13. Recital 13.12.24

Recap

Time Delays:

Mathematically we can express a time delay as:

$$y(t) = u(t-T)$$

The time delay is a linear operator that transforms an input $u(t)$ into a delayed output $y(t)$. T is the ammount of delay.

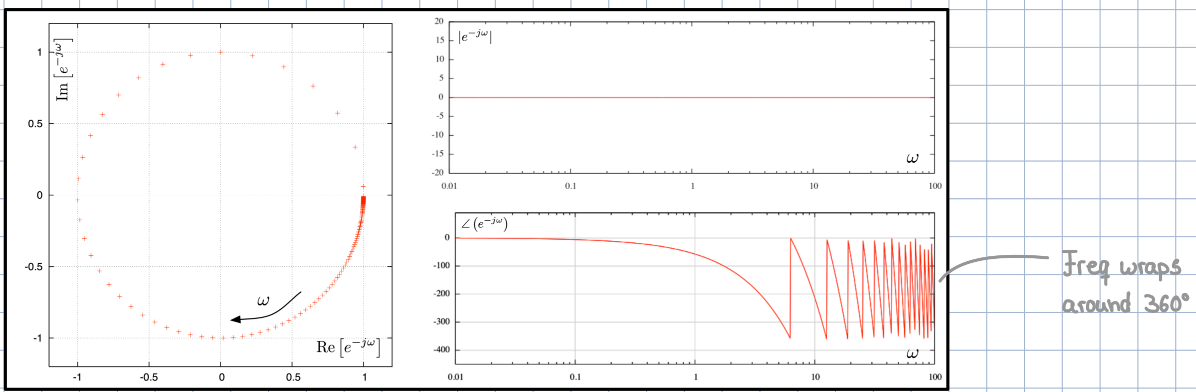
The TF of a time delay is thus given by: e^{-sT}

This not a rational function! We have no poles or zeros! Root locus doesn't work to asses closed loop behavior.

Let's take a closer look at frequency response of a time delay. Remember that for the frequency response we plug in $s = j\omega$ and look at the resulting phase and magnitude. Assume $T = 1$

$$|G(j\omega T)| = |e^{-j\omega T}| = 1$$

$$\angle G(j\omega T) = \angle e^{-j\omega T} = -\omega T$$

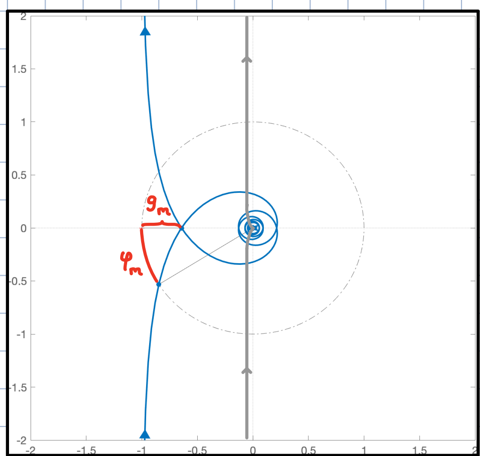


Thus we can summarize the effect of a time delay as a phase shift of $-\omega T$.

Gain and phase margins:

To get the gain margin we look at the first time we cross $-180°$, i.e.: $\angle G(j\omega) - \omega T = -180°$

Similarly we can look at the effect on the phase margin and see that: $\varphi_m = \varphi_{m,0} - \omega_c T$



Padé approximation

In the Padé approximation we represent the time delay as a ratio of two polynomials. A first order approximation would look like this:

$$e^{-sT} \approx k\,\frac{s+p}{s+q}$$

To get the values of the coefficients we can compare this term to the taylor series.

