10. Recital 22.11.24

Recap

Frequency Response:

Back in week 5 we introduced the TF and how, for LTI systems, it can modell the steady-state output to some general input e^{st}. The TF is given by:

$$y_{ss} = G(s)\, e^{st} \quad \text{with } G(s) = C(sI-A)^{-1}B + D \ , \ s \in \mathbb{C}$$

We chose the general input e^{st} since virtually evrery input can be generated as a linear combination of terms e^{st}. Since our system is LTI, we can thus calculate the response to complex inputs by breaking it down, and summing up the individual outputs. If the input is a cosine we saw that:

$$u(t) = \cos(\omega t) = \tfrac{1}{2} e^{j\omega t} + \tfrac{1}{2} e^{-j\omega t}$$

The output is then given by

$$y(t) = G(j\omega)\tfrac{1}{2} e^{j\omega t} + G(-j\omega)\tfrac{1}{2} e^{-j\omega t}$$

we can rewrite $G(j\omega)$ as $Me^{j\phi}$ with $M = |G(j\omega)|$ Magnitude of $G(j\omega)$
$\phi = \angle G(j\omega)$ Phase of $G(j\omega)$

and then

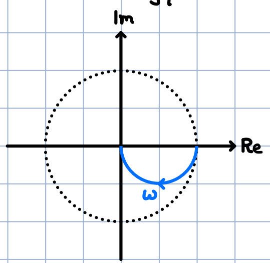
$$y(t) = Me^{j\phi}\tfrac{1}{2} e^{j\omega t} + Me^{-j\phi}\tfrac{1}{2} e^{-j\omega t}$$

$$y(t) = M\cos(\omega t + \phi)$$

The output is another sinusoid with a different amplitude and phase, but same frequency. This means that, in order to analyze how a sinusoid affects our system we only have to know how the magnitude and phase change. These changes are given by $M = |G(j\omega)|$ and $\phi = \angle G(j\omega)$. So by pluging in $s = j\omega$ we can completley define the steady-state response to a sinusoidal input, this is also called frequency response.

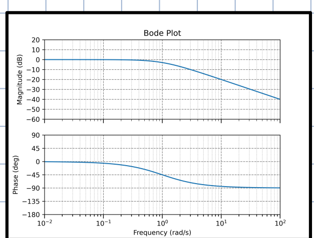
There are two ways to represent this:

→ Polar Plot (Nyquist Plot)



as parametric curve, ω is implicit

→ Bode Plot



two separet plots as a function of ω

Bode Plot:

We want to plot $M = |G(j\omega)|$ and $\phi = \angle G(j\omega)$ as a function of ω. We will use two plots with a shared frequency axis. The frequency axis is shown on a $\log_{10}$ scale in $\frac{rad}{s}$. The magnitude is plotted in deciBells, i.e.

$$|G(j\omega)|[dB] = 20\log_{10}|G(j\omega)|,$$

x	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	2	10	100
x_{dB}	-60	-40	-20	≈ -6	≈ -3	0	≈ 3	≈ 6	20	40

and the phase usually in degrees. To sketch a Bode plot we can follow these rules.

Rules for Making Bode Plots

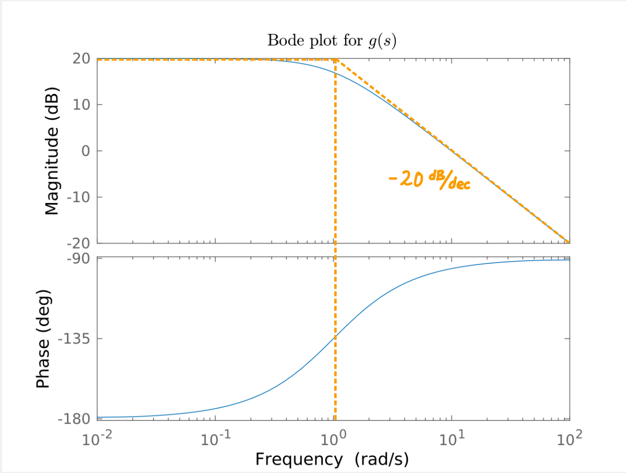
Term	Magnitude	Phase
Constant: K	$20\cdot\log_{10}(\lvert K\rvert)$	K>0: 0° K<0: ±180°
Real Pole: $\frac{1}{\frac{s}{\omega_0}+1}$	• Low freq. asymptote at 0 dB • High freq. asymptote at -20 dB/dec • Connect asymptotic lines at ω_0,	• Low freq. asymptote at 0°. • High freq. asymptote at -90°. • Connect with straight line from $0.1\cdot\omega_0$ to $10\cdot\omega_0$.
Real Zero*: $\frac{s}{\omega_0}+1$	• Low freq. asymptote at 0 dB • High freq. asymptote at +20 dB/dec. • Connect asymptotic lines at ω_0.	• Low freq. asymptote at 0°. • High freq. asymptote at +90°. • Connect with line from $0.1\cdot\omega_0$ to $10\cdot\omega_0$.
Pole at Origin: $\frac{1}{s}$	• -20 dB/dec; through 0 dB at ω=1.	• Line at -90° for all ω.
Zero at Origin*: s	• +20 dB/dec; through 0 dB at ω=1.	• Line at +90° for all ω.
Underdamped Poles: $\frac{1}{\left(\frac{s}{\omega_0}\right)^2+2\zeta\left(\frac{s}{\omega_0}\right)+1}$	• Low freq. asymptote at 0 dB. • High freq. asymptote at -40 dB/dec. • Connect asymptotic lines at ω_0. • Draw peak† at freq= ω_0, with amplitude $H(j\omega_0)=-20\cdot\log_{10}(2\zeta)$	• Low freq. asymptote at 0°. • High freq. asymptote at -180°. • Connect with line from $\omega=\omega_0\cdot 10^{-\zeta}$ to $\omega_0\cdot 10^{\zeta}$

