90°.<br>• Connect with line from $0.1 \cdot \omega_0$ to $10 \cdot \omega_0$. |
| **Pole at Origin:** $\frac{1}{s}$ | • -20 dB/dec; through 0 dB at $\omega=1$. | • Line at -90° for all $\omega$. |
| **Zero at Origin*:** s | • +20 dB/dec; through 0 dB at $\omega=1$. | • Line at +90° for all $\omega$. |
| **Underdamped Poles:** $\dfrac{1}{\left(\frac{s}{\omega_0}\right)^2+2\zeta\left(\frac{s}{\omega_0}\right)+1}$ | • Low freq. asymptote at 0 dB.<br>• High freq. asymptote at -40 dB/dec.<br>• Connect asymptotic lines at $\omega_0$.<br>• Draw peak† at freq= $\omega_0$, with amplitude $H(j\omega_0)=-20 \cdot \log_{10}(2\zeta)$ | • Low freq. asymptote at 0°.<br>• High freq. asymptote at -180°.<br>• Connect with line from $\omega=\omega_0 \cdot 10^{-\zeta}$ to $\omega_0 \cdot 10^{\zeta}$ |

**Notes:**
$\omega_0$ is assumed to be positive. If $\omega_0$ is negative, magnitude is unchanged, but phase is reversed.

## Example:

$$G(s) = \frac{(s+0.5)(s+12)}{(s+2)(s+0.1)(s+9)} = \frac{10}{3}\,\frac{\left(\frac{s}{0.5}+1\right)\left(\frac{s}{12}+1\right)}{\left(\frac{s}{2}+1\right)\left(\frac{s}{0.1}+1\right)\left(\frac{s}{9}+1\right)}$$



Until now we only considered real poles. But what if the TF has a complex-conjugate pair of poles?

we can generally write a corresponding TF as:

$$G(s) = \frac{1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1} = \frac{\omega_n^2}{s^2 + 2\zeta s \omega_n + \omega_n^2}$$

for these systems the damping ratio $\zeta$ influences the form of the Bode plot.

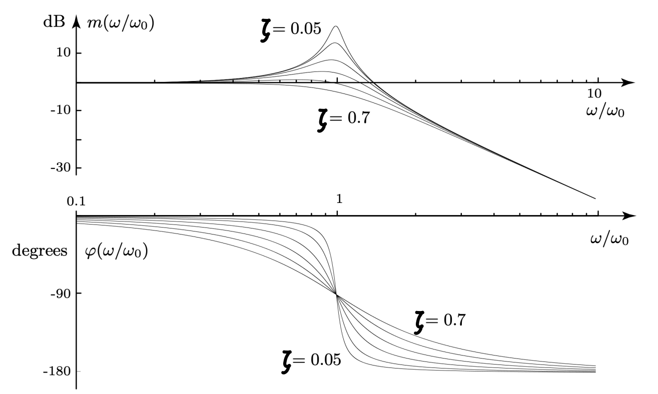


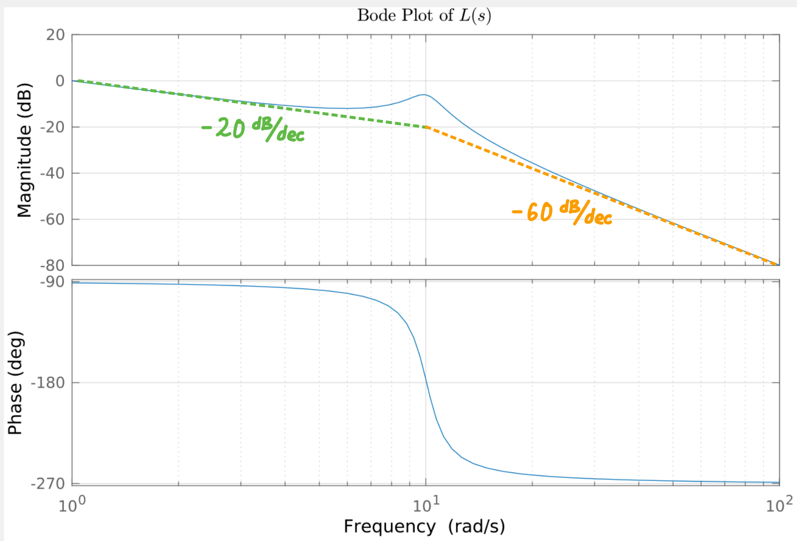

**Figure 8.3.** Bode diagram of the second-order system (8.15) for six different damping ratios $\zeta$ = [0.05, 0.1, 0.2, 0.33, 0.5, 0.7].

Despite this they can be treated like normal poles.