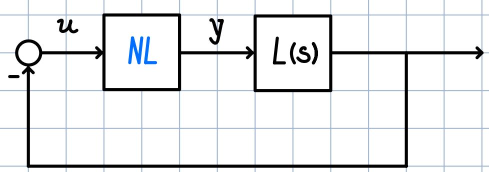
$$k\,\frac{s+p}{s+q} = 1 - sT + \frac{1}{2}(sT)^2 - \frac{1}{6}(sT)^3 \ldots$$

And considering therms upto order 2 we get:

$$e^{-sT} \approx \frac{\frac{2}{T} - s}{\frac{2}{T} + s}$$

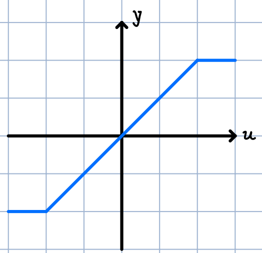
Describing Functions

As briefly mentioned last week we want to start to understand nonlinear systems. Today we will briefly introduce some concepts that can help us. We will manliy consider systems like this:

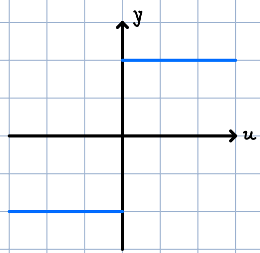


where $L(s)$ is still a linear function, and NL is a non-linear gain. NL can represent some important nonlinearity like:

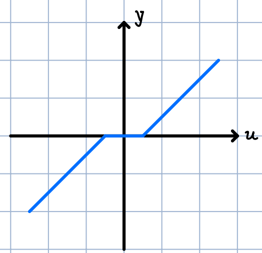
Static, memoryless:



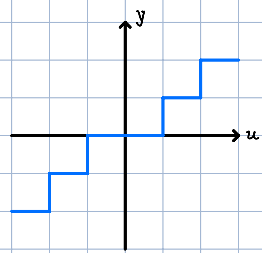
Saturation



Switch, or Relay

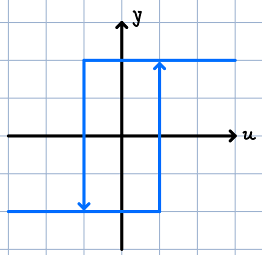


Deadzone



Quantizer

Dynamic, with memory:



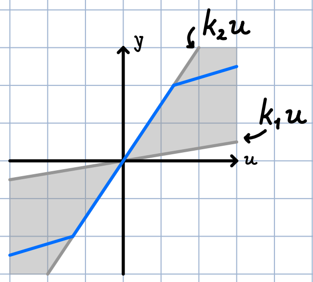
Schmitt Trigger

We can also represent them mathematically. The saturation, e.g.:

$$y = \begin{cases} & \text{if } u \geq 1; \\ & \text{if } -1 < u < 1; \\ & \text{if } u \geq -1. \end{cases}$$

Stability:

For this we will consider the general case from above where where $L(s)$ is still a linear function, and NL is a non-linear gain. We can now define some boundaries that contain all possible values of NL.



Mathematically:

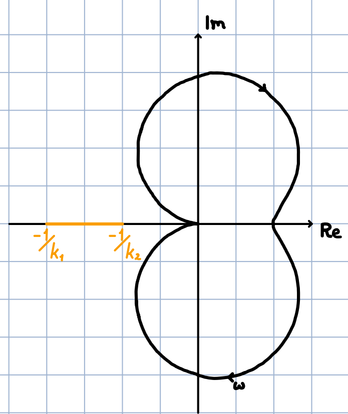
$$k_1 u \leq NL(u) \leq k_2 u$$

$$NL(0) = 0$$

A system with such a non-linear gain is said to be absolutley stable, for any choice of NL within the boundaries, if for $u=0$, $y \to 0$. So $u=0$ is a globally stable equilibrium, i.e. every initial condition with zero input converges to zero. We can also check the CL stability of the system with the NL in the Nyquist plot.

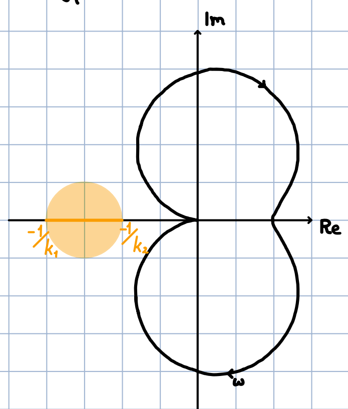
Necessary condition:

The way we defined the NL, it can include all linear gains $k_1 \leq k \leq k_2$. For the Nyquist critereon this means that we have to consider not only the point $-1/k$ but all points in [,]. Graphically you have to consider



Sufficient condition:

