Control Systems I
PVK Script

Nicolas Bartzsch
nbartzsch@ethz.ch

Fall 2024

0 Preface

This script is based on the materials from my recitals from Fall 2024. Additionally, lecture slides were used, and this script is heavily based on the lecture content. The script was made in the scope of the PVK in January 2025 and is intended to serve as a study aid and review for the material. It contains both theory and example problems.

The exam questions are mainly taken from past exams from Prof. Frazzoli, especially *HS16, FS17, HS17, FS18.* Since the newer exams have solutions with exaplnations, I decided to focus more on the older exams, leaving the newer ones for self study. A complete collection of old exams, with solutions, can be found on `https://exams.amiv.ethz.ch/category/controlsystemsi`

Despite revisions of the script, I **cannot guarantee either completeness or correctness**. It is possible that small errors are present. If you notice such an error, I would be grateful if you could inform me by email so that the script can be corrected.

You can find the latest version of this script and other materials on my website: `n.ethz.ch/~nbartzsch/`.

Thank you and good luck with Control Systems I.

Nicolas Bartzsch

Version: February 14, 2025

Contents

0 Preface **i**

1 Introduction **1**

2 Modeling **2**
2.1 Example Problems . . . 6
2.1.1 Example from Lecture . . . 6
2.1.2 FS 2017, Questions 7 and 8 . . . 7
2.1.3 HS 2016, Questions 6 and 7 . . . 8
2.1.4 HS 2017, Question 5 . . . 9

3 System Classification and Linearization **10**
3.1 Example Problems . . . 15
3.1.1 HS 2017, Question 1 . . . 15
3.1.2 HS 2016, Question 3 . . . 15
3.1.3 HS 2016, Question 4 . . . 16
3.1.4 FS 2017, Question 1 . . . 16
3.1.5 HS 2017, Question 3 . . . 17
3.1.6 HS 2017, Question 4 . . . 17
3.1.7 HS 2016, Question 9 . . . 18
3.1.8 FS 2018, Question 5 . . . 19

4 Time Response and Stability **20**
4.1 Example Problems . . . 24
4.1.1 FS 2018, Question 6 . . . 24
4.1.2 FS 2016, Questions 11 and 13 . . . 25
4.1.3 FS 2018, Question 7 . . . 26
4.1.4 FS 2017, Question 12 . . . 26
4.1.5 FS 2016, Question 13 . . . 27

5 Transfer Functions **28**
5.1 Example Problems . . . 34
5.1.1 HS 2016, Questions 18 and 19 . . . 34
5.1.2 FS 2018, Question 8 . . . 35
5.1.3 FS 2018, Question 11 . . . 36
5.1.4 HS 2017, Question 19 . . . 37
5.1.5 HS 2016, Question 21 . . . 38

6 Root Locus **39**
6.1 Example Problems . . . 44
6.1.1 FS 2018, Question 16 . . . 44
6.1.2 HS 2017, Question 25 . . . 45
6.1.3 FS 2018, Question 15 . . . 46

6.1.4 FS 2018, Question 17 . . . 47

7 Time Domain Specifications **48**
7.1 Example Problems . . . 53
7.1.1 HS 2016, Question 31 . . . 53

8 Frequency Response and Bode Plots **54**
8.1 Example Problems . . . 61
8.1.1 FS 2018, Question 18 . . . 61
8.1.2 FS 2018, Question 19 . . . 62
8.1.3 FS 2018, Question 30 . . . 63
8.1.4 HS 2018, Question 18 . . . 64
8.1.5 HS 2017, Question 28 . . . 64

9 Nyquist Plot **65**
9.1 Example Problems . . . 71
9.1.1 HS 2016, Question 45 . . . 71
9.1.2 HS 2016, Question 26 . . . 72
9.1.3 HS 2016, Question 30 . . . 73
9.1.4 HS 2017, Question 29 . . . 74
9.1.5 FS 2018, Question 24 . . . 75

10 Frequency Domain Specifications and Loop Shaping **76**
10.1 Example Problems . . . 81
10.1.1 HS 2018, Question 33 . . . 81
10.1.2 HS 2022, Question 37 . . . 82
10.1.3 HS 2022, Question 39 . . . 82

11 Time Delays **83**
11.1 Example Problems . . . 85
11.1.1 FS 2018, Question 34 . . . 85
11.1.2 HS 2017, Question 44 . . . 85
11.1.3 HS 2017, Question 45 . . . 86

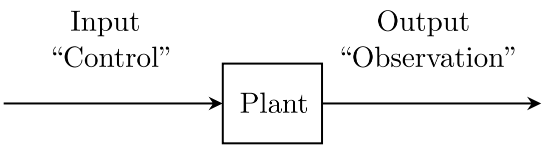
12 Nonlinearities and Describing Functions **87**
12.1 Example Problems . . . 91
12.1.1 HS 2023, Questions 40 and 41 . . . 91

1 Introduction

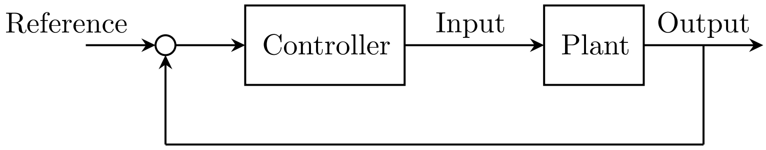
In this course we learn the basics of feedback control systems. Broadly the course can be divided into three main parts:

- **Modeling**: represent real world systems with mathematical equations
- **Analysis**: understand how a given system behaves; how the input affects the output, and how feedback influences the system
- **Synthesis**: change the system, so that it behaves in a desired way

In control systems, we examine physical systems, often referred to as the Plant, that we aim to control. Examples of such systems include cars, planes, drones, and robots. These systems can be represented using block diagrams, where the system itself is depicted as a block, and the influences on the system or observable outputs are represented with arrows.



Our goal will be to generate the right input to the plant such that the output behaves in the desired way. This is done using a controller. A good example is cruise control on a car. We tell the cruise control that we want to go at a certain speed (reference), and the cruise control generates the corresponding input (e.g. throttle position). A big part of control systems is introducing feedback into the system we want to control. This feedback is necessary since doing control without feedback would be similar to driving a car blindfolded. For cruise control, feedback would correspond to a sensor giving the current speed of the car. In block diagrams it is usually represented like this:

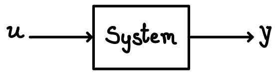


This quickly illustrates what we want to achieve in this course. Let's get started then!

2 Modeling

<u>Introduction:</u>

We want to learn how to mathematically represent dynamic systems. Specifically we want to write down equations that express the output as a function of the input, and some internal parameters.



Important: All models are wrong, but some are useful.

Inputs can be:

- Endogenous: can be manipulated by the designer, e.g. control inputs
- Exogenous: generated by the environment and can't be controlled, e.g. disturbances

they encompass everything that affects the system over time.

The outputs can be classified as:

- Measured outputs: what we can measure (sensors), e.g. speed of car
- Performance outputs: not directly measurable, but we want to control, e.g. avg. fuel consumpt.

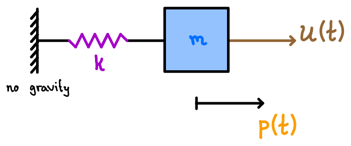
they encompass everything that we observe about the system over time.

Internal parameters are system specific and do not change over time.

All systems that we want to describe, can be represented by differential equations. That means that the way the system is changing, is related to the current state

how the system changes = f(current state)

<u>Example:</u>



Newtons second law : $F = m \cdot a$

$$u(t) - k\,p(t) = m\,\ddot{p}(t)$$

$$\rightarrow \ddot{p}(t) = -\frac{k}{m}\,p(t) + \frac{1}{m}\,u(t) \quad 2^{nd} \text{ order ODE}$$

State Space Representation

If we take closer look at the 2[nd] order ODE we can regocnize that it can be re-written as a system of 1[st] order ODEs (see Linear Algebra II).

With the substitution:

$$x_1(t) = p(t)$$
$$x_2(t) = \dot{p}(t)$$

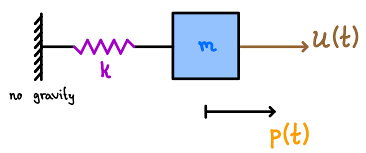
we obtain: which can also be re-written in matrix form.

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -\tfrac{k}{m}\,x_1(t) + \tfrac{1}{m}\,u(t)\end{aligned} \iff \begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{k}{m} & 0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{pmatrix}0\\ \frac{1}{m}\end{pmatrix}u(t)$$

This is something we already know and can solve. The only thing left to do is defining what we want to measure in our system, i.e. what the output is. Here we can measure, for example, the velocity of the mass. The output $y(t)$ is then equal to $\dot{p}(t)$ and therefore $x_2(t)$. Now we can write everything together as:

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{k}{m} & 0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{pmatrix}0\\ \frac{1}{m}\end{pmatrix}u(t)$$
$$y(t) = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}$$

$\iff$



This system of equations now represents the original mass-spring system.

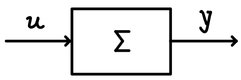
The vector $x = \begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}$ contains all state variables of the system. The states describe how a system changes internally over time. It can be thought of as a memory, containing a summary of how the system behaved in the past. Given the internal states and the current input, we can uniquley predict any future behavior. We can now generalize this to a standard form that we will generally use to describe dynamic systems.

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

This is called the state-space form, since we are observing how the state vector x changes.

Block Diagrams

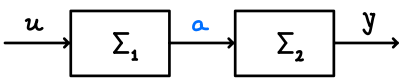
Block diagrams are an effective way to visually show how different systems are connected. It is the standard way to illustrate the interconnection of different systems and control architectures. Lets begin with the simple case given by:



Here Σ maps an input u to an output y. We can write that:

$$y = \Sigma u$$

We can also have two systems, one after the other, like so:

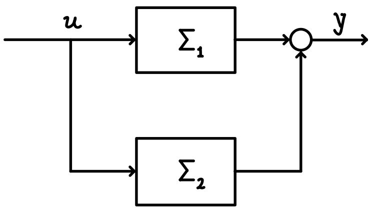


To help us find the input-output relation we can define an intermediate signal a, and analyze both blocks seperatley:

1. $y = \Sigma_2 a$
2. $a = \Sigma_1 u$

combining both results in: $y = \Sigma_2 \Sigma_1 u$

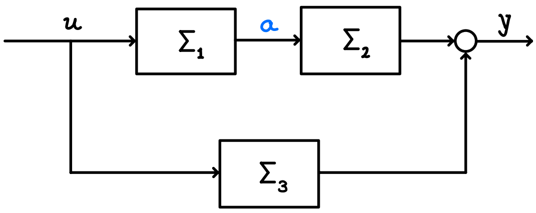
We can also have two systems in parallel:



$$y = \Sigma_1 u + \Sigma_2 u$$

$$y = (\Sigma_1 + \Sigma_2) u$$

If we combine both we obtain



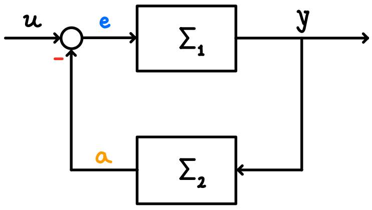
$$y = \Sigma_2 a + \Sigma_3 u$$

$$a = \Sigma_1 u$$

$$y = \Sigma_2 \Sigma_1 u + \Sigma_3 u$$

$$y = (\Sigma_2 \Sigma_1 + \Sigma_3) u$$

We can also introduce (negative) feedback:



We follow the same recipe

$y = \Sigma_1 e$

$e = u - a \qquad e = u - \Sigma_2 y \rightarrow y = \Sigma_1 (u - \Sigma_2 y)$

$a = \Sigma_2 y$

so $y = \Sigma_1 (u - \Sigma_2 y)$, rearanging we obtain:

$$y = \Sigma_1 u - \Sigma_1 \Sigma_2 y$$

$$y + \Sigma_1 \Sigma_2 y = \Sigma_1 u$$

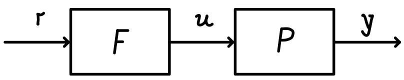
$$(1 + \Sigma_1 \Sigma_2) y = \Sigma_1 u$$

$$\boxed{y = (1 + \Sigma_1 \Sigma_2)^{-1} \Sigma_1 u} \quad \text{or} \quad \boxed{y = \frac{\Sigma_1 u}{1 + \Sigma_1 \Sigma_2}}$$

only if Σ_1 & Σ_2 are scalars!

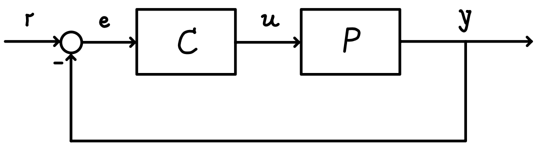
With the help of block diagrams we can also vizualize some basic control architectures.

Feed-forward:



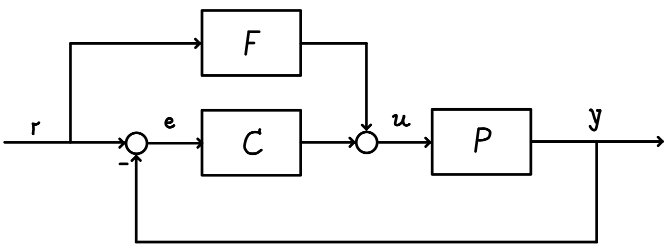
requires precise knowledge of plant

Feedback:



can handle disturbances and uncertainties, but can introduce instability

To degrees of freedom:



good transient behavior and good tracking of fast changing references.

2.1 Example Problems

2.1.1 Example from Lecture

Problem: Consider the swing/pendulum system shown in the figure 1. The system consists of a massless rod of length l and a point mass m at the end of the rod. The angle between the rod and the vertical axis is denoted by θ. The point mass is subject to gravity and an external force $u(t)$. Further, there is a friction force acting on the system. The friction force is proportional to the angular velocity $\dot{\theta}$ and its magnitude is given by the friction coefficient c.

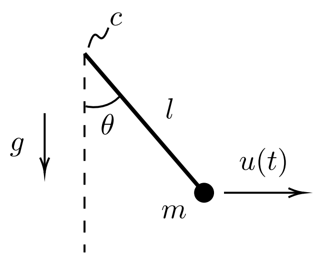


Figure 1: Pendulum System

Question: Given equation 1 for the angular momentum balance, find the state space representation of the system that has the angle θ as output.

$$M = J \cdot \ddot{\theta} \tag{1}$$

Hint: The moment of inertia is given by $J = ml^2$.

2.1.2 FS 2017, Questions 7 and 8

Problem: You have passed the Control Systems 1 exam, and now you are very bored. To do something more exciting, you decide to build a jet-kart. However, before the fun starts you need to model it and design a controller for it. Assume that the cart moves in one direction only (1D motion). To control your vehicle you use thrust from the jet engine (this is your control input) and you are interested in controlling the kart's position. Assume that there are only three kinds of forces acting on the kart:

- Thrust Force $F_{\text{TH}}(t) = k_{\text{TH}}T(t)$ where $T(t)$ is the thrust from the jet engine (can be both positive or negative) and K_{TH} is a constant.
- High velocity drag force $F_{\text{DH}} = -k_{\text{DH}}v^2(t)$ where k_{DH} is a constant and $v(t)$ is the linear velocity of the vehicle.
- Viscous drag force $F_{\text{D}} = -k_{\text{D}}v(t)$ where k_{D} is a constant and $v(t)$ is the velocity.

Question *Choose the correct answer. (1 Point)*
Let m be the mass of the vehicle and $x(t)$ its position. Which differential equation models your system?

[A] $m\ddot{x}(t) - k_{\text{DH}}\dot{x}^2 - k_{\text{D}}\dot{x} = k_{\text{TH}}T(t)$

[B] $m\ddot{x}(t) + k_{\text{DH}}\dot{x}^2 + k_{\text{D}}\dot{x} = k_{\text{TH}}T(t)$

[C] $m\ddot{x}(t) - k_{\text{DH}}\dot{x}^2 - k_{\text{D}}\dot{x} = -k_{\text{TH}}T(t)$

[D] $m\ddot{x}(t) + k_{\text{DH}}\dot{x}^2 + k_{\text{D}}\dot{x} = 0$

Question *Choose the correct answer. (1 Point)*
If you represent your differential equation of the jet kart in state space representation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, $\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u})$, what is the dimension of the state vector, i.e. if $\mathbf{x} \in \mathbb{R}^n$, what is n?