Notes:
ω_0 is assumed to be positive. If ω_0 is negative, magnitude is unchanged, but phase is reversed.

Exam problem:

Box 12: Question 25

Consider the following Bode plot of a transfer function $g(s)$:



Question 25 *Choose the correct answer. (1 Point)*

Select the transfer function which matches the Bode plot.

- [ ] $g(s) = \frac{-20}{s+1}$
- [x] $g(s) = \frac{10}{s-1}$
- [ ] $g(s) = \frac{20}{s-1}$
- [ ] $g(s) = \frac{10}{s+1}$

We can read from the plot: arround $w=1$, $-20\,^{dB}/_{dec}$ and $+90°$ $\longrightarrow$ unstable pole at $s=1$

DC gain = 20dB $\longrightarrow$ $k_{Bode} = 10$

Note that phase starts at $-180°$ $\rightarrow$ negative DC gain. $\quad g(s) = -10\,\dfrac{1}{\frac{s}{-1}+1} = \dfrac{10}{s-1}$

If the TF has a complex-conjugate pair of poles we can generally write a corresponding TF as:

$$G(s) = \frac{1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1} = \frac{\omega_n^2}{s^2 + 2\zeta s \omega_n + \omega_n^2}$$

for these systems the damping ratio ζ influences the form of the Bode plot.

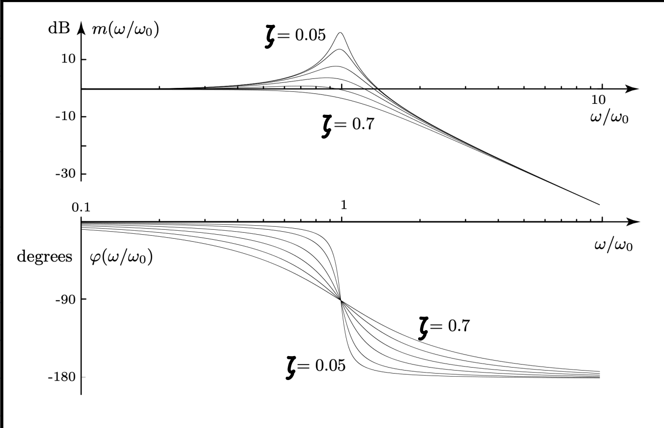
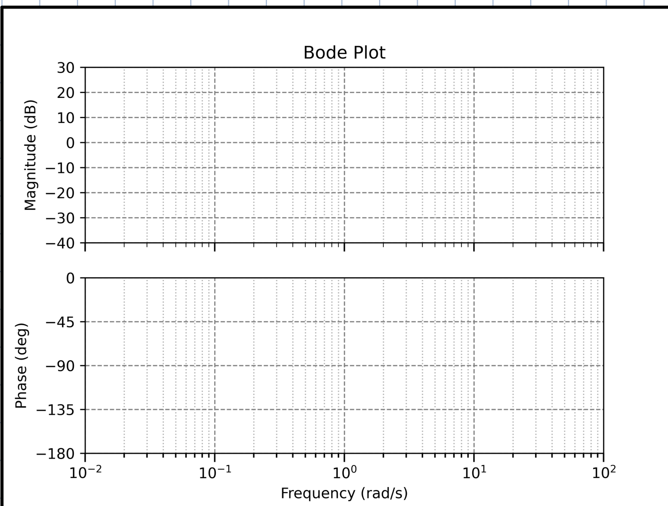


Figure 8.3. Bode diagram of the second-order system (8.15) for six different damping ratios ζ= [0.05, 0.1, 0.2, 0.33, 0.5, 0.7].