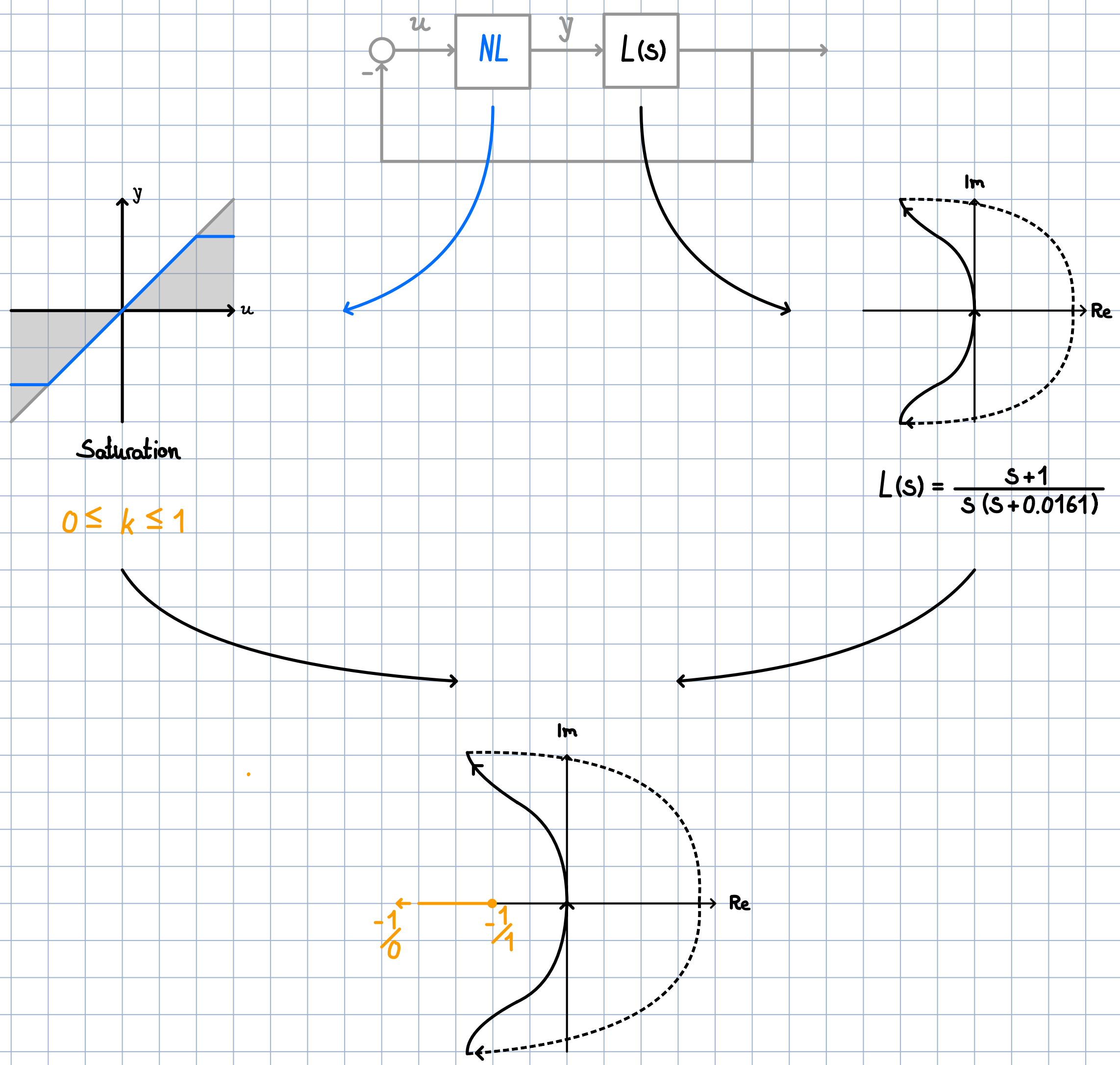
We go even further and extend the Nyquist condition to count encirclements of a with diameter $[-1/k_1, -1/k_2]$.



Example: Cruise control

This is an example of a saturation NL, since the throttle can't be actuated more than a certain value.

We can simplify the system to:



In this case we can not conclude what the stability is, since the sufficient condition is not fullfilled. But what happens here? Can we further understand how the NL influences the system?