[A] 3

[B] Cannot be determined from the information given

[C] 1

[D] 2

2.1.3 HS 2016, Questions 6 and 7

Problem: Consider an electric motor which you would like to operate at a constant rotational speed ω_0. Applying a voltage $U(t)$ results in a change in the circuit current $I(t)$, which is governed by the differential equation

$$L \cdot \frac{d}{dt} I(t) = -R \cdot I(t) - \kappa \cdot \omega(t) + U(t),$$

whereby L is the circuit inductance, R its resistance and κ a constant relating the motor speed $\omega(t)$ to an electro motor-force (EMF). The dynamics of the motor speed are given by

$$\Theta \cdot \frac{d}{dt} \omega(t) = -d \cdot \omega(t) + T(t),$$

where Θ represents its mechanical inertia, d a friction constant and $T(t) = \kappa \cdot I(t)$ the current-dependent motor torque.

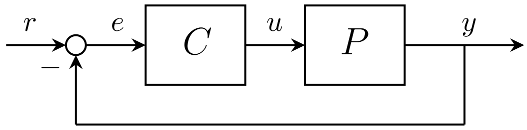


Figure 2: Control Architecture

Question *Choose the correct answer. (1 Point)*
Relate the variables in the block diagram above to the correct signals.

A $u(t) = U(t), x(t) = \begin{bmatrix} \omega(t) \\ I(t) \end{bmatrix}, y(t) = \omega(t), r(t) = \omega_0$

B $u(t) = U(t), x(t) = \begin{bmatrix} U(t) \\ T(t) \end{bmatrix}, y(t) = T(t), r(t) = \omega_0$

C $u(t) = U(t), x(t) = \begin{bmatrix} \omega(t) \\ I(t) \end{bmatrix}, y(t) = I(t), r(t) = \omega_0$

D $u(t) = I(t), x(t) = \begin{bmatrix} \omega(t) \\ U(t) \end{bmatrix}, y(t) = I(t), r(t) = \omega_0$

E $u(t) = I(t), x(t) = \begin{bmatrix} \omega(t) \\ T(t) \end{bmatrix}, y(t) = \omega(t), r(t) = \omega_0$

Question *Choose the correct answer. (1 Point)*
A colleague tells you that the circuit inductance is very small and can actually be neglected. The arising motor model can now be represented as...

A ...a second-order system

B ...a first-order system

C ...an integrator

D ...a static system

2.1.4 HS 2017, Question 5

Question *Choose the correct answer. (1 Point)*

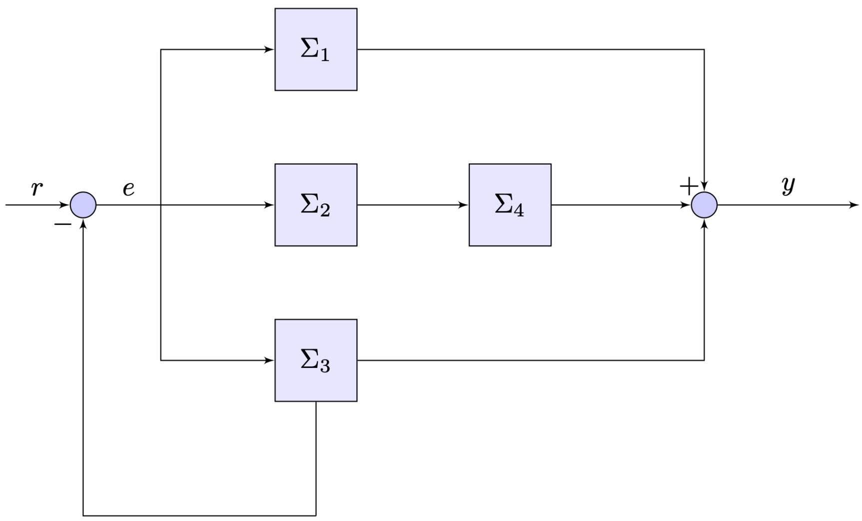


Figure 3: System Diagram

You are given the above system diagram. What is the associated transfer function from $r \to y$.

A $\Sigma_{r\to y} = (\Sigma_1 + \Sigma_2\Sigma_4 + \Sigma_3)$

B $\Sigma_{r\to y} = \frac{\Sigma_1+\Sigma_2\Sigma_4+\Sigma_3}{1+\Sigma_2\Sigma_4}$

C $\Sigma_{r\to y} = \frac{\Sigma_1+\Sigma_2\Sigma_4+\Sigma_3}{1+\Sigma_1+\Sigma_2\Sigma_4+\Sigma_3}$

D $\Sigma_{r\to y} = \frac{\Sigma_1+\Sigma_2\Sigma_4+\Sigma_3}{1+\Sigma_3}$

3 System Classification and Linearization

System Classification

Now we know how to describe physical systems with equations. With that we can also classify them in different ways. This classification is important for us, since we will only consider one specific type of system in this class. (So called linear time invariant or LTI systems)

But generally we classify system in these categories:

- → Linear vs. Nonlinear
- → Causal vs. Non-causal
- → Static (memoryless) vs. Dynamic
- → Time invariant vs. Time-varying

Linearity:

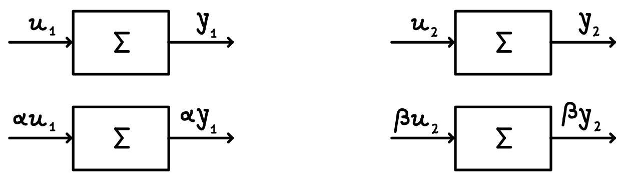
For a system to be linear two conditions have to be fullfilled.

- → Additivity: $\Sigma(u_1+u_2) = \Sigma u_1 + \Sigma u_2$
- → Homogeneity: $\Sigma k u = k \Sigma u \quad , \quad k \in \mathbb{R}$

Differentiation and integration are linear operations!

We can summarize both to:

$$\Sigma(\alpha u_1 + \beta u_2) = \alpha \Sigma u_1 + \beta \Sigma u_2 = \alpha y_1 + \beta y_2 \qquad \alpha, \beta \in \mathbb{R}$$



This implies the idea of superposition. That means that, when a system is linear, we can:

- → Break down "complicated" input signals into simpler components
  $u = u_1 + u_2$
- → Compute the output for each simple input seperatley
  $y_1 = \Sigma u_1 \; ; \; y_2 = \Sigma u_2$
- → Sum all of the simple outputs together to obtain the response to the complicated input
  $y = y_1 + y_2$

		Linear	Nonlinear
1.	$y(t) = t^2 u(t-1)$	×	
2.	$y(t) = \sin(u(t))$		×

1. $y_a(t) = t^2 u_a(t-1) = \Sigma(u_a)$, $y_b(t) = t^2 u_b(t-1) = \Sigma(u_b)$

$$\Sigma(\alpha u_a + \beta u_b) = t^2(\alpha u_a(t-1) + \beta u_b(t-1))$$
$$= \alpha \Sigma(u_a) + \beta \Sigma(u_b) \longrightarrow \text{linear}$$

2. $y_a(t) = \sin(u_a(t)) = \Sigma(u_a)$, $y_b(t) = \sin(u_b(t)) = \Sigma(u_b)$

$$\Sigma(\alpha u_a + \beta u_b) = \sin(\alpha u_a(t) + \beta u_b(t))$$
$$\neq \alpha \sin(u_a(t)) + \beta \sin(u_b(t)) \longrightarrow \text{nonlinear}$$

Causality:

A system is said to be causal, iff (if and only if) the future input does not affect the present output. All practically realizable systems are causal. Otherwise you could predict the future.

	Causal	Non-causal	
$y(t) = u(t-a), \forall a < 0$		×	← Future inputs
$y(t) = u(t-\tau), \forall \tau > 0$	×		
$y(t) = \cos(3t+1)u(t-1)$	×		

Static vs. Dynamic

An input-output system Σ is static or memoryless if for all t, $y(t^*)$ is only a function of $u(t^*)$.

In other words: the present output depends only on the present input and not on past or future inputs.

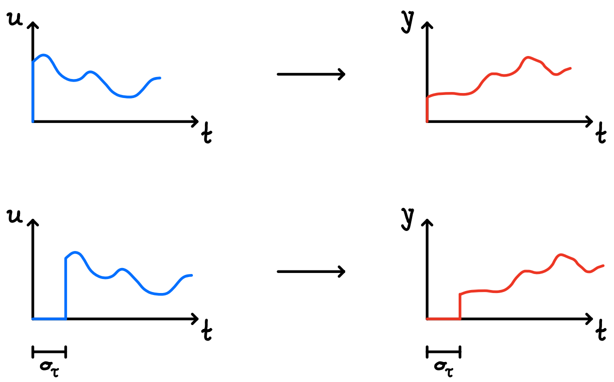
Systems described by ODEs are always dynamic. Static systems are usually described by algebraic equations. You can think of systems from Mech I as static, and those from Mech III as dynamic.

	Static	Dynamic	
$y(t) = 2^{-(t+1)} u(t)$	×		
$y(t) = \int_{-\infty}^{t} u(\tau)\, d\tau$		×	"Sums up" everything that happend
$y(t) = \dot{u}(t)$		×	$\dot{u}(t) = \lim_{h \to 0} \frac{u(t) - u(t-h)}{h}$

Time invariant vs. Time-varying:

A time invariant system will always have the same output to a certain input, independent of when the input is applied.

Formally this means that we can shift the input in time and the output will also be shifted.



	Variant	Invariant
$y(t) = u(t-1)\,u(t+2)$		×
$y(t) = \cos(t)\,u(t)$	×	

But what kind of systems do we care about?

- → Linear
- → Time invariant
- → Causal
- → Single input, Single output

} LTI SISO Systems

very restrictive class of systems
many systems can be well approximated by LTI SISO systems

One characteristic of LTI systems, is that we can write the state space model in the form:

$$\dot{x}(t) = A x(t) + B u(t)$$

$$y(t) = C x(t) + D u(t)$$

where A, B, C, and D are constant matrices or vectors. One example of such system is the mass-spring system from before. The state space form can be re-written in matrix form.

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{k}{m} x_1(t) + \frac{1}{m} u(t) \quad \Longleftrightarrow \quad \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u(t)$$
$$y(t) = x_2(t) \qquad\qquad y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

In this example A, B, C, and D are defined as follows:

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad D = 0$$

The vector $x = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ is the state of the system. It contains the information needed, together with the current input, to uniquely predict future outputs.

The dimension n of the state vector $x \in \mathbb{R}^n$ is equal to the dimension of the system. The choice of a systems state is not unique. Every different choice of states is called realization (there are infinitley many realizations). If we choose a realization with the smallest possible state vector, it is called minimal realization. A static system has a zero-dimensional state vector.

Linearization:

When linearizing, we take some finite-dimensional, time-invariant, causal nonlinear system and approximate it as a LTI system. This approximation works very well and let's us use control systems designed for LTI systems even on nonlinear systems.

The main idea is to pick an equilibrium point of the system and then make a linear approximation arround that equilibrium point. For a system modeled with ODE s $\dot{x}(t) = f(x(t), u(t))$, we can define an equilibrium point (x_e, u_e) to be at

$$f(x_e, u_e) = 0$$

Let's find some equilibrium points of the swing example. We will look at equilibria where $u_e = 0$. Intuitevley, where would those be ?

Let's take a closer look:

$u_e = 0$, $f(x_e, 0) = 0$

$$\begin{cases} \dot{x}_1(t) = x_2(t) = 0 \\ \dot{x}_2(t) = \frac{1}{ml^2}\left[-lmg \sin x_1(t) - c\, x_2(t) + l \cos x_1(t)\, u(t)\right] \\ y(t) = x_1(t) \end{cases}$$

(with $x_2(t) \to 0$ and $u(t) \to 0$)

$$\dot{x}_2(t) = -\frac{g}{l} \sin x_1(t) = 0 \implies x_1 = 0 \text{ or } x_1 = \pi$$

So for $u_e = 0$ there are two equilibrium points:

$x_e = (0, 0)$, $u_e = 0$ stable

$x_e = (\pi, 0)$, $u_e = 0$ unstable

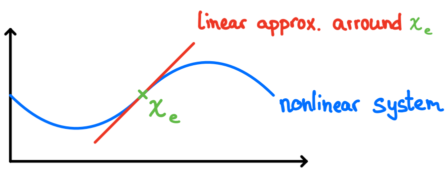
This also aligns with expectations since the first eq. is just pointing down, and the second is pointing vertically up.

In order to linearize arround an equilibrium we use the Jacobian Linearization procedure. Where we do a Taylor series expansion arround (x_e, u_e) of the nonlinear system's dynamic. The linearized LTI system matracies are then given by:

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{n\times n} \qquad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{n\times m}$$

$$C = \left.\frac{\partial h(x,u)}{\partial x}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{p\times n} \qquad D = \left.\frac{\partial h(x,u)}{\partial u}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial u_1} & \cdots & \frac{\partial h_p}{\partial u_m} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{p\times m}$$

Graphically you can think of this approach as follows:



3.1 Example Problems

3.1.1 HS 2017, Question 1

Question *Choose the correct answer. (1 Point)*
Your thesis supervisor hands you a quadrocopter and asks you to design a control algorithm for it. Is the quadrocopter system causal or not?

[A] Causal

[B] Not causal

3.1.2 HS 2016, Question 3

Question *Mark all correct statements. (2 Points)*
All signals are scalars. The system $\frac{d}{dt}y(t) = (u(t+1))^2$ is:

[A] Causal

[B] Memoryless / Static

[C] Time-Invariant

[D] Linear

3.1.3 HS 2016, Question 4

Question *Mark all correct statements. (2 Points)*
All signals are scalars. The system $y(t) = t^2 \cdot u(t-1)$ is:

- A Linear
- B Time-Invariant
- C Causal
- D Memoryless / Static

3.1.4 FS 2017, Question 1

Question *Mark all correct statements. (2 Points)*
Which of these systems is not causal?

- A $y(t) = u^2(t) + 10$
- B $y(t-\sigma) = 2e^{u(t)}, \quad \sigma > 0$
- C $y(t) = u(t+\sigma), \quad \sigma > 0$
- D $y(t-\sigma) = 1, \quad \sigma > 0$

3.1.5 HS 2017, Question 3

Question *Mark all correct statements. (2 Points)*
All signals are scalars. The system $y(t) = n^3 t^3 u(t)$, with $n \in \mathbb{R}_{>0}$ is:

- [A] Causal
- [B] Linear
- [C] Memoryless / Static
- [D] Time-Invariant

3.1.6 HS 2017, Question 4

Question *Mark all correct statements. (2 Points)*
All signals are scalars. Input $u(t)$, output $y(t)$, state $x(t)$.
The system $x(t) = \frac{u(t)}{t}$, $y(t) = 3x(t) - x^2(t)$, with $t > 0$ is:

- [A] Memoryless / Static
- [B] Causal
- [C] Linear
- [D] Time-Invariant

3.1.7 HS 2016, Question 9

Problem: Given the system

$$\frac{d}{dt}x(t) = (x(t))^2 + 5 \cdot u(t) - 10$$
$$y(t) = \frac{4 \cdot x(t) - 12}{u(t)},$$

you have to linearize it around the equilibrium $x_e = 0, u_e = 2$.

Question *Choose the correct answer. (1 Point)*
Which are the state-space matrices A, b, c and d?

- [A] $A = 0, b = 5, c = 4, d = -3$.
- [B] $A = 5, b = 5, c = 2, d = -3$.
- [C] $A = -10, b = 5, c = 2, d = 3$.
- [D] $A = 0, b = 5, c = 2, d = 3$.

3.1.8 FS 2018, Question 5

Problem: Given the system

$$\dot{x}_1(t) = x_1^2(t) - \sin(3x_2(t)) + u^3(t)$$
$$\dot{x}_2(t) = x_2 - u(t) + x_1(t)e^{-x_2(t)},$$

you have to linearize it around the equilibrium point $\mathbf{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{u}_e = 0.$

Question *Choose the correct answer. (1 Point)*
What are the state-space matrices A and B?

A $A = \begin{bmatrix} -3 & 0 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

B $A = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

C $A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

D $A = \begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

4 Time Response and Stability

Time response:

We know now how to represent physical systems with ODEs and how to uniformally represent them in the LTI state space form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

The solution for the state space LTI system is given by:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$
$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

To obtain this solution we made use of the linearity of the equations, decomposing it in simpler bits.