<u>Exam Question:</u>

**Box 11: Question 24**

Consider the following Bode plot for a loop transfer function $L(s)$:




**Question 24** *Choose the correct answer. (1 Point)*

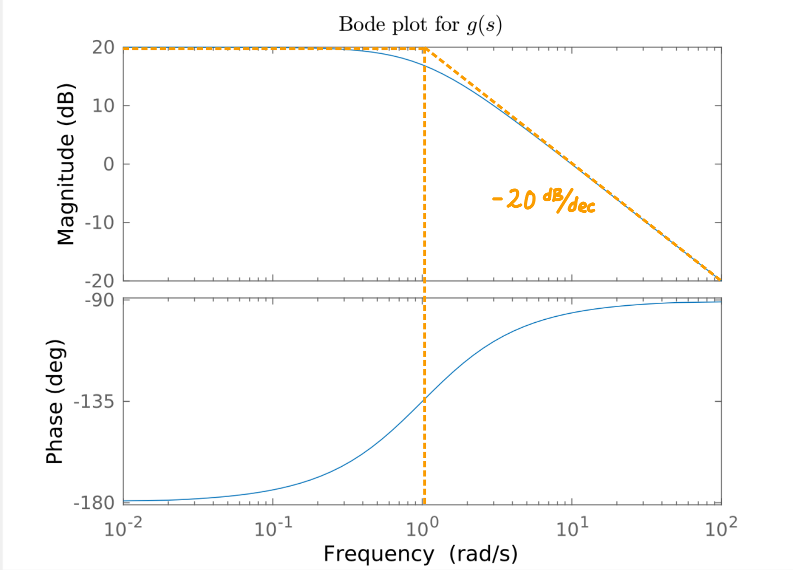
How many poles and zeros does $L(s)$ have?

- ☐ $L(s)$ has 2 poles and 0 zeros.
- ☐ $L(s)$ has 4 poles and 2 zeros.
- ☐ $L(s)$ has 3 poles and 2 zeros.
- ☒ $L(s)$ has 3 poles and 0 zeros.

- Looking towards $\omega \to 1$ we see a slope of $-20\,^{dB}/_{dec}$, and a phase of $-90^\circ$ $\longrightarrow$ Integrator. 1 pole
- At $\omega = 10$ there is a peak in magnitude and sudden drop in phase $\longrightarrow$ Pair of complex-conjugate poles. 2 poles
- Checking the slopes of the magnitude plot and we can conclude that there are no other components.

**Box 12: Question 25**

Consider the following Bode plot of a transfer function $g(s)$:




**Question 25** *Choose the correct answer. (1 Point)*

Select the transfer function which matches the Bode plot.

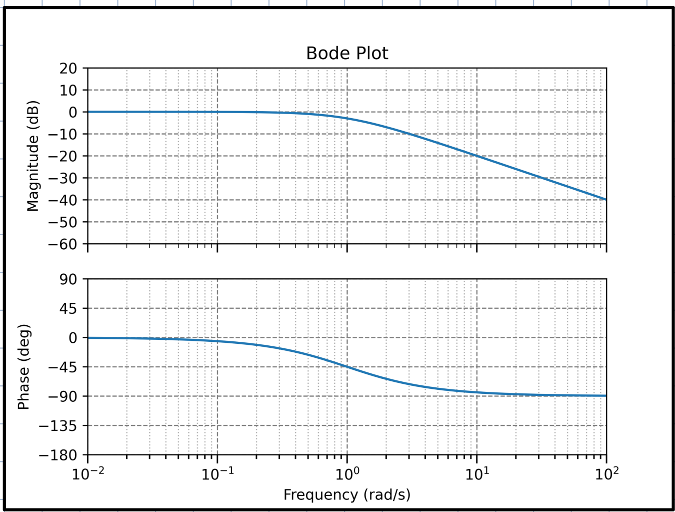
- ☐ $g(s) = \frac{-20}{s+1}$
- ☒ $g(s) = \frac{10}{s-1}$
- ☐ $g(s) = \frac{20}{s-1}$
- ☐ $g(s) = \frac{10}{s+1}$

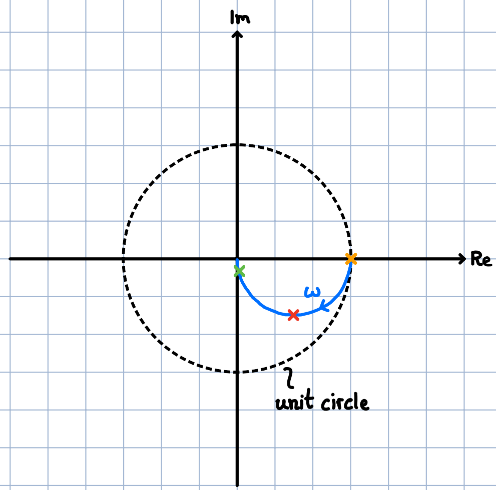
We can read from the plot: arround $\omega = 1$, $-20\,^{dB}/_{dec}$ and $+90^\circ$ $\longrightarrow$ unstable pole at $s=1$

DC gain = 20 dB $\longrightarrow$ $k_{Bode} = 10$

# Polar Plot

We can also represent the frequency response as a parametric curve, where $\omega$ is implicit. We can then plot one curve in the complex plane. Lets start with a simple example. $G(s) = \dfrac{1}{s+1}$