Describing Functions:

As with the frequency response for linear systems, we will consider what happens when we apply

$$u(t) = A\sin(\omega t).$$

The output will now be of the form

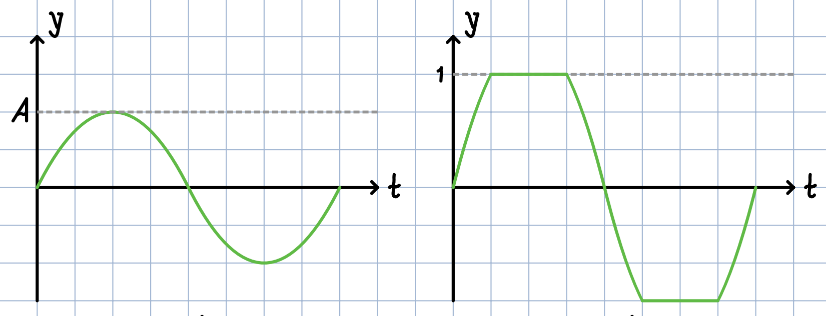
$$y(t) =$$

That means some periodic function with the same frequency as the input. The output is now also dependend on the input amplitude A.

Example: Saturation

$$y = \begin{cases} 1 & \text{if } u \geq 1; \\ u & \text{if } -1 < u < 1; \\ -1 & \text{if } u \geq -1. \end{cases}$$

$$u(t) = A\sin(\omega t).$$



$A \leq 1$ $A > 1$

Since $y(t) = f(A\sin(\omega t))$ is a periodic function we can write the corresponding Fourier series expansion

$$y(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty}[a_i\cos(i\omega t) + b_i\sin(i\omega t)],$$

with the coefficents a_i and b_i given by

$$a_i = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t)\cos(i\omega t)\,d(\omega t) \quad \text{and} \quad b_i = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t)\sin(i\omega t)\,d(\omega t)$$

i is an index not $\sqrt{-1}$

for odd functions $a_i = 0 \ \forall i \in \mathbb{N}$. We can use this and approximate the output with the first harmonic, i.e. only consider $i = 1$. For an odd NL this would mean

$$y(t) \approx b_1\sin(\omega t).$$

This approximation works since physical systems usually attenuate high frequencies, acting as a low pass filter. The value of b_1 is a function of A, the input amplitude. We now define the describing function to be

$$N(A) = \frac{b_1(A)}{A} = \frac{1}{\pi A}\int_{-\pi}^{\pi} y(t)\sin(i\omega t)\,d(\omega t),$$

the ratio $\frac{b_1}{A}$. We can use this function to approximate NL as an amplitude-dependend gain. We can generalize to any input $u(t) = Ae^{j\omega t}$, for which the output can be approximated using the complex Fourier series

$$y(t) \approx c_1(A,\omega)\,e^{j(\omega t + \phi_1(A,\omega))}$$

in this general case the first complex harmonic coefficient can be expressed as $c_1(A,\omega)\,e^{j\phi_1(A,\omega)}$. The rusulting describing function can then also be complex. We write it as

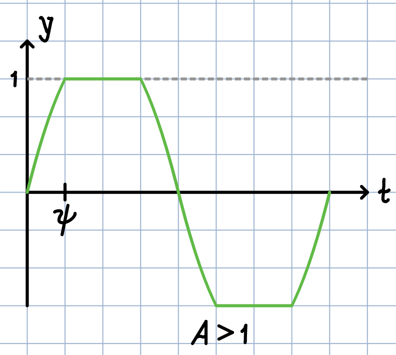
$$N(A,\omega) = \frac{c_1(A,\omega)}{A}\,e^{j\phi_1(A,\omega)}.$$

Example: Saturation

The saturation is odd and thus $a_i = 0 \ \forall i \in \mathbb{N}$. If we assume we're in saturation, i.e. $A > 1$ we define ψ as $A\sin(\psi) = 1$, the point where the saturation kicks in.