Despite this they can be treated like normal poles. So for a TF given by:

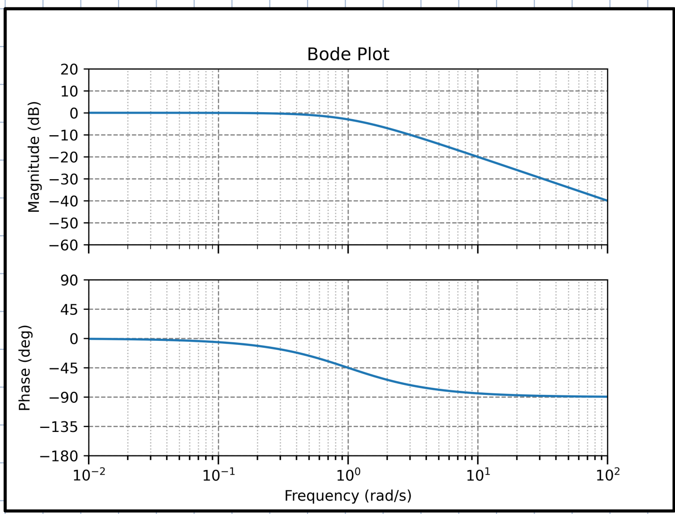
$$G(s) = \frac{1}{s^2 + 0.1s + 1}$$

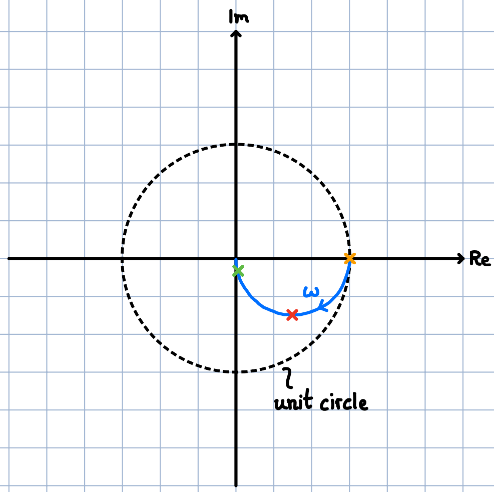
through comparison we can obtain the following values: $\omega_n = 1$, $\zeta = 0.05$. To draw the plot we proceed as usual. Draw a line at 0dB up to ω_n. At ω_n we encounter two poles, i.e. -40 dB/dec. If we now consider $\zeta = 0.05$, like in the plot above, we will see some resonance on the magnitude plot and a very quick phase-shift.



Polar plot:

We can also represent the frequency response as a parametric curve, where ω is implicit. We can then plot one curve in the complex plane. For example $G(s) = \frac{1}{s+1}$