→ Initial-condition response: $\begin{cases} x_{IC}(0) = x_0 \\ u_{IC}(t) = 0 \quad , t \geq 0 \end{cases} \rightarrow y_{IC}$

→ Forced response: $\begin{cases} x_F(0) = 0 \\ u_F(t) = u(t) \quad , t \geq 0 \end{cases} \rightarrow y_F$

We can later identify the different components in the output.

$$y(t) = \underbrace{Ce^{At}x_0}_{y_{IC}} + \underbrace{C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{y_F} + \underbrace{Du(t)}_{\text{Feedthrough}}$$

If we take a closer look, we see that some terms contain the matrix exponential e^{At}.
But how do we compute e^{At}?

Throwback: Linear Algebra II

The matrix exponential can be defined through a Taylor-Series: (also valid for scalars)

$$e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n = I + At + \frac{1}{2}(At)^2 + \ldots + \frac{1}{n}(At)^n$$

Therefore we would have to calculate infinetley many terms. But for some matrices we can drastically simplify the calculations:

→ Diagonal: $\exp\left(\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}t\right) = \begin{bmatrix}\exp(\lambda_1 t) & 0\\ 0 & \exp(\lambda_2 t)\end{bmatrix}$

→ Jordan Form: $\exp\left(\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}t\right) = \begin{bmatrix}\exp(\lambda t) & t\exp(\lambda t)\\ 0 & \exp(\lambda t)\end{bmatrix}$

Where λ_i are the eigenvalues of the respective matrix.

To facilitate calculations we can therefore do a coordinate transformation, $x = T\tilde{x}$ such that e^{At} is easier to compute Note that the time response remains unchanged. Through the coordinate transformation, we simply use a different realization of the system i.e. a different state vector.

Initial condition (homogeneous) responses:

Let us now take a closer look at systems where A is diagonal. More specific we will look at the initial condition response, i.e. $u(t) = 0$.

→ For a diagonal, real matrix:

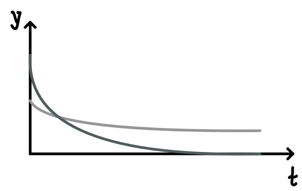
$$A = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}, \quad \lambda_i \in \mathbb{R}$$

$$y(t) = Ce^{At}x_0$$

where we can write out all terms and simplify for A being diagonal.

$$y(t) = \begin{bmatrix}c_1 & c_2\end{bmatrix}\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$$

$$y(t) = c_1 e^{\lambda_1 t}x_1(0) + c_2 e^{\lambda_2 t}x_2(0)$$



So for diagonal, real matrices the initial condition response is the linear combination of two exponentials.

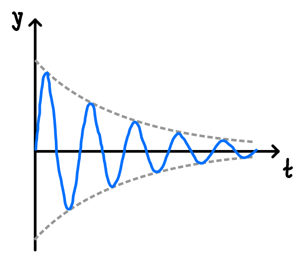
→ For a diagonal, complex matrix:

$$A = \begin{bmatrix}\sigma + j\omega & 0\\ 0 & \sigma - j\omega\end{bmatrix}$$

$$y(t) = Ce^{At}x_0$$

where we can write out all terms and simplify for A being diagonal.

$$\begin{aligned} y(t) &= c_1 e^{\sigma t} e^{j\omega t} x_1(0) + c_2 e^{\sigma t} e^{-j\omega t} x_2(0) \\ &= e^{\sigma t}\left(c_1 e^{j\omega t} x_1(0) + c_2 e^{-j\omega t} x_2(0)\right) \\ &= e^{\sigma t}\left(\alpha_1 \sin(\omega t) + \alpha_2 \cos(\omega t)\right) \\ &= \alpha e^{\sigma t}\sin(\omega t + \phi) \end{aligned}$$



Forced response:

We have seen how systems react to initial conditions, i.e. how the homogeneous solution $Ce^{At}x_0$ behaves. It gives us a feeling about the natural dynamics of the system. But what about the forced response, given by the convolution integral:

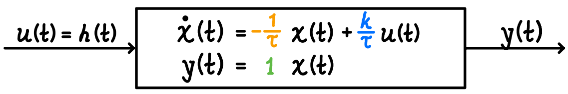
$$C\int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau + D\,u(t)$$

This is harder to interpret. To gain some intuition, we will look at how elementary inputs affect the system.

Step-response of 1st order system:

Let's look at a simple example where we apply a step input, given by the Heaviside function

$$u(t) = h(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$ as input to the first order system: $$\dot{x}(t) = -\frac{1}{\tau}x(t) + \frac{k}{\tau}u(t)$$ $$y(t) = 1\,x(t)$$



we can compute the output $y(t)$ as follows:

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\rho)}B\,\cancel{u(\rho)}^{\,1}\,d\rho + \cancel{D}^{\,0}u(t)$$

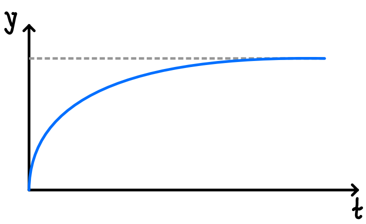
$$e^{-\frac{1}{\tau}t}x_0 + \int_0^t e^{-\frac{1}{\tau}(t-\rho)}\frac{k}{\tau}\,d\rho$$

$$= e^{-\frac{1}{\tau}t}x_0 + \frac{k}{\tau}e^{-\frac{1}{\tau}t}\int_0^t e^{\frac{1}{\tau}\rho}\,d\rho$$

$$= e^{-\frac{1}{\tau}t}x_0 + \frac{k}{\tau}e^{-\frac{1}{\tau}t}\left[\tau\, e^{\frac{1}{\tau}\rho}\right]_0^t$$

$$= e^{-\frac{1}{\tau}t}x_0 + k\,e^{-\frac{1}{\tau}t}\left(e^{\frac{1}{\tau}t} - 1\right)$$

$$= e^{-\frac{t}{\tau}}x_0 + k\left(1 - e^{-\frac{t}{\tau}}\right)$$



Stability:

In the two cases above we can see that the output is somehow linked to exponential terms.

$$y(t) = c_1 e^{\lambda_1 t} x_1(0) + c_2 e^{\lambda_2 t} x_2(0)$$

$$y(t) = \alpha e^{\sigma t} \sin(\omega t + \phi)$$

The growth of these terms is dictated by the real part of the eigenvalues of A. We can see that if the eigenvalues λ have a positive real part, the output will grow exponentially over time, i.e. become unstable. ($y \to \infty$). There are few ways to classify stability:

→ Lyapunov Stability: a system is Lyapunov stable if, for any bounded initial condition, and zero input, the state remains bounded, i.e.:

$$\|x_0\| < \epsilon, \text{ and } u = 0 \Rightarrow \|x(t)\| < \delta \quad \forall t \geq 0$$

→ Asymptotic Stability: a system is asymptotically stable if, for any bounded initial condition, and zero input, the state converges to zero, i.e.:

$$\|x_0\| < \epsilon, \text{ and } u = 0 \Rightarrow \lim_{t \to \infty} \|x(t)\| = 0$$

→ Bounded-Input, Bounded-Output Stability: a system is BIBO-stable if, for any bounded input, the output remains bounded, i.e.:

$$\|u(t)\| < \epsilon \quad \forall t \geq 0, \text{ and } x_0 = 0 \Rightarrow \|y(t)\| < \delta \quad \forall t \geq 0$$

Every system that is not stable, is called unstable.

We can check stability by looking at the eigenvalues of A!

→ Lyapunov stable if $Re(\lambda_i) \leq 0 \quad \forall i$

→ Asymptotically stable if $Re(\lambda_i) < 0 \quad \forall i$

4.1 Example Problems

4.1.1 FS 2018, Question 6

Problem: Consider a system with the following dynamics,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Question *Choose the correct answer. (1 Point)*
If $u(t) = e^{-t}, t \geq 0$, and $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, find $x(t)$ for $t \geq 0$.

A $x(t) = \begin{bmatrix} -1 + t + e^{-t} \\ 1 - e^{-t} \end{bmatrix}$

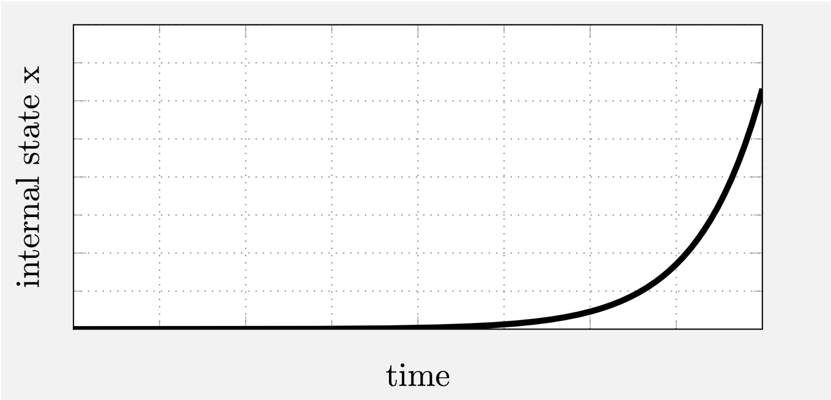
B $x(t) = \begin{bmatrix} 1 + t + e^{-t} \\ -1 + e^{-t} \end{bmatrix}$

C $x(t) = \begin{bmatrix} -1 + e^{-t} \\ 1 + t - e^{-t} \end{bmatrix}$

D $x(t) = \begin{bmatrix} -1 - e^{-t} \\ -1 + t + e^{-t} \end{bmatrix}$

4.1.2 FS 2016, Questions 11 and 13

Problem: The figure below shows the response of an internal state x of a system as an impulse is applied to the systems input.



Question *Choose the correct answer. (1 Point)*

- [A] No conclusion about the stability is possible
- [B] The system is unstable
- [C] The system is asymptotically stable
- [D] The system is Lyapunov stable

Question *Choose the correct answer. (1 Point)*

- [A] The system is BIBO stable
- [B] No conclusion about the BIBO stability is possible
- [C] The system is BIBO unstable

4.1.3 FS 2018, Question 7

Problem: Consider the two systems with output signal $y(t)$ and input signal $u(t)$, described below:

1. $y(t) = \sin(t)u(t)$
2. $y(t) = \int_0^t \sin(\tau)u(\tau)d\tau$

Question *Choose the correct answer. (1 Point)*
Which system is BIBO stable?

- A None of the systems
- C Both of the systems
- B System 2
- D System 1

4.1.4 FS 2017, Question 12

Question *Choose the correct answer. (1 Point)*
Consider a linear time-invariant SISO system. Pick a correct logical relation between different stability criteria.

- A Asymptotically stable $\Longleftrightarrow$ BIBO stable
- B Asymptotically stable $\Longrightarrow$ BIBO stable
- C Asymptotically stable $\Longleftrightarrow$ BIBO stable $\Longleftarrow$ Lyapunov stable
- D Asymptotically stable $\Longrightarrow$ BIBO stable $\Longleftrightarrow$ Lyapunov stable

4.1.5 FS 2016, Question 13

Problem: Given a system with the state space representation

$$\dot{x}(t) = \begin{bmatrix} -1 & 10 \cdot \alpha^2 \\ 0 & \alpha - 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u(t)$$
$$y(t) = \alpha \cdot x(t) + u(t),$$

where $\alpha \in \mathbb{R}$.

Question *Choose the correct answer. (1 Point)*
For which values of α is the system asymptotically stable?

- A $\alpha < 1$
- B $\alpha < 0$
- C $\alpha \leq 1$
- D $\alpha \geq 1$

5 Transfer Functions

General response:

Since we are working with linear systems, we can make use of superposition. So we re-write any input u as $u = u_1 + \ldots + u_n$. We can then apply $u_1, \ldots, u_n$ seperatley to the system and sum all outputs $y = y_1 + \ldots + y_n$.
In this case it would be convinient if we could find some input v that when linearly combined with itself could represent all, or most other signals, i.e. $u = a_1 v + \ldots + a_n v$. Luckily we can use a mathematical tool from Analysis III to help us with that.
The inverse Laplace transform tells us, how to write a function $f(t)$ as a linear combination of terms e^{st} weighted by $F(s)$, the Laplace transform of $f(t)$. Where s is a complex variable of the form: $s = \sigma + j\omega$. So now we know that if we compute the output to some general e^{st} we can later easily compute the output to any input, since it will be a linear combination of e^{st} terms. Let's see how the response to e^{st} will generally look like:

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)} B u(\tau) d\tau + D u(t)$$

with $u(t) = e^{st}$:

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)} B e^{s\tau} d\tau + D e^{st}$$

rearrange:

$$y(t) = Ce^{At}x_0 + Ce^{At}\int_0^t e^{(sI-A)\tau} B d\tau + D e^{st}$$

if $(sI-A)$ is invertible:

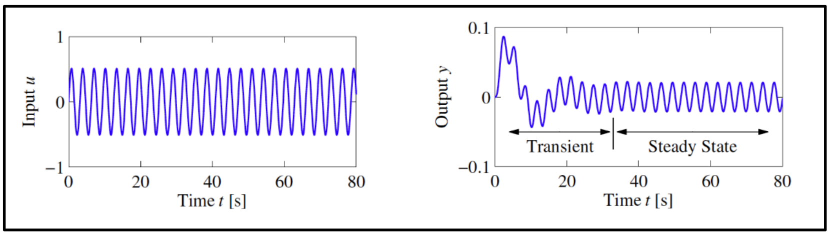
$$y(t) = Ce^{At}x_0 + Ce^{At}\left((sI-A)^{-1} e^{(sI-A)\tau} B\right)\Big|_0^t + D e^{st}$$

rearrange:

$$y(t) = Ce^{At}x_0 + Ce^{At}(sI-A)^{-1}\left(e^{(sI-A)t} - I\right)B + D e^{st}$$

and finally:

$$y(t) = \underbrace{Ce^{At}\left[x_0 - (sI-A)^{-1}B\right]}_{\text{Transient response } (\to 0 \text{ if as. stable})} + \underbrace{\left[C(sI-A)^{-1}B + D\right]e^{st}}_{\text{Steady-state response } y_{ss}}$$



We can now generally say that the steady state response to the input e^{st} is given by

$$y_{ss} = G(s)\, e^{st} \qquad \text{with } G(s) = C(sI-A)^{-1}B + D \quad \in \mathbb{C}$$

The complex function $G(s)$ is known as the transfer function and describes how a stable system G transforms an input e^{st} into the output $G(s)\, e^{st}$. You can thing of it like the Σ in the block diagrams.

But how exactly do we decompose some arbitrary input? Let's see how this looks if our input is a sinusoid, e.g.:

$$u(t) = \cos(\omega t) = \tfrac{1}{2} e^{j\omega t} + \tfrac{1}{2} e^{-j\omega t}$$

We notice that we can decompose $u(t)$ as follows:

$$u(t) = \sum_i U_i\, e^{s_i t} \qquad \text{with } U_{1,2} = \tfrac{1}{2} \text{ and } s_{1,2} = \pm j\omega$$

The output is then given by

$$y(t) = G(j\omega)\, \tfrac{1}{2} e^{j\omega t} + G(-j\omega)\, \tfrac{1}{2} e^{-j\omega t}$$

or

$$y(t) = \sum_i G(s_i)\, U_i\, e^{s_i t}$$

In general we can say that:

$$u(t) = \sum_i U_i\, e^{s_i t} \implies y(t) = \sum_i G(s_i)\, U_i\, e^{s_i t}$$

We can now make use of the inverse Laplace transform that generalizes this sum, such that we can represent all inputs:

$$Y(s) = G(s)\, U(s)$$

By using the Laplace transform the output can be computed by muliplying the input with the transfer function!

Relationsphip between state-space representation and TF:

→ If A is diagonal:

$$G(s) = \frac{P_1}{s-\lambda_1} + \frac{P_2}{s-\lambda_2} + \dots + \frac{P_n}{s-\lambda_n} + d \iff A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} \sqrt{P_1} \\ \vdots \\ \sqrt{P_n} \end{bmatrix}$$

$$C = \begin{bmatrix} \sqrt{P_1} & \cdots & \sqrt{P_n} \end{bmatrix}, \quad D = d$$

→ If A is in the controllable canonical form:

$$G(s) = \frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\dots+b_0}{\underbrace{s^n+a_{n-1}s^{n-1}+\dots+a_0}_{\text{characteristic polynomial of } A}} \iff A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \ddots & \\ & & & & & 1 \\ -a_0 & -a_1 & \dots & & & a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [b_0 \; b_1 \dots b_{n-1}], \quad D = d$$

The roots of the denominator of $G(s)$ are called poles of the system. They are also the eigenvalues of A. So if we want do analyze the stability of a system, we have to look at it's poles.