$$\psi = \arcsin\left(\frac{1}{A}\right)$$

Now we have to calculate b_1



$$b_1 = \frac{4}{\pi}\int_0^{\pi/2} y(t)\sin(\psi)\,d\psi$$

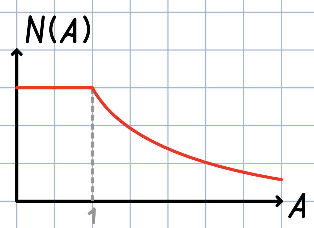
$$b_1 = \frac{4}{\pi}\int_0^{\psi} A\sin(\psi)\sin(\psi)\,d\psi + \frac{4}{\pi}\int_{\psi}^{\pi/2} 1\,\sin(\psi)\,d\psi$$

$$= \dots$$

$$= \frac{2}{\pi}A\left[\arcsin\left(\frac{1}{A}\right) + \frac{1}{A}\sqrt{1-\left(\frac{1}{A}\right)^2}\right]$$

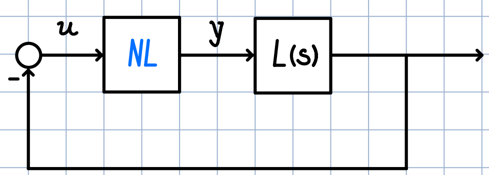
Hence

$$N(A) = \begin{cases} \dfrac{b_1(A)}{A} = \dfrac{2}{\pi}\left[\arcsin\left(\dfrac{1}{A}\right) + \dfrac{1}{A}\sqrt{1-\left(\dfrac{1}{A}\right)^2}\right] & \text{, for } A>1 \\ 1 & \text{, for } A\le 1 \end{cases}$$



How is this useful?

Consider again the starting point:



We can now say for some $u(t) = A\sin(\omega t)$ we can approximate $y(t) \approx b_1 \sin(\omega t)$. Since $N(A) = \frac{b_1}{A}$

we can also say

$$y(t) \approx A\,N(A)\sin(\omega t)$$

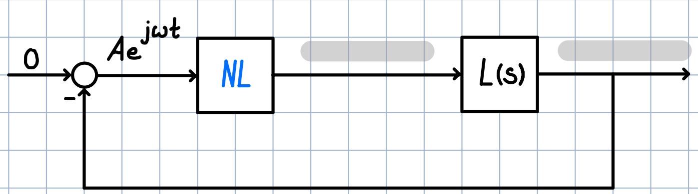
in other words, an approximation for the OL TF of the system with the NL is given by

$$L'(A,s) \approx N(A)\,L(s).$$

L' is now dependend on A, the input amplitude. We can now see what happens for different A.

Limit Cycles:

Consider the case of the input to your NL to be of the form $Ae^{j\omega t}$. Let's see how this signal propagates.



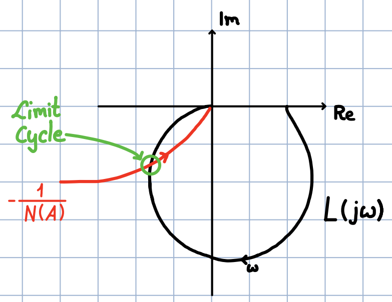
we can observe that if

then the feedback loop is self-sustaining. The input $Ae^{j\omega t}$ produces an output of $-Ae^{j\omega t}$ which is negativley fed-back. This causes the system to oscilate indefinetley, causing a so called limit cycle. A condition for the exsistance of limit cycles can be derived from above:

Checking for limit cycles:

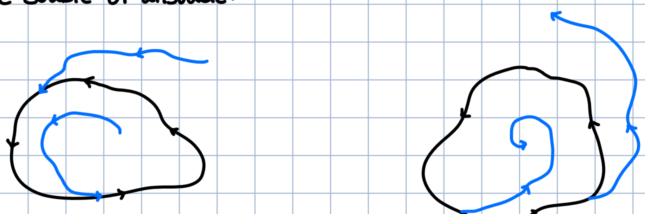
There is a nice way to verify graphically wether there are limit cycles present:

- → Sketch polar plot of $L(j\omega)$
- → Sketch polar plot of $-\frac{1}{N(A)}$
- → Check for intersections.



Stability of limit cycles:

Limit cycles can be stable or unstable:

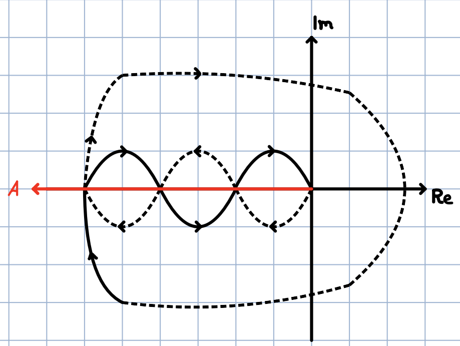


stable "attracts"

unstable "repels"

How does this look on the Nyquist plot? (Assume OL stability).