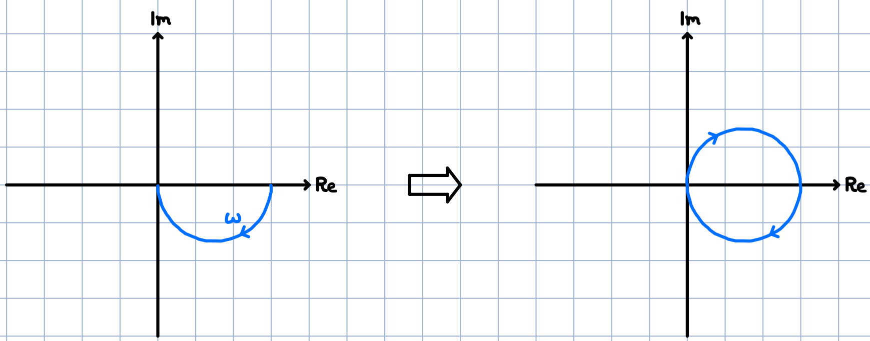




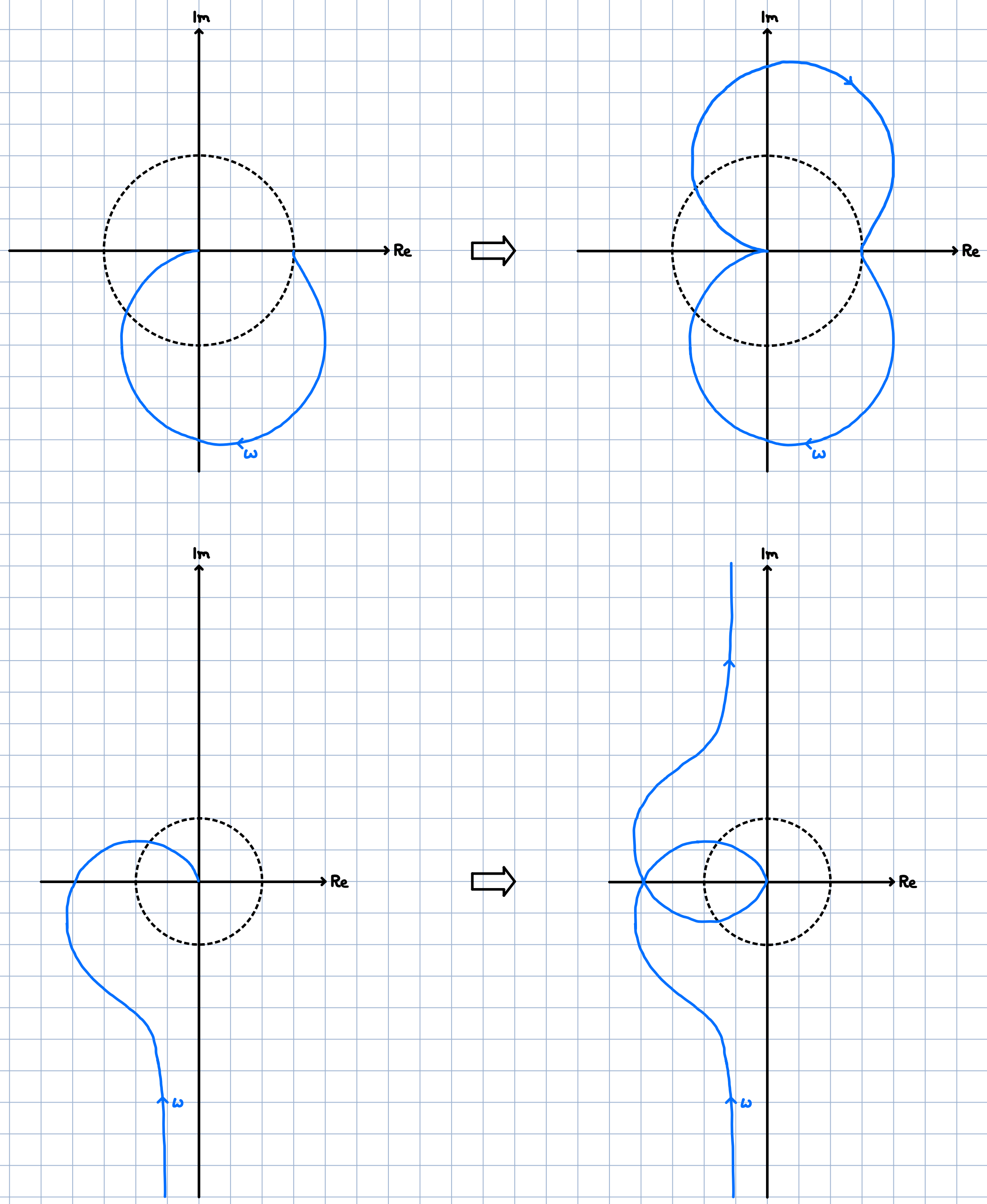
To draw these plots it is usefull to look at special points e.g. $\omega \to 0$ and $\omega \to \infty$, also the intersection(s) with the unit circle and imaginary and real axes. Today we will expand and go further with the polar plot.

Nyquist Plot

To draw this plot, just take the polar plot and mirror it along the real axis!



We will understand why, in just a second. Here are some more examples:



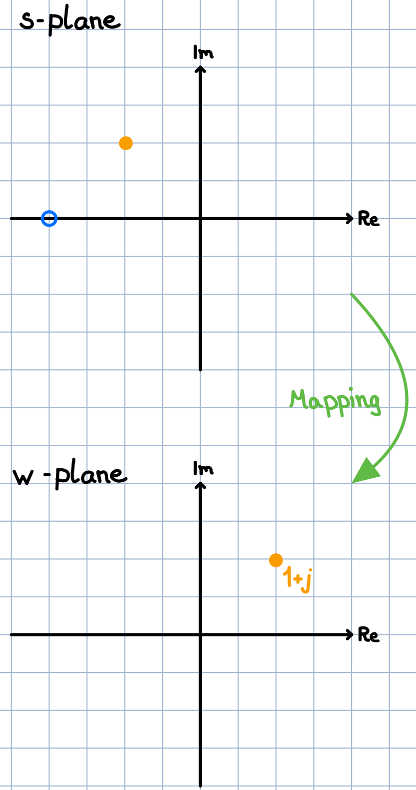
With these plots we will be able to asses stability and robustness of our closed loop system!

First let's take a step back and see how this plot comes about.

For that we will have to take a quick detour to complex analysis. We will look at the principle of varation of the argument (we won't prove it).

Principle of variation of the argument (Cauchy's Argument Principle)

Consider a simple TF given by $G(s) = \frac{s+2}{1}$, it's poles and zeros in the s-plane:



Let's now take some random point, e.g. $s = -1+j$, and plug it into $G(s)$.