Ways to write TFs

Mostly we can write TFs as rational functions.

$$G(s) = \frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\dots+b_0}{s^n+a_{n-1}s^{n-1}+\dots+a_0}$$

We can do partial fraction expansion and get:

$$G(s) = \frac{r_1}{s-p_1}+\frac{r_2}{s-p_2}+\dots\frac{r_n}{s-p_n}$$

We can also factorize the numerator and denominator in different ways to obtain:

→ Root-locus form:

$$G(s) = \frac{k_{rl}}{s^q}\frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_{n-q})}$$

→ Bode form:

$$G(s) = \frac{k_{Bode}}{s^q}\frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\dots\left(\frac{s}{-z_m}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\dots\left(\frac{s}{-p_{n-q}}+1\right)}$$

in all cases $p_1,\dots,p_{n-q}$ are called poles and $z_1,\dots,z_m$ zeros. Poles are the roots of the denominator and the zeros the roots of the numerator. There are other ways to write the TF, depending on the application some are more usefull then others. What insides can we gain form the TFs ?

Steady state response to a unit step:

Given a TF of a stable system, we can obtain the stead state response by looking at how the system reacts to a unit step, i.e. $u(t)=h(t)=e^{0t}=1, t\geq 0$

Example: $G(s) = 3\,\frac{s+8}{s^2+5s+4}$

$x(t)$	$X(s)$
$impulse: \delta(t)$	1
$step: h(t)$	$\frac{1}{s}$

we know that: $y_{ss} = G(s)\, e^{st}$

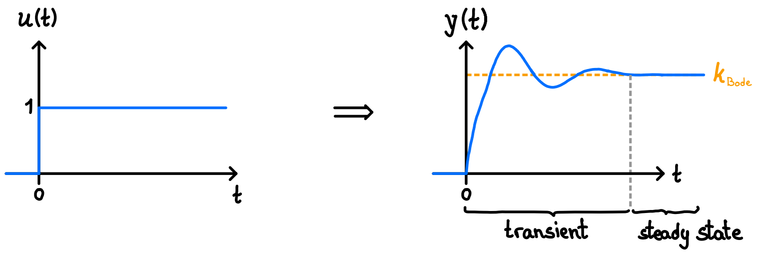
pluging in: $y_{ss} = G(0)\, e^{0t} = G(0) = 6$

You can also use the final value theorem:

$\mathcal{L}\{h(t)\}$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\, G(s)U(s) = \lim_{s\to 0} s\, G(s)\frac{1}{s} = \lim_{s\to 0} G(s) = 6$$

This is also called the Bode gain (k_{Bode})



Poles:

We know that we can write TFs as rational functions in the form:

$$G(s) = \frac{b_m s^m + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

the roots of the nominator (zeros) and denominator (poles) are very important and tell us a lot about the behavior of a system. To understand poles, lets look at the impulse response.

Impulse response:

Let us consider the response to a unit impuls, i.e. $u(t) = \delta(t)$

$\int_0^{\varepsilon} \delta(t)\,dt = 1 \quad \forall\ \varepsilon > 0$ and $\int_0^{t} f(\tau)\delta(\tau)\,d\tau = f(0) \quad \forall\ t > 0$

we can solve the general equation for $y(t)$ by pluging in: $D = 0$, $x_0 = 0$, and $u(t) = \delta(t)$

$$y(t) = Ce^{At}\cancel{x_0}^{\,0} + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + \cancel{D}^{\,0}u(t)$$

$\int_0^t f(\tau)\delta(\tau)\,d\tau = f(0)$

$$y_{imp}(t) = C\int_0^t e^{A(t-\tau)}B\,\delta(\tau)\,d\tau = Ce^{At}\int_0^t e^{-A\tau}B\,\delta(\tau)\,d\tau$$

$$y_{imp}(t) = Ce^{At}B$$

now we can apply our knowledge from the Laplace transform. Since $Y(s) = G(s)U(s)$, we can apply the Laplace transform to y_{imp} and u to obtain $G(s)$. Lets consider a first-order system.

$$y_{imp}(t) = ce^{at}b \quad \overset{\mathcal{L}}{\Longrightarrow} \quad Y(s) = \frac{cb}{s-a} \overset{\tau = cb}{=} \frac{\tau}{s-a}$$

$$u(t) = \delta(t) \quad \overset{\mathcal{L}}{\Longrightarrow} \quad U(s) = 1$$

$x(t)$	$X(s)$
$impulse: \delta(t)$	1
$step: h(t)$	$\frac{1}{s}$
$h(t) \cdot t^n$	$\frac{n!}{s^{n+1}}$
$h(t) \cdot e^{at}$	$\frac{1}{s-a}$

now $G(s) = Y(s) = \frac{\tau}{s-a}$ and $y(t) = \tau e^{at}$

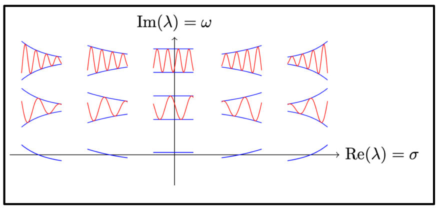
We can extend this to higher order systems, and our TF will take the form

$$G(s) = \frac{\tau_1}{s-P_1} + \frac{\tau_2}{s-P_2} + \ldots + \frac{\tau_n}{s-P_n}$$

which translates to

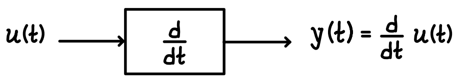
$$y(t) = \tau_1 e^{P_1 t} + \tau_2 e^{P_2 t} + \ldots + \tau_n e^{P_n t}$$

So each pole in our TF generates an exponential in the time domain. For real poles these are exponentials and for complex-conjugate pole pairs, sinusoids.



Zeros:

To understand the effects of zeros consider the following system:



This is a differentiator. Let's see what happens to some general $u(t) = e^{st}$. Since $y(t) = d/dt\, u(t)$, in this case $y(t) = se^{st}$.

We can conclude that the TF of a differentiator is given by:

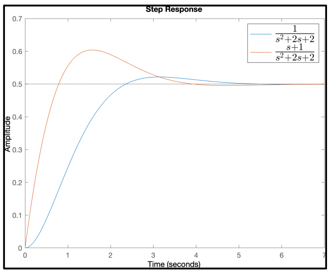
$$G(s) = s$$

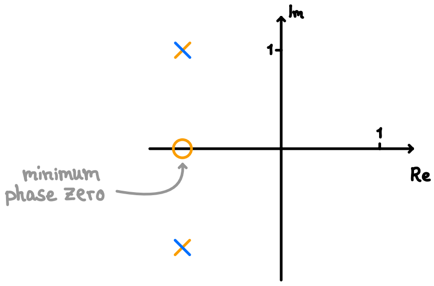
Side note: Integrators are given by $G(s) = s^{-1}$

So by multiplying $G(s)$ with s, essentially adding a zero to the original TF, introduces some derivative action. This usually has an "anticipatory effect".

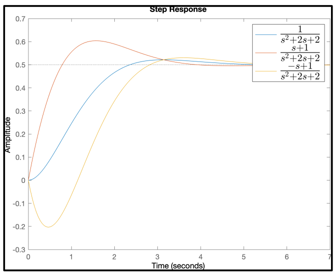
Step response:

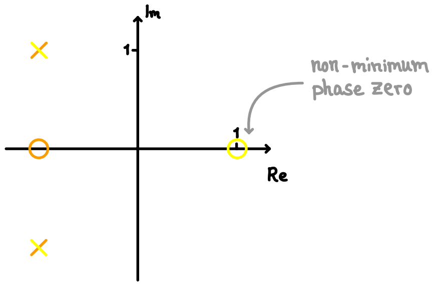
The step response is the systems output given a step input i.e.: $u(t) = h(t)$. The resulting graphs give good insights. Consider the plot below, showing the step response of two systems. Both have the same poles, but the orange system has an additional zero at $s = -1$





Zeros in the left half plane are called minimum phase zeros. They add some derivative action to the response. What happens when the zero is in the right half plane, i.e. positive real part?





We observe that the stability is conserved, but the non-minimum phase zero causes the output to initially move in the wrong direction, i.e. we have some sort of "negative" derivative action.

Pole-Zero Cancellation:

What happens if we add a zero near a pole? If the zero coincides with the pole they cancel out!

$$G(s) = \frac{\cancel{s+1}}{(\cancel{s+1})(s+1+j)(s+1-j)}$$

Since the TF describes input-output behavior, we can cancel out poles, such that we can't observe the asociated behavior, or we cannot influence it through the input. If the pole that is being cancelled is stable this is of no concern, but if the pole is unstable this becomes a big problem.

5.1 Example Problems

5.1.1 HS 2016, Questions 18 and 19

Problem: Given a linear time-invariant system in state-space description.

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & -8 \\ 1 & 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x(t)$$

Question *Choose the correct answer. (1 Point)*
The transfer function $g(s)$ is...

A $g(s) = \frac{2s+10}{s(s+4)(s+1)}$

B $g(s) = \frac{2s+10}{s^2+5s+4}$

C $g(s) = \frac{s+5}{s^2+5s+4}$

D $g(s) = \frac{2(s+5)}{s^2+5s+4}$

Question *Choose the correct answer. (1 Point)*
The system is...

A Asymptotically stable

B Lyaupnov stable

C Unstable

5.1.2 FS 2018, Question 8

Problem: You are given a linear time-invariant system in state-space form.

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

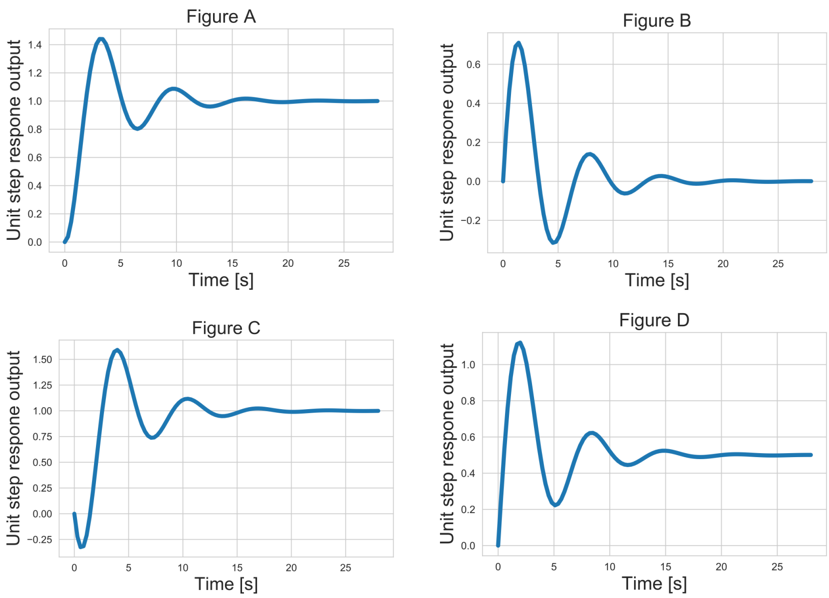
Question *Choose the correct answer. (1 Point)*
The transfer function $g(s)$ is:

- [A] $g(s) = \frac{3+2s}{s^2(s+1)}$
- [B] $g(s) = \frac{1}{s+1}$
- [C] $g(s) = \frac{3-2s}{s(s+1)}$
- [D] $g(s) = \frac{3}{s(s+1)}$

5.1.3 FS 2018, Question 11

Problem: You are given the following set of input to output transfer functions:

1. $G(s) = \frac{s+0.5}{s^2+0.5s+1}$
2. $G(s) = \frac{1-s}{s^2+0.5s+1}$
3. $G(s) = \frac{s}{s^2+0.5s+1}$
4. $G(s) = \frac{1}{s^2+0.5s+1}$



Question *Choose the correct answer. (1 Point)*
Choose the correct assignment of transfer function to unit step responses.

A $1 \to D, 2 \to A, 3 \to B, 4 \to C$

B $1 \to D, 2 \to C, 3 \to B, 4 \to A$

C $1 \to A, 2 \to C, 3 \to D, 4 \to B$

D $1 \to A, 2 \to B, 3 \to C, 4 \to D$

5.1.4 HS 2017, Question 19

Problem: A system given by its transfer function $g(s) = \frac{s-1}{s^2+s-2}$.

Question *Choose the correct answer. (1 Point)*
Which of the following state space representations is equivalent to the system given by its transfer function?

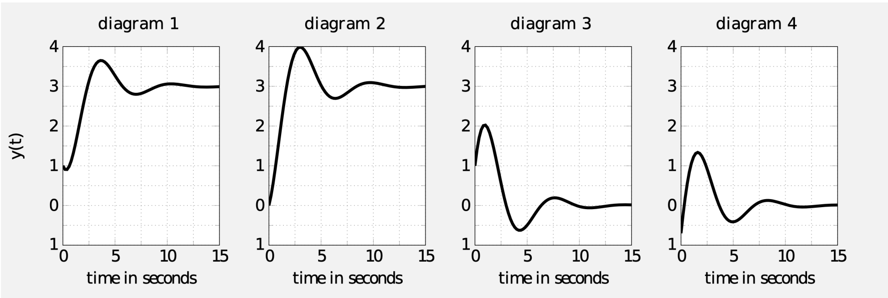
- [A] $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 1$
- [B] $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = 0$
- [C] $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$
- [D] $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = 1$

5.1.5 HS 2016, Question 21

Problem: Given the transfer function $g(s)$ of a system.

$$g(s) = \frac{s+3}{s^2 + 0.7s + 1}$$

In addition, the figure below shows the four time responses A, B, C, and D.



Question *Choose the correct answer. (1 Point)*
Which of the four diagrams shows the correct step response of the system?

- A Diagram 3
- B Diagram 4
- C Diagram 1
- D Diagram 2

6 Root Locus

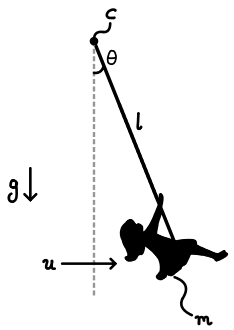
Remember course objectives:

Modeling: represent real world systems with mathematical equations.

Analysis: understand how a given system behaves; how the input affects the output, how feedback influences the system.

Synthesis: Change the system, so that it behaves in a desirable way.

What we did so far



Physical system

Modeling ⇘

$$\dot{x}_1(t) = x_2(t)$$

$$m l^2 \dot{x}_2(t) = -lmg \sin x_1(t) - c\, x_2(t) + l \cos x_1(t)\, u(t)$$

$$y(t) = x_1(t)$$

Differential equations

Linearization ⇙

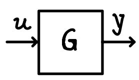
$x_e = (\pi, 0)$, $u_e = 0$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{c}{ml^2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{ml} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

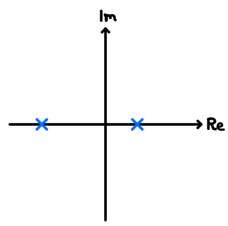
LTI State space form

S-Domain ⇘



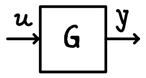
$$G(s) = \frac{0.2}{s^2 + s - 2}$$

Transfer Functions



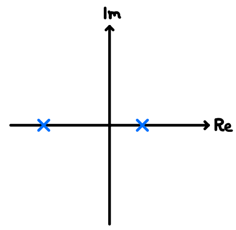
Root Locus:

We arrived at a point where we can start to look at how different controler influence our system. But how do we know how to chose our controler? Let's take our inverted swing example. First of all, we want our system to be stable, i.e. no poles in the right half plane. Our original system has two poles, which can be represented as follows:

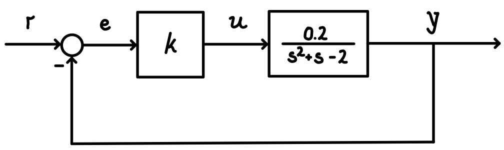


$$G(s) = \frac{0.2}{s^2+s-2} = \frac{0.2}{(s+2)(s-1)} = L(s)$$

open Loop TF



In order to control this system, let's add some controller $C(s)$ and introduce some feedback. The block diagram would look as follows:



For now, the controller C will just be a constant that multiplies the error e. This also called a gain and we will denote it with the letter k. The TF mapping r to y, in this closed loop system, is now given by:

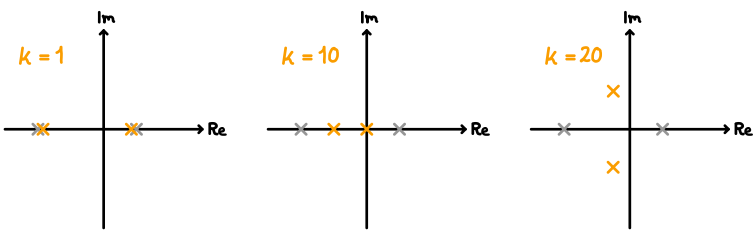
$$T(s) = \frac{k\frac{0.2}{s^2+s-2}}{1+k\frac{0.2}{s^2+s-2}} = \frac{0.2k}{s^2+s-2+0.2k}$$

$T(s)$ is also referred to as the complementary sensitivity.

