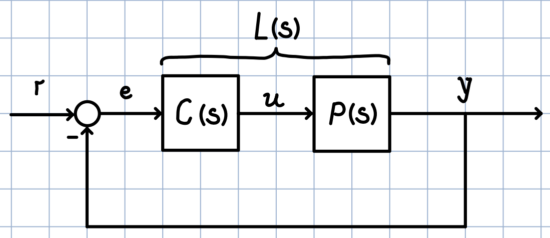
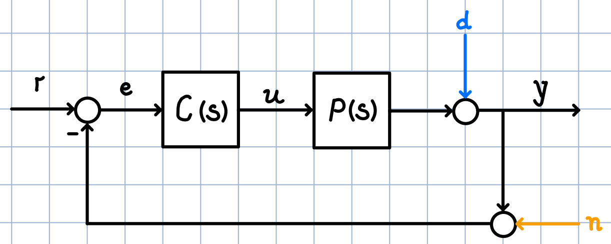
12. Recital 06.12.24

Recap

Frequency domain specifications:

Similar to the time-domain specs. that resulted in feasible areas in the s-plane, we can define frequency domain specs, that dictate/shape how our Bode plot should look like. Recall the sensitivity/CL TFs we introduced a while back.





→ Open loop TF:

$$L(s) = C(s)\,P(s)$$

→ Complementary Sensitivity:

maps $r \to y$, $n \to y$

$$T(s) = \frac{L(s)}{1 + L(s)}$$

→ Sensitivity:

maps $r \to e$, $d \to y$

$$S(s) = \frac{1}{1 + L(s)}$$

If we have disturbances d and/or noise n entering our CL system, we can use T and S to map the noise n and disturbances d to the output y. Usually disturbances have low frequencies and noise has high frequencies. The commands we input to our system usually also have a relativley low frequency. Knowing this we can constrain the magnitudes of $S(j\omega)$ and $T(j\omega)$ for specific frequencies.

→ $|S(j\omega)| \ll 1$ at low frequencies for disturbance rejection and good command tracking.

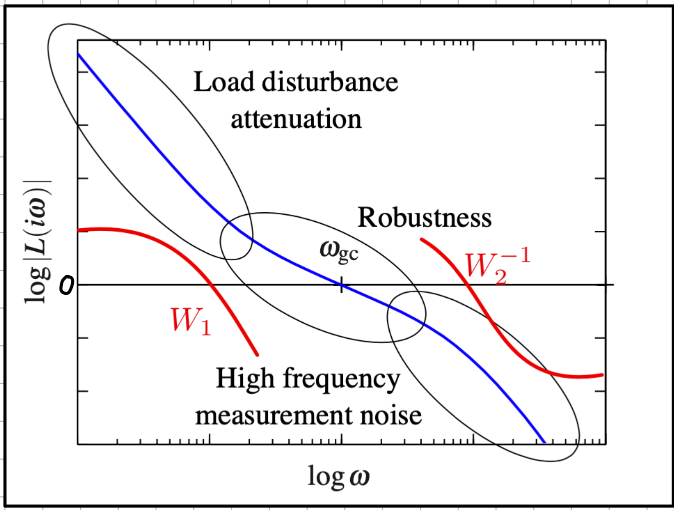
→ $|T(j\omega)| \ll 1$ at high frequencies for noise rejection.

We can rewrite thas as functions of $L(s)$ and with some function $W(j\omega)$.

→ $|L(j\omega)| > |W_1(j\omega)|$ at low frequencies

→ $|L(j\omega)| < |W_2(j\omega)|^{-1}$ at high frequencies

This results in the following "obstacle course" for the Bode plot of $L(j\omega)$.



Next to high- and low-frequency behavior we can also constrain the bandwidth of CL system. The bandwidth tells us the maximum frequency for which the output can track commands within a factor ≈ 0.7.

We can usually approximate the CL bandwidth with the OL crossover frequency ω_{gc}.

Loop Shaping:

To design our controler $C(s)$ given the desired behavior of $L(s)$ we can use some basic building blocks to steer $L(s)$ through the Bode obstacle course. That way we construct a dynamic compensator $C(s)$ that fullfills our requirements.

Proportional compensation:

In this case $C(s) = k$, where k is a simple gain.

→ Shifts the magnitude, while phase is unaffected.

→ Improves command tracking (higher magnitude at low ω) and CL bandwidth (moves crossover f req. to right)

→ Stability can be compromized!!

Lead compensator

A lead compensator is a pole zero pair where the zero always comes before the pole.

$$C_{lead}(s) = \frac{\frac{s}{a}+1}{\frac{s}{b}+1} = \frac{b}{a}\,\frac{s+a}{s+b} \qquad 0 < a < b$$

Main use: increase phase margin. $C_{lead}(s) = k\,\frac{b}{a}\frac{s+a}{s+b}$ optional to adjust ω_c

i. Pick $\sqrt{ab}$ at desired ω_c

ii. Pick $\frac{b}{a}$ depending on desired phase increase.

iii. Adjust k to put ω_c at desired freq.