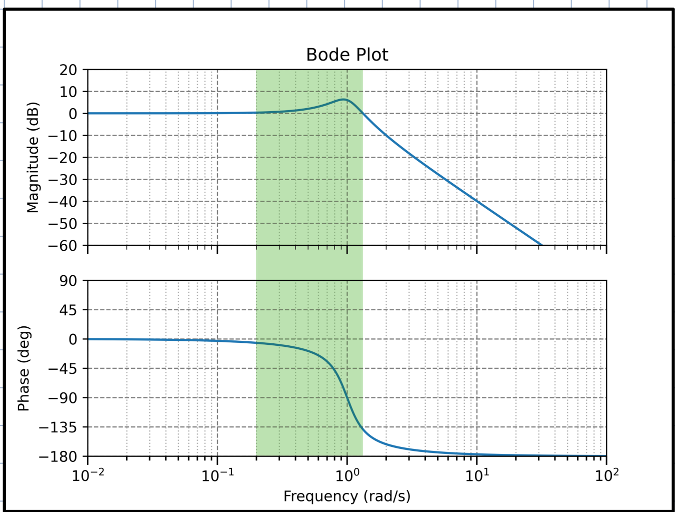

We will go through some $\omega$ and note down magnitude and phase: $\omega = 0$, $|\cdot| = 0\,dB = 1$, $\measuredangle = 0°$

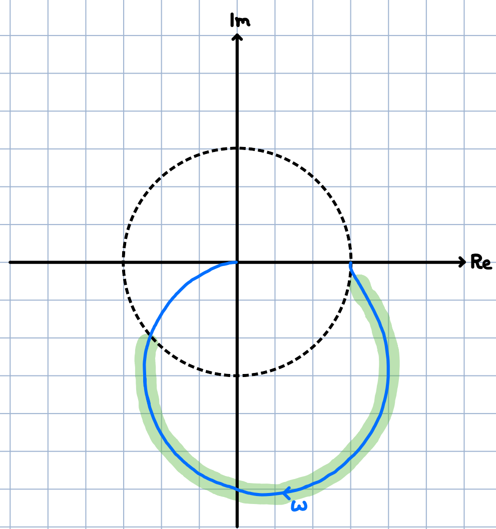
$\omega = 1$, $|\cdot| = -3\,dB = 1/\sqrt{2}$, $\measuredangle = -45°$

$\omega = 10$, $|\cdot| \cong -20\,dB = 0.1$, $\measuredangle \cong -90°$

Based on this we can make a rough sketch.

Another example: $G(s) = \dfrac{1}{s^2+0.5s+1}$

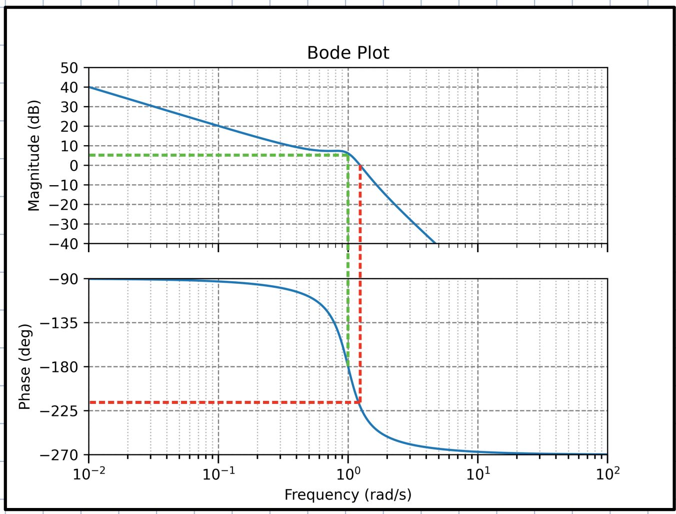


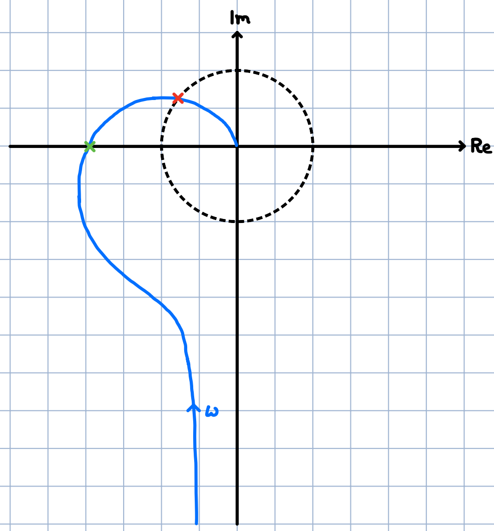




Outside of unit circle between ~ 10° - 135°

## Last example:

$$G(s) = \frac{1}{s}\,\frac{1}{s^2+0.5s+1}$$




Point at -180° approx 5dB ~ 2 , Point at 0 dB ~ -210°