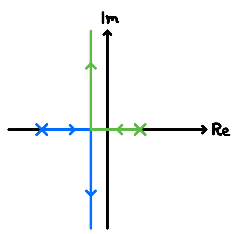
So ideally, by changing k, we can bring the unstable pole to the left half plane, stabalizing the system. Now instead of looking at $L(s)$ we have to consider $T(s)$. Luckily the poles of $T(s)$ are dependend on k, i.e. by adjusting k we influence the position of the closed loop poles. Let's see how they behave.

Solve: $s^2+s-2+0.2k = 0$

for different k



We can trace out the paths that the poles take and obtain:



We can now see that for some value k both poles are in the LHP and the system is stable. The plot we just created is called root locus, and it shows how the location of the closed loop poles, based on the open loop poles. It allows us to quickly see if some controler is feasible or not. Ideally we don't want to calculate the position of the poles for all values of k to be able to sketch a root locus. Luckily there are some rules we can follow:

Sketching Rules

1. Root loci start at poles → go to zeros
2. There are n lines (loci) where n is the degree of Poles or Zeros (whichever is greater).
3. As k increases from 0 to ∞, the roots move from the poles of $G(s)$ to the zeros of $G(s)$.
4. When roots are complex, they occur in conjugate pairs.
5. At no time will the same root cross over its path.
6. The portion (Anteil) of the real axis to the left of an odd number of open loop poles and zeros are part of the loci. → **Roots are always sketched from the right to the left.**
7. Lines leave and enter the real axis at 90°.
8. If there are not enough poles or zeros to make a pair, then the extra lines go to / come from infinity.
9. Lines go to infinity along asymptotes.
10. If there are at least two lines to infinity, then the sum of all roots is constant.
11. K going from 0 to $-\infty$ can be drawn by reversing rule 5 and adding 180° to the asymptote angles.

Asymptotes

Contact point / Centroid of asymptotes

$$S_{com} = \frac{\sum x_{Poles} - \sum x_{Zeros}}{\#Poles - \#Zeros}$$

$$x_i \rightarrow Coordinates\ on\ the\ Real\ axis$$

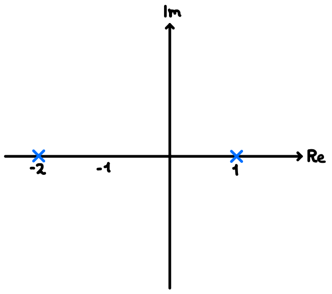
Angle of asymptotes

$$\alpha_n = \frac{2n+1}{\#Poles - \#Zeros} \cdot 180°$$

$$n = \{0; 1; \ldots; (\#Poles - \#Zeros - 1)\}$$

Examples:

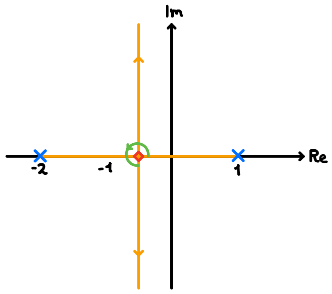
Let's take the example from above, and follow the rules.



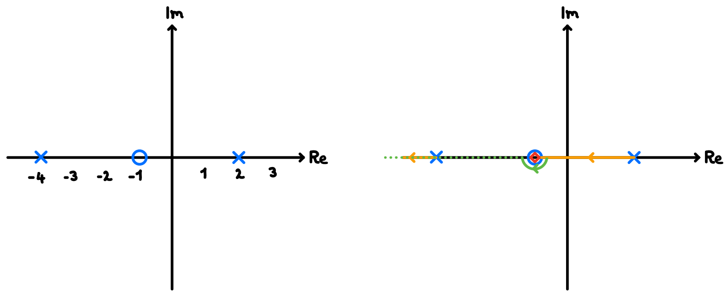
The portion (Anteil) of the real axis to the left of an odd number of open loop poles and zeros are part of the loci. → **Roots are always sketched from the right to the left.**

Contact point of asymp. : $\frac{-2+1}{2} = -0.5$

Angle of asymp.: $\frac{3}{2} \cdot 180° = 270°$



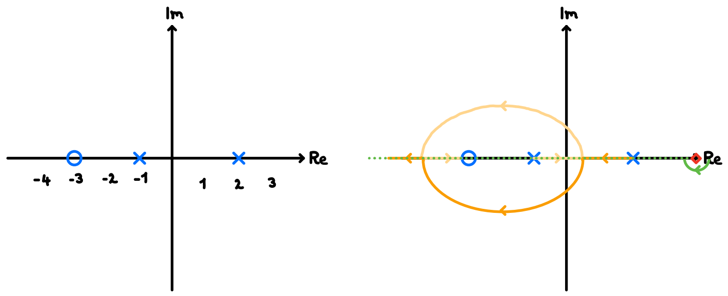
Let's add a zero, and look at some more examples. The centroid is the intersection point of the asymptotes



\# Poles: 2 # Zeros: 1

Asymptote centroid: $\frac{-4+2+1}{2-1} = -1$

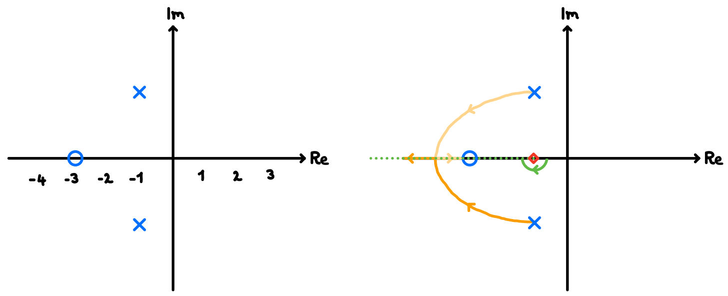
Asymptote angle: $\frac{2\cdot 0+1}{2-1}\cdot 180^\circ = -180^\circ$



\# Poles: 2 # Zeros: 1

Asymptote centroid: $\frac{-1+2+3}{2-1} = 4$

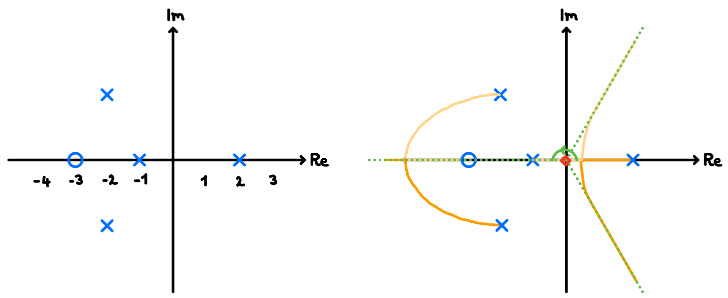
Asymptote angle: $\frac{2\cdot 0+1}{2-1}\cdot 180^\circ = -180^\circ$



\# Poles: 2 # Zeros: 1

Asymptote centroid: $\frac{-1-1+3}{2-1} = 1$

Asymptote angle: $\frac{2\cdot 0+1}{2-1}\cdot 180^\circ = -180^\circ$



\# Poles: 4 # Zeros: 1

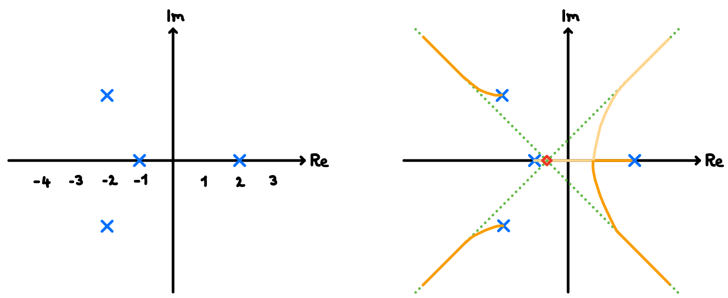
Asymptote centroid: $\frac{-1-2-2+2+3}{4-1} = 0$

Asymptote angles: $\frac{2\cdot 0+1}{4-1}\cdot 180^\circ = 60^\circ$

$n=\{0;1;2\}$

$\frac{2\cdot 1+1}{4-1}\cdot 180^\circ = 180^\circ$

$\frac{2\cdot 2+1}{4-1}\cdot 180^\circ = 300^\circ$



\# Poles: 4 # Zeros: 0

Asymptote centroid: $\frac{-1-2-2+2}{4} = -\frac{3}{4}$

Asymptote angles: $\frac{2\cdot 0+1}{4}\cdot 180^\circ = 45^\circ$

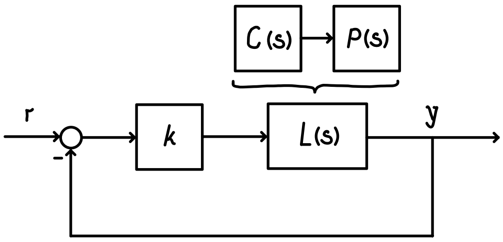
$n=\{0;1;2;3\}$

$\frac{2\cdot 1+1}{4}\cdot 180^\circ = 135^\circ$

$\frac{2\cdot 2+1}{4}\cdot 180^\circ = 225^\circ$

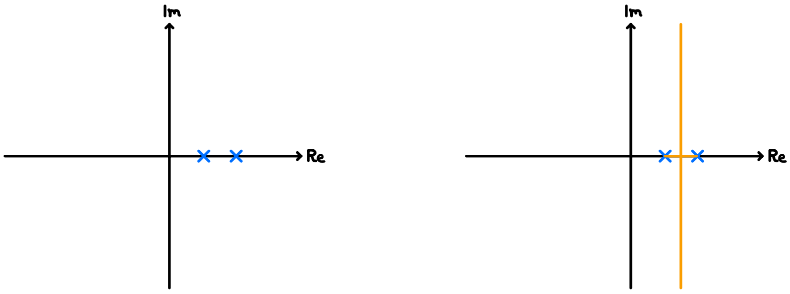
$\frac{2\cdot 3+1}{4}\cdot 180^\circ = 315^\circ$

If we want to implement more complex cortroller, that is itself a dynamic system, the open loop TF is the product of the plant with controler, i.e. $L(s) = C(s)\,P(s)$

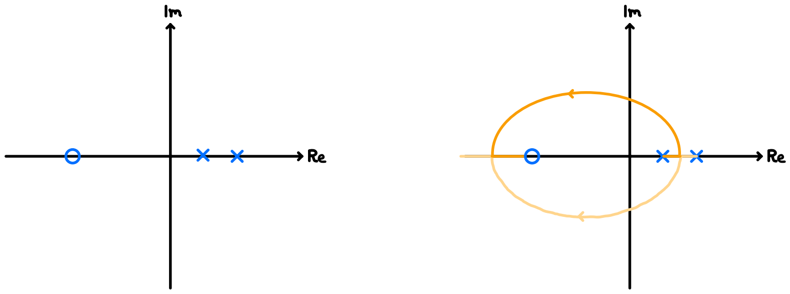


The controler $C(s)$ is often called dynamic compensator. We can use dynamic compensators to e.g. stabalize unstable systems.

Lets look at a simple example:



now we can design the dynamic compensator to have a minimum phase zero.



By adding a zero we are able to make the feedback connection stable. This is also called a PD controler.

6.1 Example Problems

6.1.1 FS 2018, Question 16

Question *Mark all correct statements. (2 Points)*
Which of the following statements are *true*?

- A For $k = 0$ the open-loop poles are equal to the closed-loop zeros.
- B The closed-loop poles are symmetric with respect to the real axis
- C For the root locus of a causal system, the number of closed-loop poles is equal to the number of open-loop poles.
- D The angle rule of the positive root locus states that $\sum_{z_i} \angle(s - z_i) - \sum_{p_i} \angle(s - p_i) = 0°(\pm q360°), q \in \mathbb{Z}$.

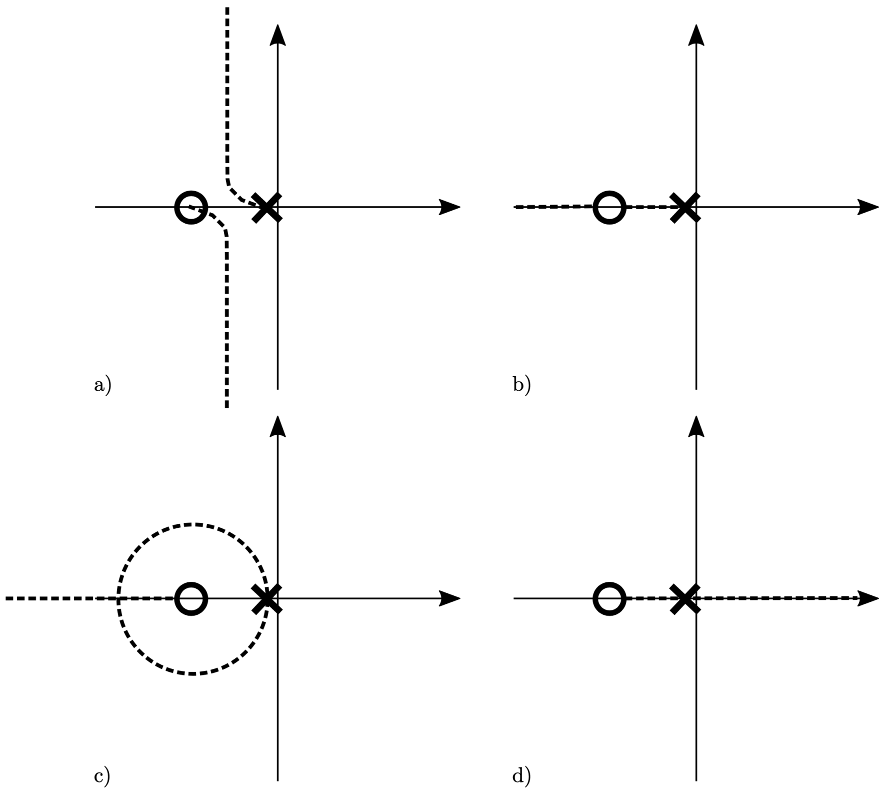
6.1.2 HS 2017, Question 25

Problem: The transfer function of a blood pressure regulator is given by:

$$P(s) = \frac{1}{10} \cdot \frac{s + \frac{21}{20}}{(s + \frac{1}{20}) \cdot (s + \frac{1}{20})}$$

Consider the following four root-locus curves:

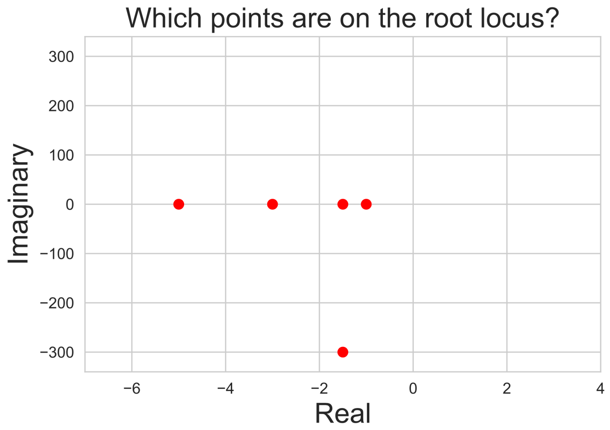
Question *Choose the correct answer. (1 Point)*
Out of the four root-locus sketches displayed, which one is corresponding to the previous transfer function and values $k \in \{0, 20\}$?



- [A] c
- [B] b
- [C] a
- [D] d

6.1.3 FS 2018, Question 15

Problem:



You are given the open-loop transfer function $L(s)$.

$$L(s) = \frac{s+2}{s(s+1)(s+4)}$$

Question *Mark all correct statements. (2 Points)*
Which of the following points p (marked in the figure) are on the positive root-locus for the above $L(s)$?

- A $p \approx -1.5 + 0.0j$
- B $p \approx -3 + 0.0j$
- C $p \approx -1 + 0.0j$
- D $p \approx -5 + 0.0j$
- E $p \approx -1.5 - 300j$

6.1.4 FS 2018, Question 17

Problem: Your coffe-enthusiast friend experimentally determined the second-order *closed-loop* poles a coffee machine should have. Both poles should have real-part $\text{Re}(\pi_{1,2}) = -2.5$ and there should be no oscillations.

Your *open-loop* coffee machine transfer function is $P(s) = \frac{1}{(s+2)(s+3)}$.