Side effects: increase magnitude at high freq.

Lag compensator:

A lag compensator is a pole zero pair where the pole always comes before the zero.

$$C_{lag}(s) = \frac{\frac{s}{a}+1}{\frac{s}{b}+1} = \frac{b}{a}\frac{s+a}{s+b} \qquad 0<b<a$$

Main use: improve command tracking / disturbance rejection. $C_{lag}(s) = k\,\frac{\frac{s}{a}+1}{\frac{s}{b}+1}$

i. Pick $\frac{a}{b}$ as the desired increase in magnitude of low freq.

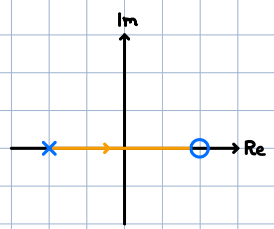
ii. Multiply k by $\frac{a}{b}$ (high freq. not affected)

iii. Pick a to be sufficiently small not to affect ω_c

Side effects: reduction of phase margin.

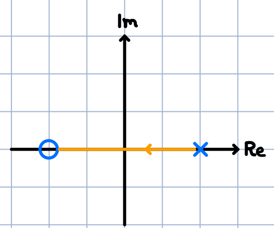
Limitations:

If one of the zeros is non-minimum-phase, the closed loop system might become unstable.



For large enough gains the closed loop will become unstable. The CL system becomes slow.

If we have a OL pole in the RHP. We need a high gain to stabalize the system.



This should give you a sense of how, for many systems, there are clear performance limitations. Certain requirements can thus never be satisfied.

Time Delays

After choosing our controler $C(s)$, we have to implement it. Usually we use computers for this task. Unfortunatley, computers have a finite compute time, which means that the control input to a certain error has some delay. Some physical systems themselves also have delays. An extreme example would be communication with e.g. a mars rover.

In this example the delay can be several minutes. How can we take this into account?

Mathematically we can express a time delay as:

$$y(t) = u(t-T)$$

The time delay is a linear operator that transforms an input $u(t)$ into a delayed output $y(t)$. T is the ammount of delay.

We can also compute the TF of a time delay. Assume $u(t) = e^{st}$.

$$y(t) = e^{s(t-T)} = e^{-sT} u(t)$$

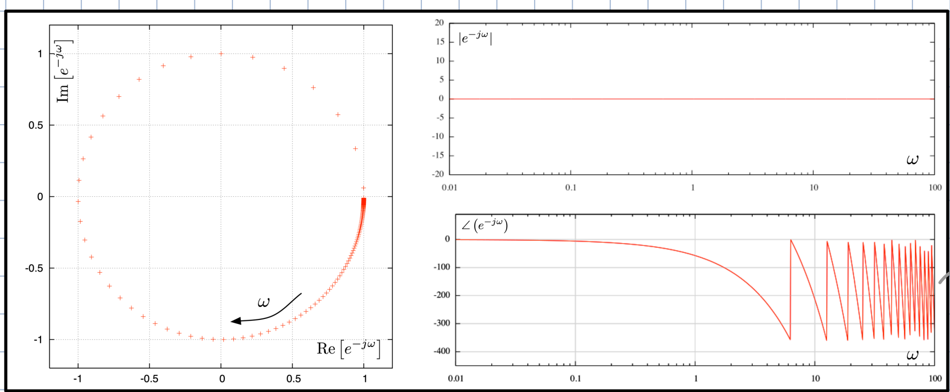
The TF of a time delay is thus given by: e^{-sT}

This not a rational function! We have no poles or zeros! Root locus doesn't work to asses closed loop behavior.

Let's take a closer look at frequency response of a time delay. Remember that for the frequency response we plug in $s = j\omega$ and look at the resulting phase and magnitude. Assume $T = 1$

$$|G(j\omega T)| = |e^{-j\omega T}| = 1$$

$$\angle G(j\omega T) = \angle e^{-j\omega T} = -\omega T$$



Freq wraps around 360°

Thus we can summarize the effect of a time delay as a phase shift of $-\omega T$. That means that for any system with a time delay we have to be very careful about the phase!

Gain and phase margins:

To get the gain margin we look at the first time we cross $-180°$, i.e.:

$$\angle G(j\omega) - \omega T = -180°$$

Similarly we can look at the effect on the phase margin and see that:

$$\varphi_m = \varphi_{m,0} - \omega_c T$$

where $\varphi_{m,0}$ is the phase margin without the time delay and ω_c the crossover frequency. Overall we see a decrease in phase margin that gets amplified as the crossover frequency increases.

Control design with time delays:

So how can we deal with these time delays and design controller that work with them? One way is to:

i. Design a feedback control system ignoring the delay

ii. Check the effective phase margin. If margin too small:

- → Increase phase at crossover (e.g. lead compensator.)
- → Decrease crossover frequency (e.g. smaller gain)

iii. Iterate until good.

We can also approximate the time delay with a rational function to then use the known methods.