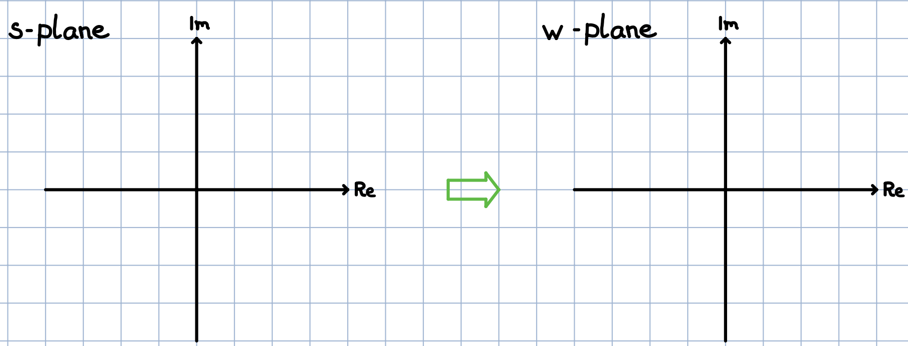
$$G(-1+j) = \frac{(-1+j)+2}{1} = 1+j$$

We get another complex number. Let's plot this new complex number in a new plane and call it the w-plane.

We can now say that our TF $G(s)$ maps a point from the s-plane to the w-plane.

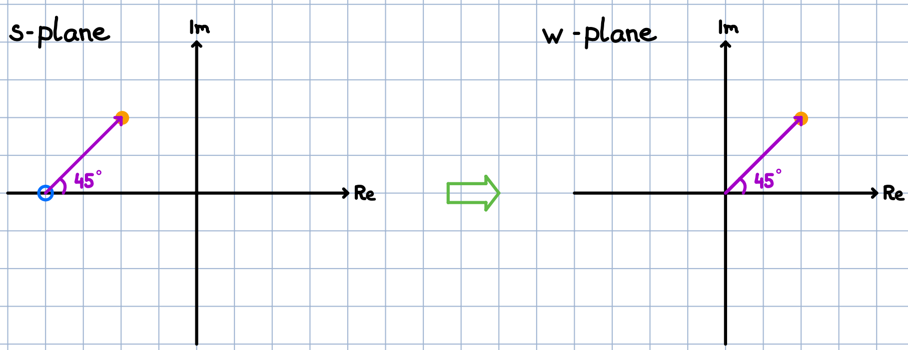
We can extend this mapping and map entire lines from s to w.

We can also look at how closed curves, so called contours, map from s to w.

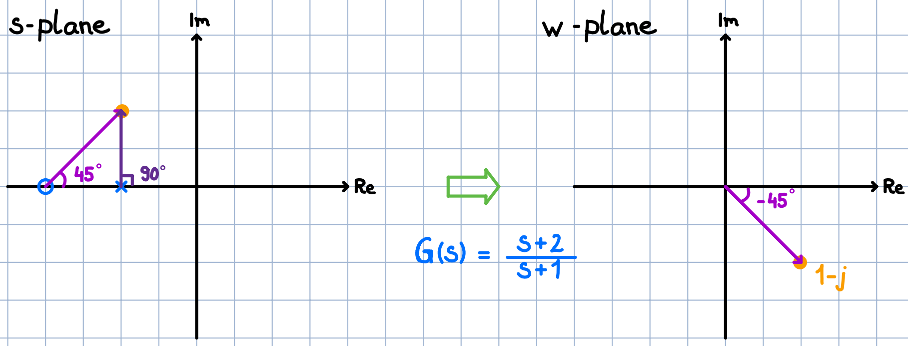


a closed curve in the s-plane will also result in a closed curve in the w-plane. This closed curve in the w-plane now also includes informations about the TF we used to map from s to w.

Let's look at the first example again.

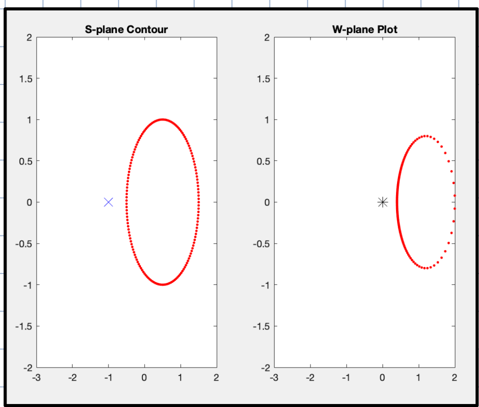


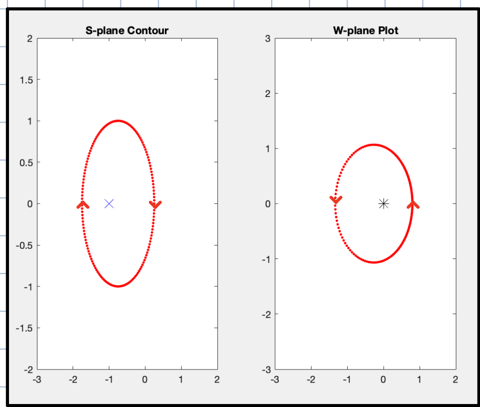
We observe that the phasor of the point in the w-plane is the same as the one from the zero to the original point in the s-plane. Let's add in a pole and see what happens:

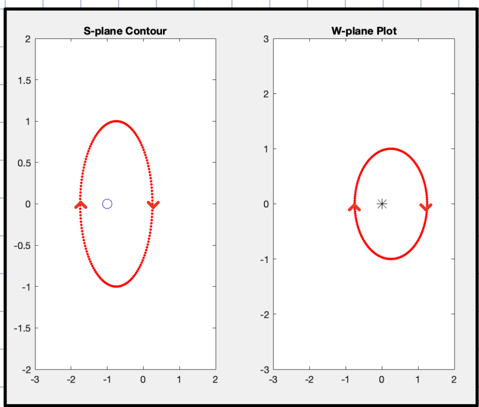


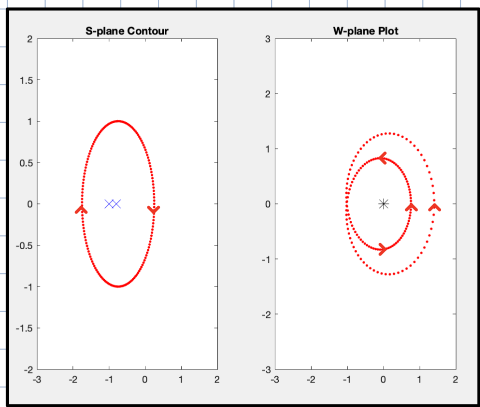
The general rule is: Add phases of zeros subtract phases of poles.

Let's look at some different contours. Matlab



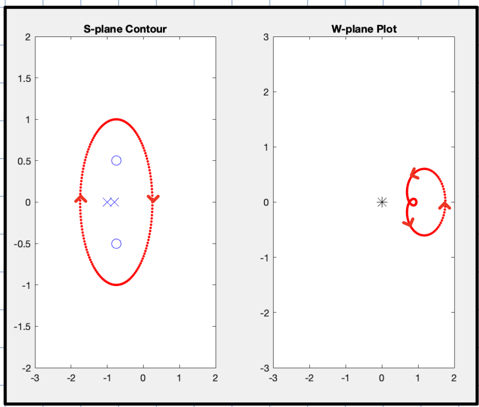


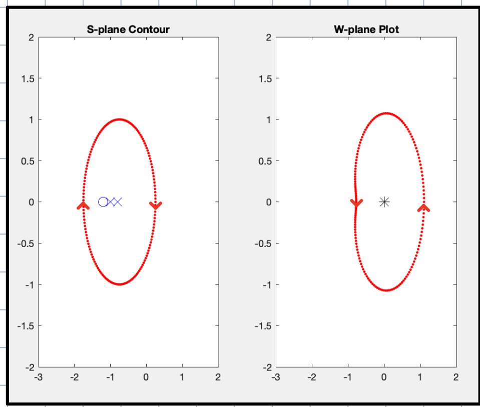




What we can see is that for every time we encircle a pole / zero in the s-plane we also encircle the origin in the w-plane. For each clockwise encirclement of zeros we get one CW encirclement of the origin, and for each CW encirclement of a pole we get one CCW encirclement of the origin. Think of a zero as adding 360° and a pole as subtracting 360°.

If we have as many poles as zeros we end up with no encirclements, and if we have one more pole than zeros, we end up with one CCW encirclement of the origin.





This means that you can tell the relative difference of poles and zeros inside a contour by how many times the plot circles the origin and in which direction.

In other words:

Theorem (Variation of the argument [Proof in A&M, pp. 277–278])

The number N of times that $G(s)$ encircles the origin of the complex plane as s moves along the boundary Γ of a bounded simply-connected region of the plane satisfies

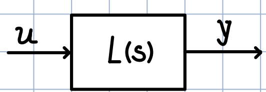
$$N = Z - P,$$

where Z and P are the numbers of zeros and poles of $G(s)$ in D, respectively. Note that the encirclements are counted positive if in the same direction as s moves along Γ, and negative otherwise.

But why is this relevant?