Question *Choose the correct answer. (1 Point)*
Can this be achieved? If yes, what is the P-controller gain k needed to reach the desired poles?

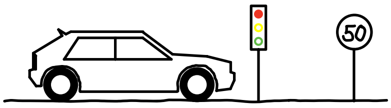
- A $k = \frac{1}{2}$
- B $k = \frac{1}{4}$
- C $k = 1$
- D It is not possible to reach the desired closed-loop poles.

7 Time Domain Specifications

So far, we assessed wether we can stabilize our system with feedback, based on the open loop TF $L(s)$.

Next to stability, there are many different requirements that are of interest when designing a controler.

Consider a car driving with cruise control:



Assume we can modell the car with a stable first-order system.

$$G(s) = \frac{1}{\tau s+1} \iff \begin{cases} \dot{x} = -\frac{1}{\tau}x + \frac{1}{\tau}u \\ y = x \end{cases}$$

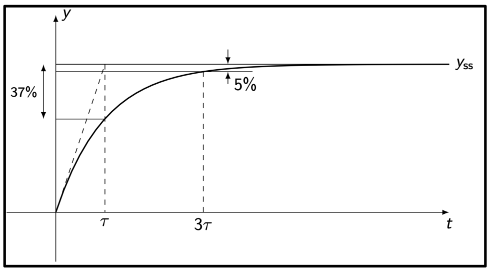
pole at $s = -\frac{1}{\tau}$

We are standing at a red traffic light and as soon as it turns green we want to accelerate to the max. allowed velocity of 50 km/h. So the reference changes from 0 to 50 in the instant that the light turns green. This corresponds to a scaled step-response to our system:

$$y(t) = Ce^{At}\cancel{x_0}^{\,0} + C\int_0^t e^{A(t-p)}B\cancel{u(p)}^{\,50}dp + \cancel{D}^{\,0}u(t)$$

$$= 50\int_0^t e^{-\frac{1}{\tau}(t-p)}\frac{1}{\tau}\,dp$$

$$= 50\left(1 - e^{-\frac{t}{\tau}}\right)$$



We can see that the behavior is dependend on τ. We can define a settling time T_d, i.e. the time it takes for the system to get within d% of the steady-state.

$$T_d = \tau \ln(100/d)$$

This measure of T_d can help us adjust the behavior of the system, such that it doesn't accelerate too fast or too slow. Since T_d is directly proportional to τ, and τ influences the location of the pole of $G(s)$, this time-domain specification translates to a constraint on the location of the poles of the system. This means that we can choose our controler in such a way that the time domain specification is met, by controling where our poles go.

Usually the first order aproximation from above is not enough to describe a system. So let's approximate with a second order system. We can define our TF to be:

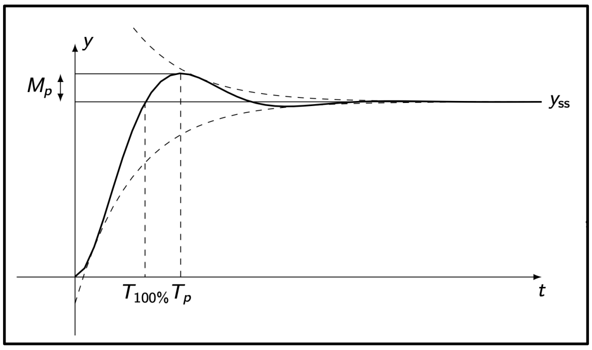
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \iff \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} x + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u \\ y = x \end{cases}$$

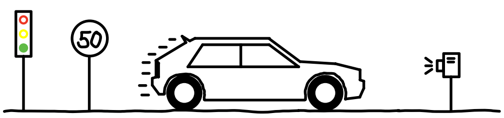
for an underdamped system ($\zeta < 1$) with zero initial condition we get:

$$y(t) = 1 - \frac{1}{\cos\varphi} e^{\sigma t} \cos(\omega t + \varphi)$$

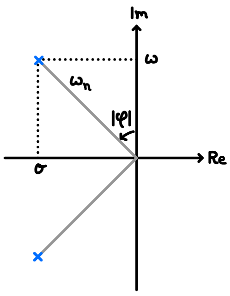
Again we can define important characteristics of the step response:

- → Time to peak: $T_p = \frac{\pi}{\omega}$
- → Peak overshoot: $M_p = e^{\frac{\sigma\pi}{\omega}}$
- → Rise time: $T_{100\%} = \frac{\frac{\pi}{2} - \varphi}{\omega} \approx \frac{\pi}{2\omega_n}$

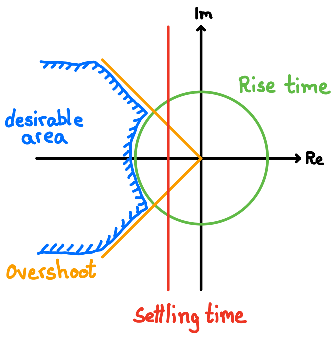




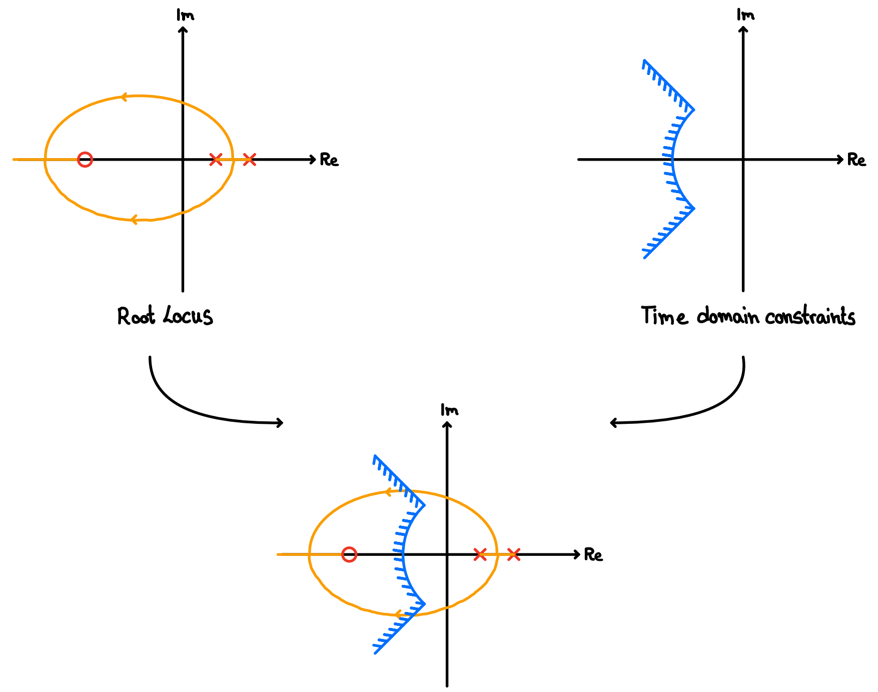
We can map out the poles in the complex plane and observe that each component of the time response affects the poles differently.



That means that we can again, take requirements in the time domain and transform them into constraints on the location of the poles of the system. When designing our closed loop system we can take these restrictions into account.



A way to relate these restrictions to the root locus is overlaying both

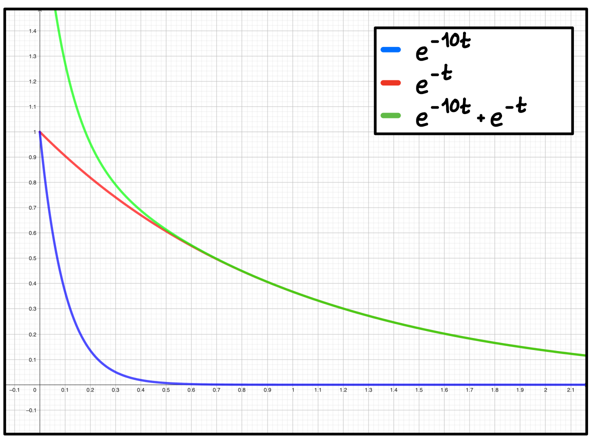


we can now choose k s.t. the poles fall in the restricted area.

<u>Dominant Pole Approximation:</u>

What if our system has more than two poles?

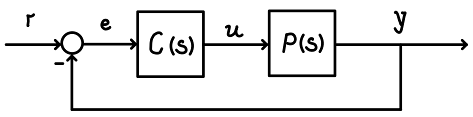
Often we can approximate the higher order system. Remember that every pole corresponds to an exponential. The real part of the pole indicates how fast the exponential grows or decays. For poles in the LHP we can say that poles further away from the imaginary axis are "faster" since they decay at a higher rate.



We observe that the be combined behavior can be well approximated by the slower pole, i.e. the pole closer to the imaginary axis.

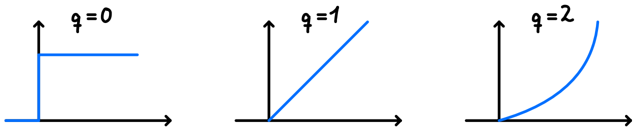
Steady-state error:

The steady-state error refers to the error e when $t \to \infty$. If the error is zero, we fully achieved the control objective.



To obtain a steady-state error of 0, we need a pole at the origin, also known as integrator. Depending on the input, we potentially need more than one integrator. When considering the steady-state error we usually look at ramp inputs. These are given by:

$$r(t) = \frac{1}{q!} t^q \quad \text{or in the s-domain} \quad R(s) = \frac{1}{s^q}$$ → order of ramp.



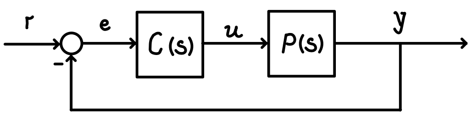
Depending on the order of the ramp and the type of the system we get different steady-state errors

e_{ss}	$q = 0$	$q = 1$	$q = 2$
Type 0	$\frac{1}{1 + k_{\text{Bode}}}$	∞	∞
Type 1	0	$\frac{1}{k_{\text{Bode}}}$	∞
Type 2	0	0	$\frac{1}{k_{\text{Bode}}}$

number of poles at the origin (Type)

PID Control:

Reall what our controler is designed to do:

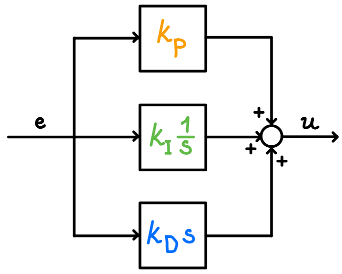


It takes the error, i.e. the difference between the current output and the reference we want to achieve, and transforms it into an input to the plant. The question is, how do we convert an error into a command? One way to do it is with PID control.

PID control is everywhere! What does PID stand for?

P → Proportional
I → Integral
D → Derivative

each term handles the error differently



The effects of each Term can be summarized.

→ <u>Proportional</u>

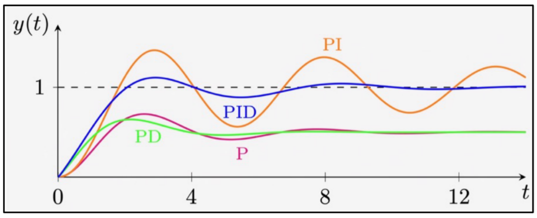
Decrease steady state error
Increase close-loop bandwidth
Increase sensitivity to noise
Can reduce stability margin for high order systems.

→ <u>Integral</u>

Eliminates the steady state error to a step (if CL is stable)
Reduces stability margins.

→ <u>Derivative</u>

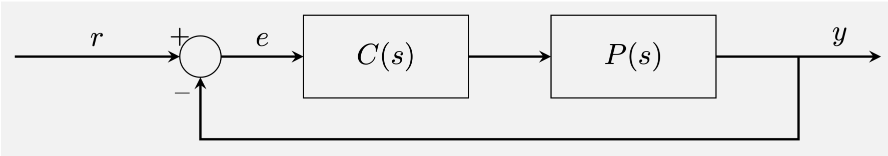
Reduces overshoot, increases damping
Improves stability margins
Increase sensitivity to noise



7.1 Example Problems

7.1.1 HS 2016, Question 31

Problem: Consider the system described by the following block diagram



where

$$P(s) = \frac{1}{10s + 1}, \qquad C(s) = k.$$

Question *Choose the correct answer. (1 Point)*
Determine the smallest positive gain k such that the feedback system is stable and, when $r(t)$ is a unit step, $\lim_{t\to\infty}|e(t)| \leq 0.1$.

The smallest positive gain k is:

- A 10
- B 2
- C 1
- D 5
- E 9

8 Frequency Response and Bode Plots

Recall that for LTI systems the TF can modell the steady-state output to some general input e^{st}. The TF is given by:

$$y_{ss} = G(s)\, e^{st} \quad \text{with } G(s) = C(sI-A)^{-1}B + D \;,\; s \in \mathbb{C}$$

We chose the general input e^{st} since virtually every input can be generated as a linear combination of terms e^{st}.

Since our system is LTI, we can thus calculate the response to complex inputs by breaking it down, and summing up the individual outputs. If the input is a cosine we saw that:

$$u(t) = \cos(\omega t) = \tfrac{1}{2} e^{j\omega t} + \tfrac{1}{2} e^{-j\omega t}$$

We notice that we can decompose $u(t)$ as follows:

$$u(t) = \sum_i U_i e^{s_i t} \quad \text{with } U_{1,2} = \tfrac{1}{2} \text{ and } s_{1,2} = \pm j\omega$$

The output is then given by

$$y(t) = G(j\omega)\tfrac{1}{2} e^{j\omega t} + G(-j\omega)\tfrac{1}{2} e^{-j\omega t}$$

we can rewrite $G(j\omega)$ as $Me^{j\phi}$ with $M = |G(j\omega)|$ Magnitude of $G(j\omega)$, $\phi = \angle G(j\omega)$ Phase of $G(j\omega)$

and then

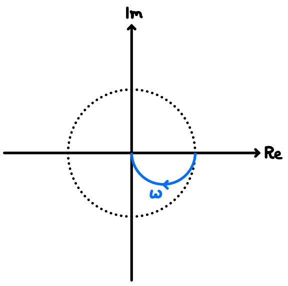
$$y(t) = Me^{j\phi}\tfrac{1}{2} e^{j\omega t} + Me^{-j\phi}\tfrac{1}{2} e^{-j\omega t}$$

$$y(t) = M\cos(\omega t + \phi)$$

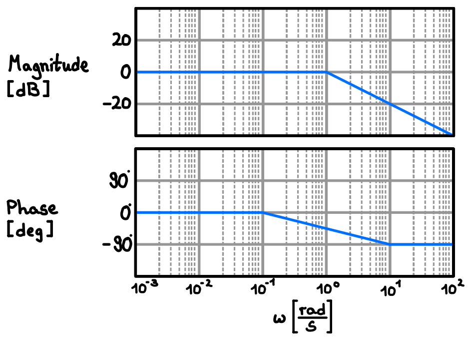
The output is another sinusoid with a different amplitude and phase, but same frequency. This means that, in order to analyze how a sinusoid affects our system we only have to know how the magnitude and phase change. These changes are given by $M = |G(j\omega)|$ and $\phi = \angle G(j\omega)$. So by pluging in $s = j\omega$ we can completley define the steady-state response to a sinusoidal input, this is also called frequency response.

When representing the frequency response we are essentially plotting a complex function $G(j\omega) \in \mathbb{C}$ with a real argument $\omega \in \mathbb{R}$. There are two ways to represent this:

as parametric curve, ω is implicit



two separate plots as a function of ω



Bode Plot:

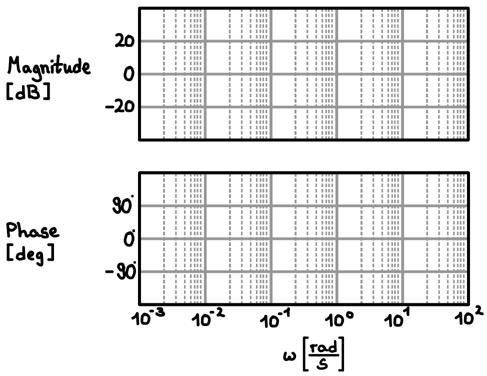
We want to plot $M = |G(j\omega)|$ and $\phi = \angle G(j\omega)$ as a function of ω. We will use two plots with a shared frequency axis.

The frequency axis is shown on a $\log_{10}$ scale in $\frac{rad}{s}$. The magnitude is plotted in deciBells, i.e.

$$|G(j\omega)|[dB] = 20\log_{10}|G(j\omega)|,$$

x	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	2	10	100
x_{dB}	-60	-40	-20	≈ -6	≈ -3	0	≈ 3	≈ 6	20	40

and the phase usually in degrees. This results in a plot of this form.