Padé approximation

In the Padé approximation we represent the time delay as a ratio of two polynomials. A first order approximation would look like this:

$$e^{-sT} \approx k \frac{s+p}{s+q}$$

To get the values of the coefficients we can compare this term to the taylor series.

$$k \frac{s+p}{s+q} = 1 - sT + \frac{1}{2}(sT)^2 - \frac{1}{6}(sT)^3 \ldots$$

And considering therms upto order 2 we get:

$$e^{-sT} \approx \frac{\frac{2}{T} - s}{\frac{2}{T} + s}$$

With this approximation we can use things like the root locus again. Note the non-minimum-phase zero! This limits the max. gain in our CL.

Nonlinear Systems

All of the tools we have learned in this course are only valid for linear system. Thus most systems in the real world are nonlinear. To bridge this gap we linearized systems around equilibrium points with the Jacobian linearization procedure that takes some system

$$\dot{x}(t) = f(x(t), u(t))$$

$$y(t) = h(x(t), u(t))$$

and produces a linearized system valid around the equilibrium point. The system is then given by:

$$A = \frac{\partial f(x,u)}{\partial x}\bigg|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{n\times n} \qquad B = \frac{\partial f(x,u)}{\partial u}\bigg|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{n\times m}$$

$$C = \frac{\partial h(x,u)}{\partial x}\bigg|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{p\times n} \qquad D = \frac{\partial h(x,u)}{\partial u}\bigg|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial u_1} & \cdots & \frac{\partial h_p}{\partial u_m} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{p\times m}$$

Given this linearization, the Hartman-Großman theorem tells us that if the linearized system is closed-loop BIBO stable, then the nonlinear system is also stable in a range around the equilibrium point. We just don't know how large that region is.

If we now want to design a controler for a nonlinear system we proceed as follows:

i. Design a linear compensator for the linear model

ii. If the system is CL stable the nonlinear system will also be stable in the region of the equilibrium

iii. Check in a nonlinear simulation if your design is robust with respect to typical deviations.

Exam problems:

HS22

Problem: Consider the closed-loop system as shown in Figure 5.

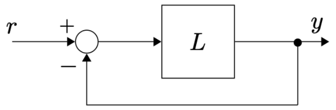


Figure 5: Closed-loop system.

Where L is given as

$$L(s) = \frac{s-5}{(s+3)(s-10)} e^{-10 \cdot s}.$$

Q18 **(0.5 Points)** Mark the correct answer for each statement.

Statement	**True**	**False**
The closed-loop stability of the above system can be assessed by using the Nyquist plot of L.		
The closed-loop stability of the above system can be assessed by relying on the root locus of L.		
The closed-loop stability of the above system can be assessed by relying on the Bode plot of L by testing for $\lvert L(j\omega_{pc})\rvert < 1$, where ω_{pc} is the the phase crossover frequency, i.e. $\angle L(j\omega_{pc}) = -180°$.		

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Question 34 *Choose the correct answer. (1 Point)*

The transfer function of a time delay is...

- linear and rational.
- nonlinear and rational.
- nonlinear and not rational.
- linear and not rational.

HS23

Problem: Consider the closed-loop system T shown in Figure 22, where L represents a linear time-invariant plant and $k \in \mathbb{R}$. We denote the transfer function corresponding to the system L by $L(s)$.

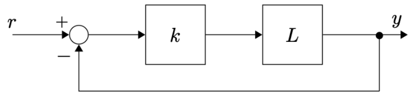

Figure 22: Closed-loop system T. Linear time-invariant system L and proportional gain k.

Q38 **(1 Points)** Mark all correct statements.

For any linear time-invariant system L, by relying on the Nyquist criterion, we can assess the stability of the closed-loop system T as a function of k.

For rational transfer functions $L(s)$, the root locus of $L(s)$ shows the poles of the closed-loop system T in the complex plane as a function of k.

For non-minimum phase and unstable linear time-invariant systems L, by relying on the Bode plot of L, we can reliably assess the stability of the closed-loop system T by checking whether $|L(j\omega_{\mathrm{pc}})| < 1$,where ω_{pc} is such that $\angle L(j\omega_{\mathrm{pc}}) = -180°$.

Both the Polar plot of $L(s)$ and the Bode plot of $L(s)$ are representations of the complex number $L(j\omega)$ for $\omega \geq 0$.

Problem: Address the question below.

Q39 **(1 Points)** Mark all correct statements.

Time delays are linear systems/operators.

Let $L(s)$ represent the transfer function of a linear time-invariant system without delays. Let $\tilde{L}(s)$ represent the transfer function of the same system as $L(s)$, where the output of the system now is delayed by 5s, i.e. $\tilde{L}(s)$ is a delayed version of the system represented by $L(s)$. It holds that, $\forall \omega \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} |\tilde{L}(j\omega)| &= |L(j\omega)|, \text{ and,} \\ \angle \tilde{L}(j\omega) &= \angle L(j\omega) - 5\omega. \end{aligned}$$

The transfer function of a time delay is a rational transfer function.

Time delays can cause closed-loop systems to become unstable.