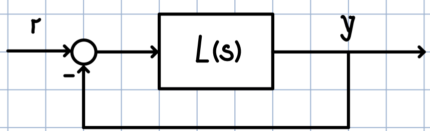
Nyquist Plot:

Recall that for an open loop system,



we can check wether the system is stable by looking at the poles of $L(s)$. If there are any poles in the RHP the system is unstable.

In closed-loop systems:

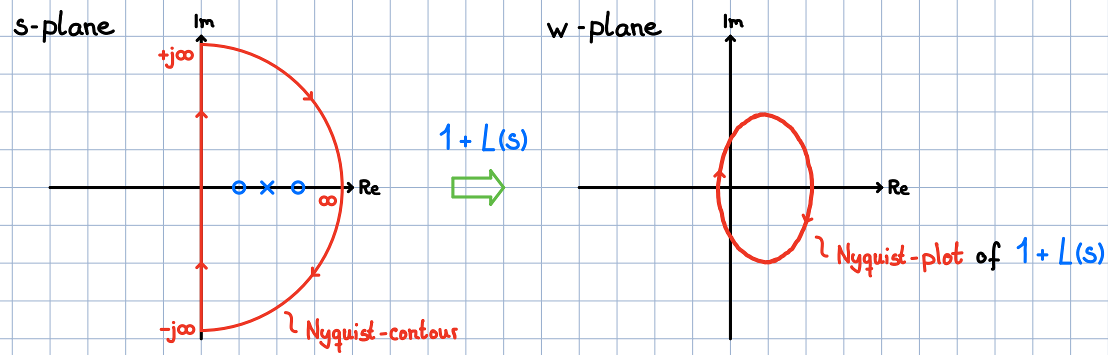


we now have to look at the poles of:

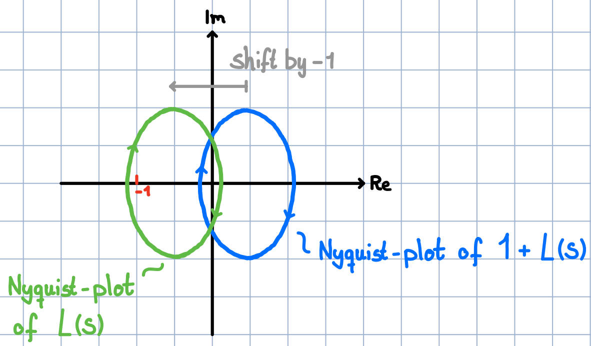
$$T(s) = \frac{L(s)}{1 + L(s)}$$

i.e. all points where $1 + L(s) = 0$. To asses the closed loop stability we have to check if any of the the zeros of $1 + L(s)$ are in the RHP.

To do this we can use the principle of variation of the argument introduced above. We can choose our contour to encircle the entire RHP:



By choosing our contour like this we can tell the relative difference of poles and zeros of $1+L(s)$ in the RHP by counting the encirclements of the Nyquist-plot of $1+L(s)$, arround the origin. A nice characteristic of the Nyquist-plot is that:



So instead of counting the encirclements of $1+L(s)$ arround zero, we can shift the coordinate system by -1 and count the encirclements of $L(s)$ arround -1. This is nice since we know how to draw the Nyquist-plot of $L(s)$.

All in all this means:

$$\boxed{\begin{aligned}\#\text{Encirclements of } 0 \text{ using } 1+L(s) &= \#\text{Zeros of } 1+L(s) \text{ in RHP} - \#\text{Poles of } 1+L(s) \text{ in RHP} \\ &\overset{!}{=} \#\text{Encirclements of } -1 \text{ using } L(s)\end{aligned}}$$

Let's take a closer look. Whats the # Poles of $1+L(s)$ in RHP ?

If $L(s)=\dfrac{N(s)}{D(s)}$, then the poles of $L(s)$ are given by $D(s)$.

We can also re-write $1+L(s)$:

$$1+L(s) = 1+\frac{N(s)}{D(s)} = \frac{D(s)+N(s)}{D(s)}$$, the poles are also given by $D(s)$

$$\boxed{L(s) \text{ and } 1+L(s) \text{ have the same poles}}$$

We can combine both and get:

$$\# \text{Encirclements of } -1 \text{ using } L(s) = \# \text{Zeros of } 1+L(s) \text{ in RHP} - \# \text{Poles of } L(s) \text{ in RHP}$$

Which finally yields the Nyquist stability theorem:

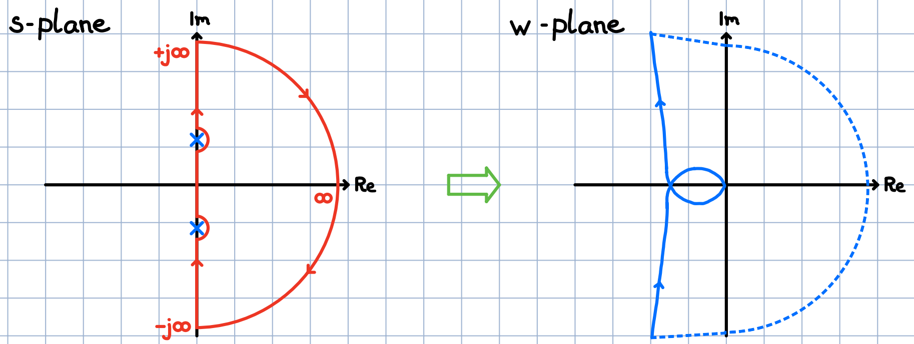
$$\boxed{Z = N + P}$$

- Z: # Zeros of $1 + L(s)$ in RHP = # unstable closed loop poles
- N: # CW Encirclements of -1 using $L(s)$
- P: # Poles of $L(s)$ in RHP = # unstable open loop poles

We can now asses wether a closed loop system is stable, by only looking at the OL poles and the Nyquist-plot

Special case:

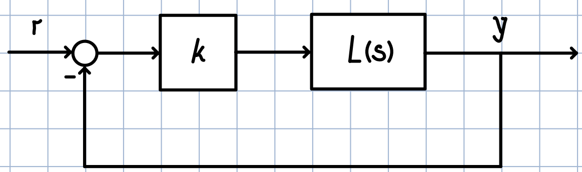
How do we treat poles and zeros of $L(s)$ on the imaginary axis?



We make little indents on the imaginary axis. If you move around the poles CCW, then you have to close the Nyquist-plot CW at infinity.

Note:

If your CL has some gain k:



$$T(s) = \frac{kL(s)}{1 + kL(s)}$$

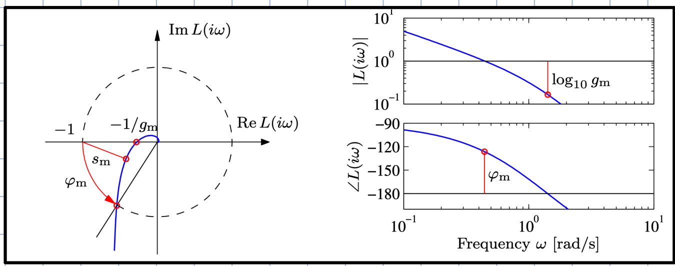
you have to count the #CW Encirclements of $-\frac{1}{k}$.

Stability Margins:

Next to stability of the CL system, the Nyquist-plot can also tell us how far away we are from being unstable.

Assume our OL to be stable, i.e. $P = 0$. For our CL system to be stable we now need $N = 0$. Then $Z = N + P = 0$.

We can now define a phase- and gain margin that tell us how "close" we are to encircling -1.



What does each margin mean?

→ Gain Margin g_m: The point at 180°. It tells us how much we can scale until reaching -1.

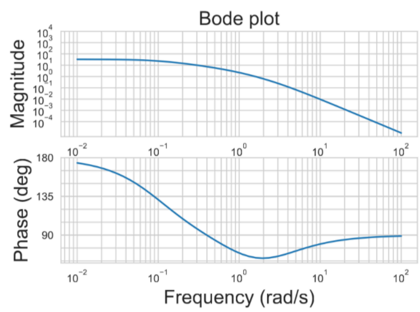
→ Phase Margin φ_m: Point at magnitude 1. It tells us how much we can change the phase until reaching -1.

For this special case of a stable OL system we can also read the margins from the Bode-plot. If the OL is unstable check the stability of the CL with Nyquist or Root locus.

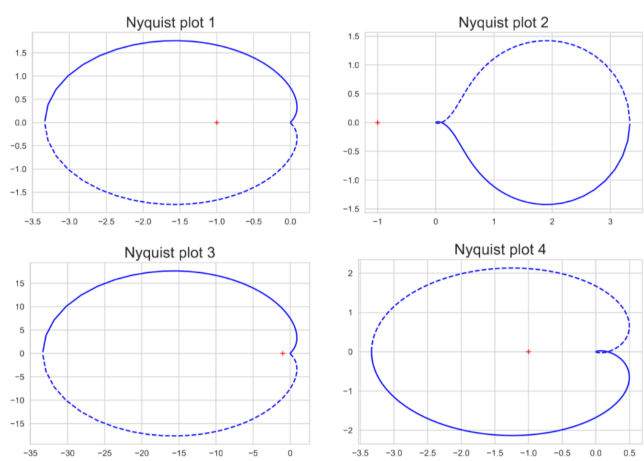
Exam Question:

Question 24 *Choose the correct answer. (1 Point)*

You are given the following Bode plot.



What is the associated Nyquist plot to this Bode plot?



[A] 3
[B] 4
[C] 1
[D] 2