One reason for choosing this scaling is that when we have multiple inputs the magnitude gets multiplied and the phases added.

Since the magnitude plot is in log scale we can then add two lines in the plot instead of multiplying.

To sketch the Bode Plot we follow these basic rules:

Phase \ Magnitude	-20 dB/dec	$+20$ dB/dec
$-90°$	stable pole	non-minimum phase zero
$+90°$	unstable pole	minimum phase zero

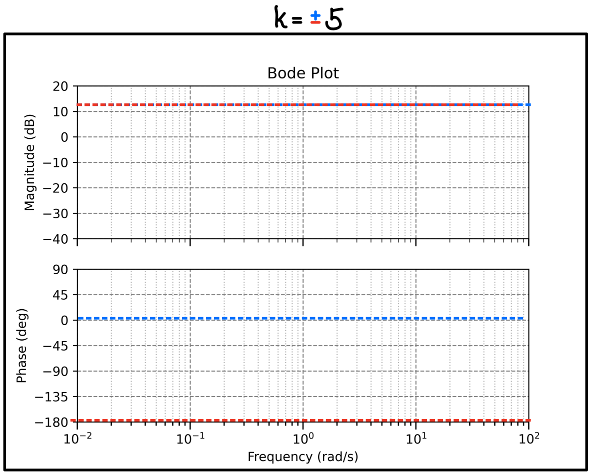
Let's try drawing some plots. To draw these plots by hand we can do a straight line approximation with the table above.

Also recall the Bode Form of a TF:

$$G(s) = \frac{k_{Bode}}{s^q}\frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\dots\left(\frac{s}{-z_m}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\dots\left(\frac{s}{-p_{n-q}}+1\right)}$$

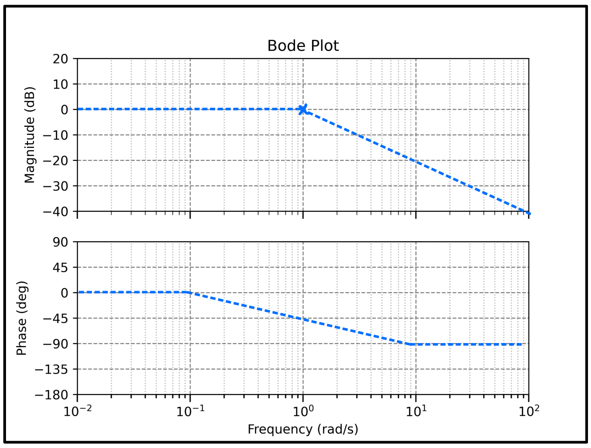
Let's look at simple examples

→ $G(s) = k$, $k>0$ and $k<0$



Magnitude : $20 \log|k| \approx 14\,dB$

→ $G(s) = \frac{1}{s+1}$ in Bode Form

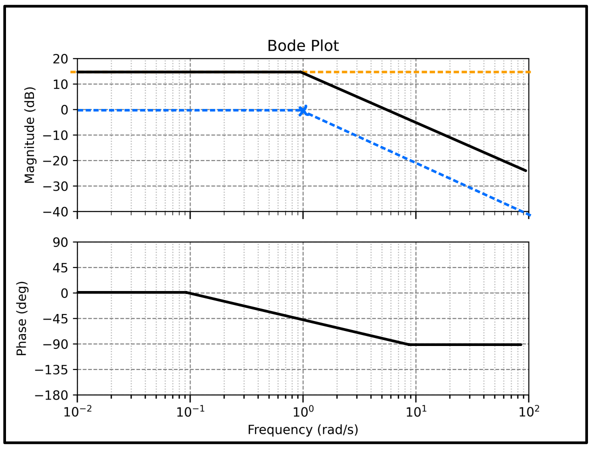


to draw this plot we take the absolute position of the pole and mark it on the plot. Draw the low freq. asymptote at 0 dB until the location of the pole. Then the high freq. asymptote from the pole until $\omega \to \infty$.

The phase usually starts changing 1 dec befor the pole location and continues until 1 dec after the pole.

we can put both together and draw the Bode plot for:

→ $G(s) = 5 \frac{1}{s+1}$ in Bode Form



Since we chose a log scale for the magnitude we can now add everything up.

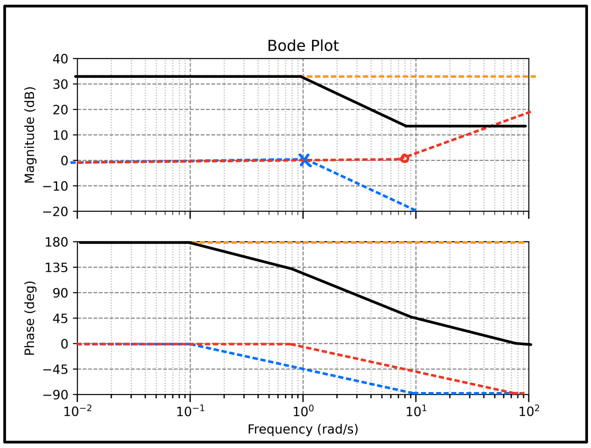
We can also add a zero:

$\rightarrow G(s) = 5\,\frac{s-8}{s+1}$

bring to Bode Form!

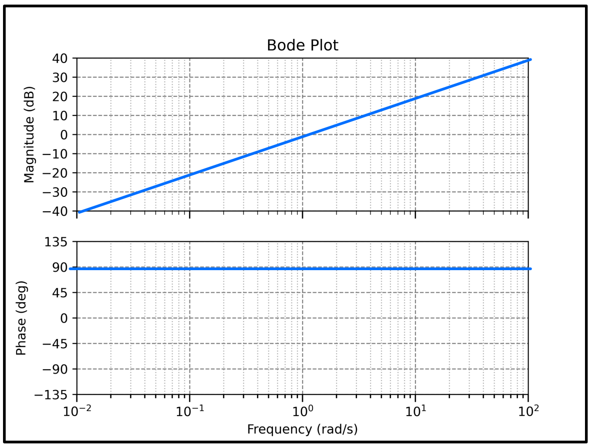
$G(s) = -40\,\frac{\frac{s}{-8}+1}{s+1}$

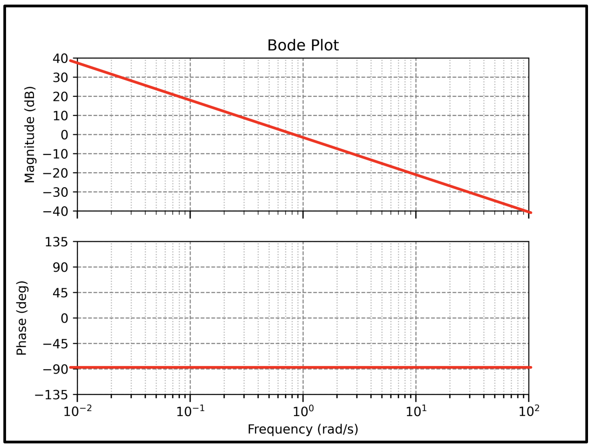
Proceed as above, but look at the table for the behavior of the non-minimum phase zero.



This can be done for an arbitrary numbers of poles and zeros. Two important cases are poles and zeros at the origin. The respective plots look like this:

$G(s) = s = j\omega$ $G(s) = \frac{1}{s}$





A more complete table with sketching rules is given by:

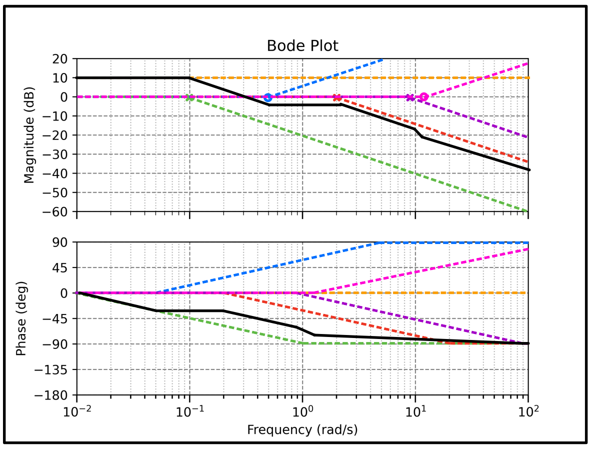
Rules for Making Bode Plots

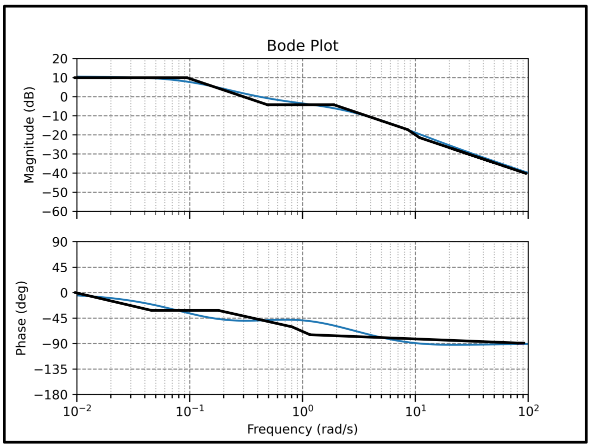
Term	Magnitude	Phase
Constant: K	$20\cdot\log_{10}(\lvert K\rvert)$	K>0: 0° K<0: ±180°
Real Pole: $\frac{1}{\frac{s}{\omega_0}+1}$	• Low freq. asymptote at 0 dB • High freq. asymptote at -20 dB/dec • Connect asymptotic lines at ω_0,	• Low freq. asymptote at 0°. • High freq. asymptote at -90°. • Connect with straight line from $0.1\cdot\omega_0$ to $10\cdot\omega_0$.
Real Zero*: $\frac{s}{\omega_0}+1$	• Low freq. asymptote at 0 dB • High freq. asymptote at +20 dB/dec. • Connect asymptotic lines at ω_0.	• Low freq. asymptote at 0°. • High freq. asymptote at +90°. • Connect with line from $0.1\cdot\omega_0$ to $10\cdot\omega_0$.
Pole at Origin: $\frac{1}{s}$	• -20 dB/dec; through 0 dB at ω=1.	• Line at -90° for all ω.
Zero at Origin*: s	• +20 dB/dec; through 0 dB at ω=1.	• Line at +90° for all ω.
Underdamped Poles: $\frac{1}{\left(\frac{s}{\omega_0}\right)^2+2\zeta\left(\frac{s}{\omega_0}\right)+1}$	• Low freq. asymptote at 0 dB. • High freq. asymptote at -40 dB/dec. • Connect asymptotic lines at ω_0. • Draw peak† at freq= ω_0, with amplitude $H(j\omega_0)=-20\cdot\log_{10}(2\zeta)$	• Low freq. asymptote at 0°. • High freq. asymptote at -180°. • Connect with line from $\omega=\omega_0\cdot 10^{-\zeta}$ to $\omega_0\cdot 10^{\zeta}$

Notes:
ω0 is assumed to be positive. If ω0 is negative, magnitude is unchanged, but phase is reversed.

Example:

$$G(s) = \frac{(s+0.5)(s+12)}{(s+2)(s+0.1)(s+9)} = \frac{10}{3}\,\frac{\left(\frac{s}{0.5}+1\right)\left(\frac{s}{12}+1\right)}{\left(\frac{s}{2}+1\right)\left(\frac{s}{0.1}+1\right)\left(\frac{s}{9}+1\right)}$$





Until now we only considered real poles. But what if the TF has a complex-conjugate pair of poles? We can generally write a corresponding TF as:

$$G(s) = \frac{1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1} = \frac{\omega_n^2}{s^2 + 2\zeta s\omega_n + \omega_n^2}$$

For these systems the damping ratio ζ influences the form of the Bode plot.

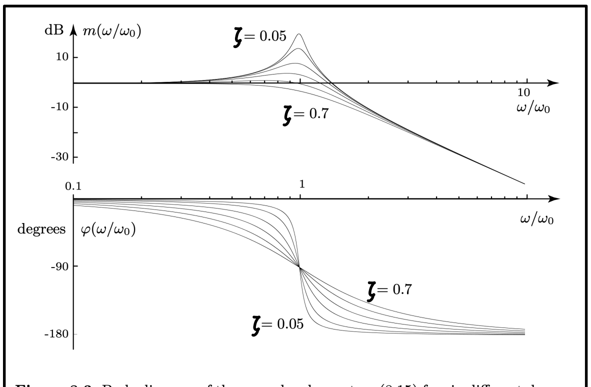
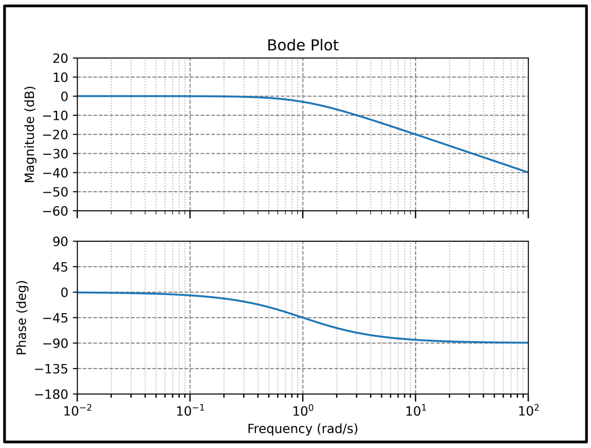


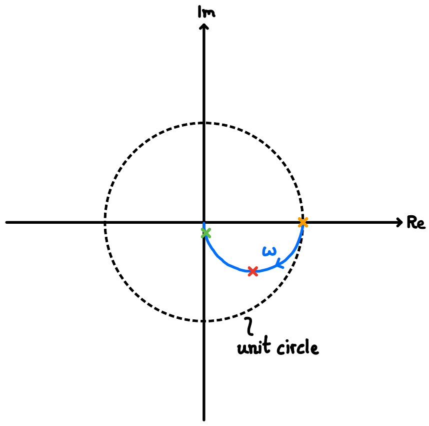
Figure 8.3. Bode diagram of the second-order system (8.15) for six different damping ratios $\zeta = [0.05, 0.1, 0.2, 0.33, 0.5, 0.7]$.

Despite this they can be treated like normal poles.

Polar Plot:

We can also represent the frequency response as a parametric curve, where ω is implicit. We can then plot one curve in the complex plane. Lets start with a simple example. $G(s) = \frac{1}{s+1}$





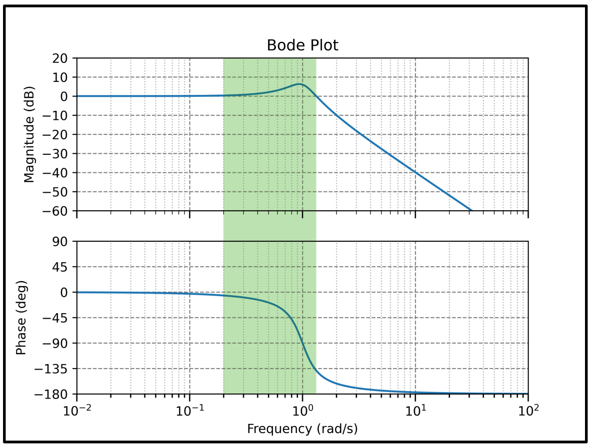
We will go through some ω and note down magnitude and phase: $\omega = 0$, $|\cdot| = 0\,dB = 1$, $\measuredangle = 0°$

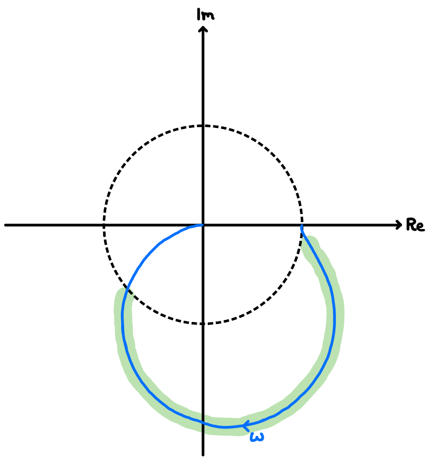
$\omega = 1$, $|\cdot| = -3\,dB = \frac{1}{\sqrt{2}}$, $\measuredangle = -45°$

$\omega = 10$, $|\cdot| \cong -20\,dB = 0.1$, $\measuredangle \cong -90°$

Based on this we can make a rough sketch.