If a point $-\frac{1}{N(A)}$ is in an unstable part of the Nyquist plot (U), the amplitude of oscilations will increase. If a point $-\frac{1}{N(A)}$ is in a stable part of the Nyquist plot (S), the amplitude of oscilations will decrease.



Exam Problem:

Problem: Consider the closed-loop system T shown in Figure 23, where L is a linear time-invariant system and where NL is a saturation non-linearity.

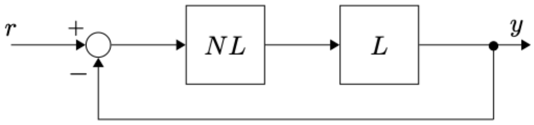


Figure 23: Standard feedback architecture. System L and saturation non-linearity denoted by NL.

The describing function $N(A)$ of the saturation non-linearity is given by,

$$N(A) = \frac{2}{\pi}\left[\arcsin\left(\frac{1}{A}\right) + \frac{1}{A}\sqrt{1 - \left(\frac{1}{A}\right)^2}\right].$$

It is known that $L(s)$ has a pole at the origin, i.e. at $s = 0$, and that $L(s)$ has one unstable pole. Figure 24 shows the Nyquist plot of $L(s)$ (solid line) together with $\frac{-1}{N(A)}$ (dashed line). Further, assume that all assumptions required for a describing function analysis are met.

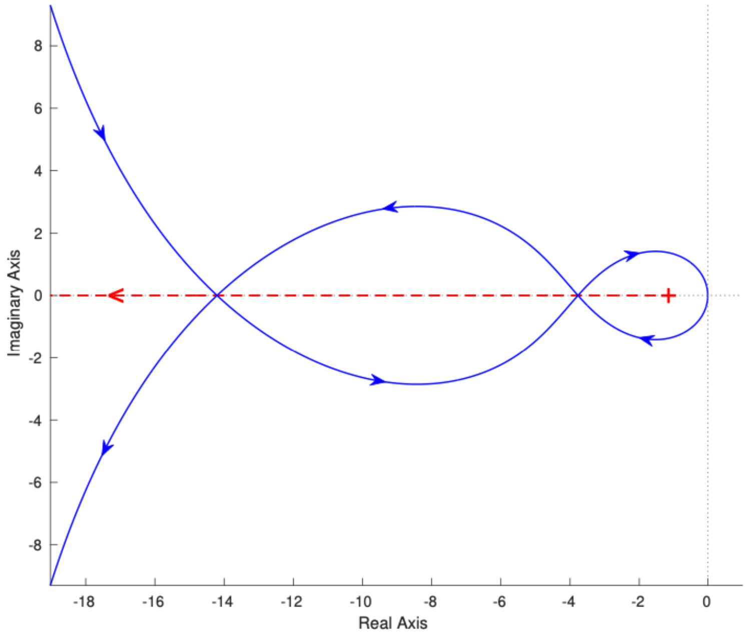


Figure 24: [Solid line] Nyquist plot of $L(s)$. [Dashed Line] Plot of $\frac{-1}{N(A)}$, where the arrow indicates the direction of $\frac{-1}{N(A)}$ as A increases.

Q40 **(0.5 Points)** Mark the correct answer.
First, without doing any stability analysis, what is the maximum number M of potential limit cycles that the closed-loop system T could support?

- ☐ $M = 0$
- ☐ $M \to \infty$
- ☐ $M = 2$
- ☐ $M = 1$

Problem: Address the question below.

Q42 **(1 Points)** Mark all correct statements.

- ☐ The circle criterion is a necessary and sufficient condition for absolute stability.
- ☐ Describing functions can be seen as approximate transfer functions that represent the transfer function from sinusoidal inputs of amplitude A and frequency ω to the first harmonic of the output of the corresponding non-linearity.
- ☐ Limit cycles are typically observed in linear systems.
- ☐ The describing function of a static non-linearity is an amplitude dependent gain $N(A)$.