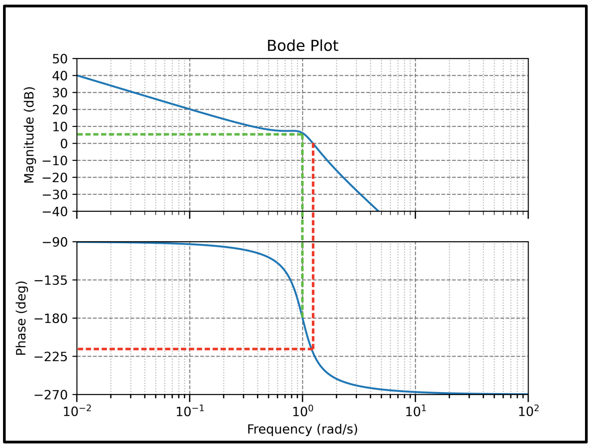
Another example: $G(s) = \frac{1}{s^2 + 0.5s + 1}$

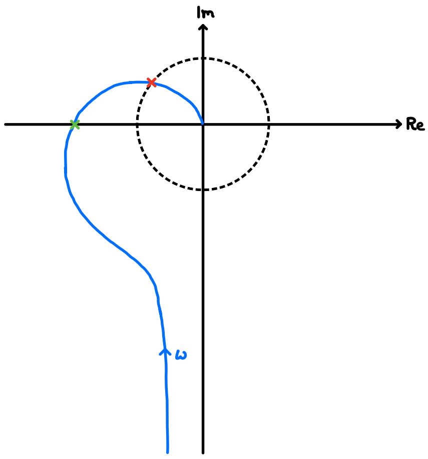




Outside of unit circle between ~ 10°-135°

$$G(s) = \frac{1}{s}\,\frac{1}{s^2+0.5s+1}$$



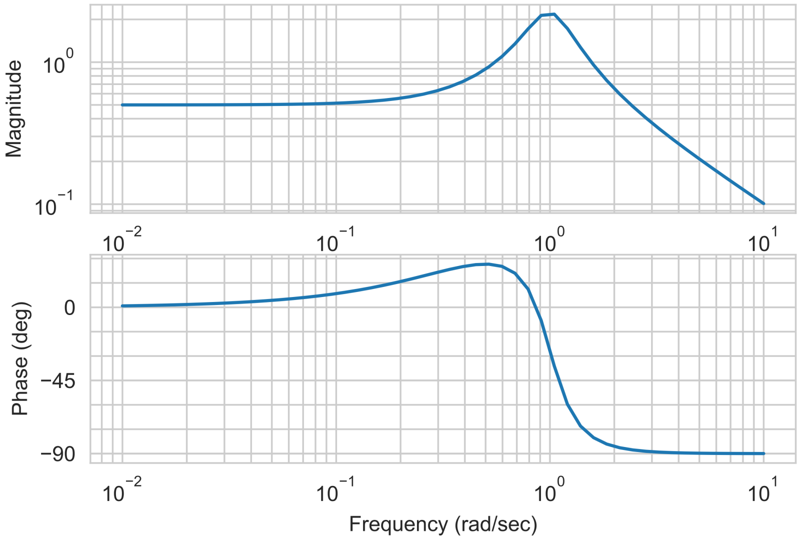


Point at -180° approx 5dB ~ 2 , Point at 0 dB ~ -210°

8.1 Example Problems

8.1.1 FS 2018, Question 18

Problem: You are given the following Bode plot



Question *Choose the correct answer. (1 Point)*
Which of the following transfer functions corresponds to the above Bode plot?

A $G(s) = \frac{1}{s^2+0.5s+1}$

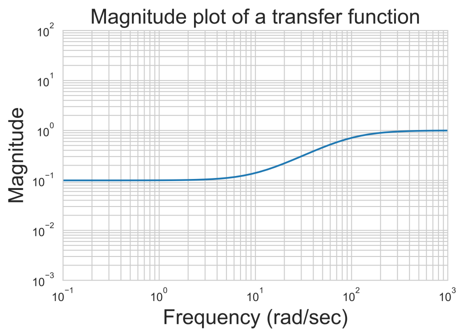
B $G(s) = \frac{s+0.5}{s^2+0.5s+1}$

C $G(s) = \frac{1-s}{s^2+0.5s+1}$

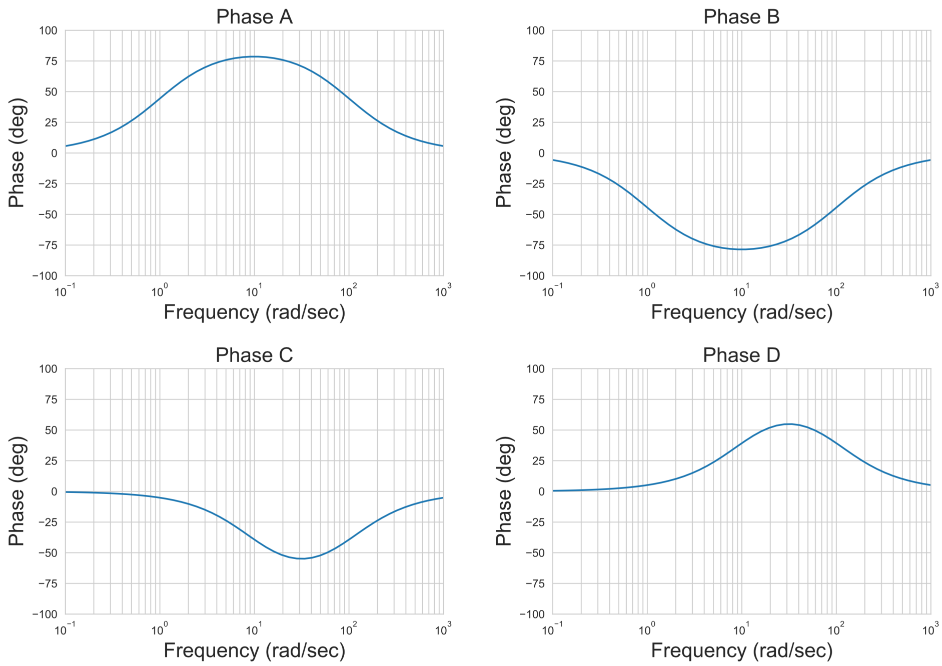
D $G(s) = \frac{s}{s^2+0.5s+1}$

8.1.2 FS 2018, Question 19

Problem: You are given the following magnitude plot of a plant. Furthermore, you know the plant **does not** have any unstable poles.



Question *Choose the correct answer. (1 Point)*
Which of the following phase plots corresponds to the above plant?



[A] Phase plot B

[B] Phase plot C

[C] Phase plot D

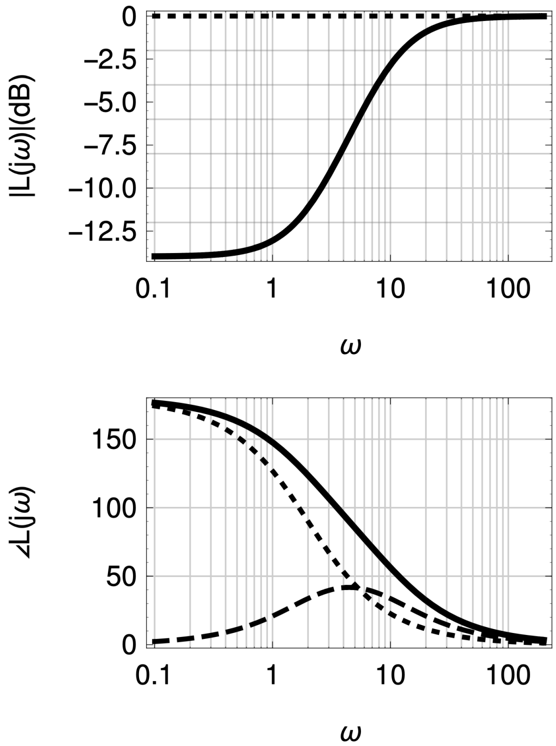
[D] Phase plot A

8.1.3 FS 2018, Question 30

Problem: A plant $P(s)$ with non-minimum phase poles or zeros has been reformulated in order to design a controller in the following way:

$$P(s) = P_{\text{mp}}(s)D(s)$$

Where $D(j\omega) = 1,\ \forall\omega$. All transfer functions $P(s), P_{\text{mp}}(s)$ and $D(s)$ are displayed in the Bode plot below:



Question *Choose the correct answer. (1 Point)*
What is the transfer function of $P(s)$?

A $P(s) = 10\frac{s-2}{s+10}$

B $P(s) = \frac{s+5}{(s-10)^2}$

C $P(s) = \frac{s-12}{s+60}$

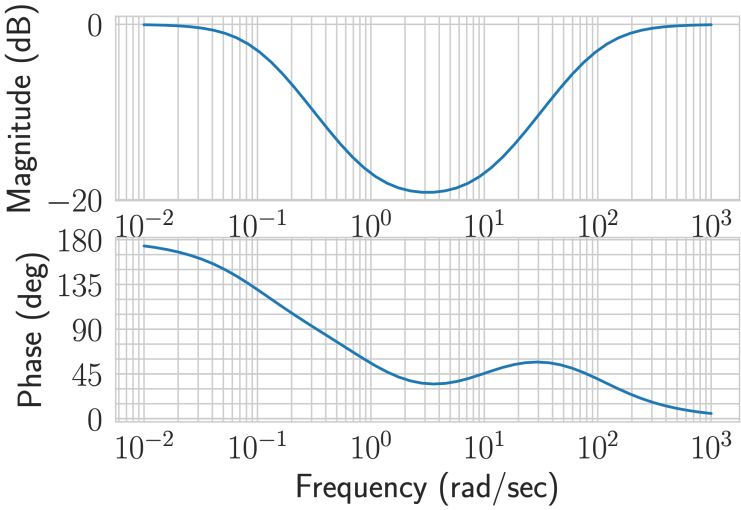
D $P(s) = \frac{s-2}{s+10}$

E $P(s) = \frac{s+2}{(s-10)^2}$

F $P(s) = \frac{s-5}{(s+10)^2}$

8.1.4 HS 2018, Question 18

Problem: You are given the following Bode plot.



Question *Choose the correct answer. (1 Point)*
What are the poles π and zeros ζ of the system?

- A $\pi = \{-0.1, -100\}, \zeta = \{1, -10\}$
- B $\pi = \{-1, -10\}, \zeta = \{-0.1, -10\}$
- C $\pi = \{1, -100\}, \zeta = \{1, -10\}$
- D $\pi = \{-0.1, -10\}, \zeta = \{-1, -100\}$

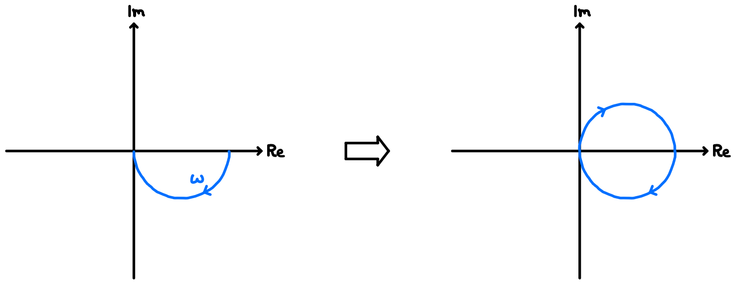
8.1.5 HS 2017, Question 28

Question *Choose the correct answer. (1 Point)*
If a Bode plot of a rational transfer function is plotted for $\omega \in \{-\infty, \infty\}$ then both the magnitude plot and phase plot are symmetrical with respect to the y-axis.

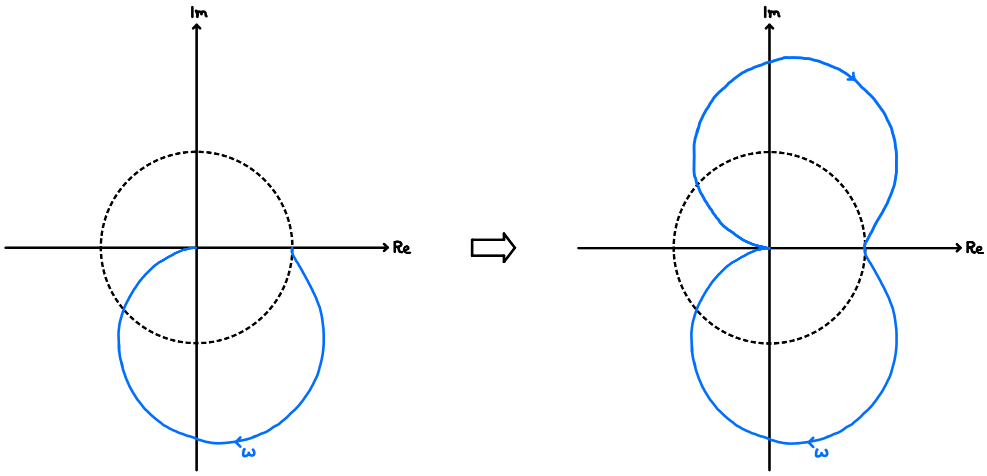
- A True
- B False

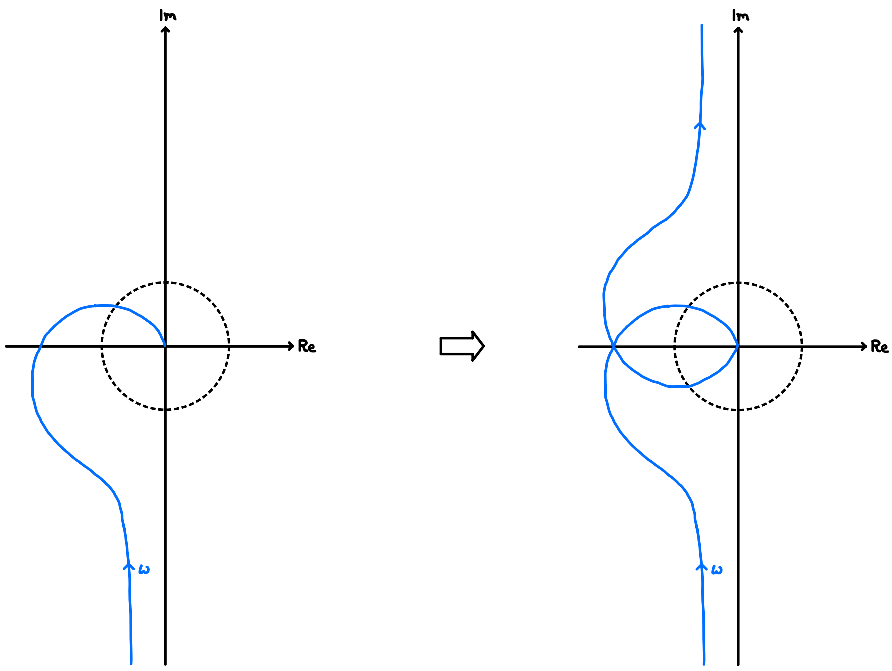
9 Nyquist Plot

To draw this plot, just take the polar plot and mirror it along the real axis!



We will understand why, in just a second. Here are some more examples:



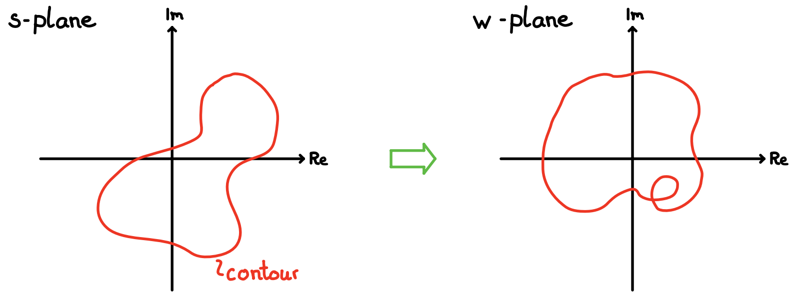


With these plots we will be able to asses stability and robustness of our closed loop system!

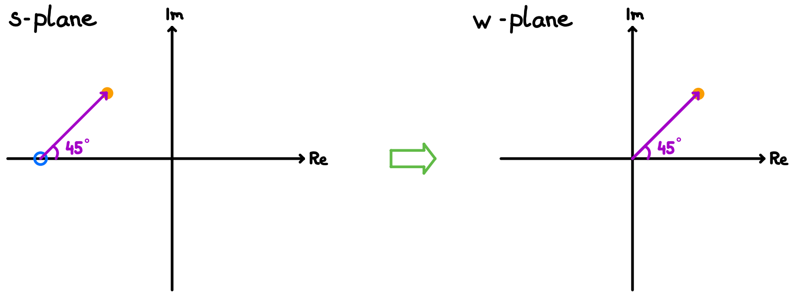
First let's take a step back and see how this plot comes about.

Principle of variation of the argument (Cauchy's Argument Principle)

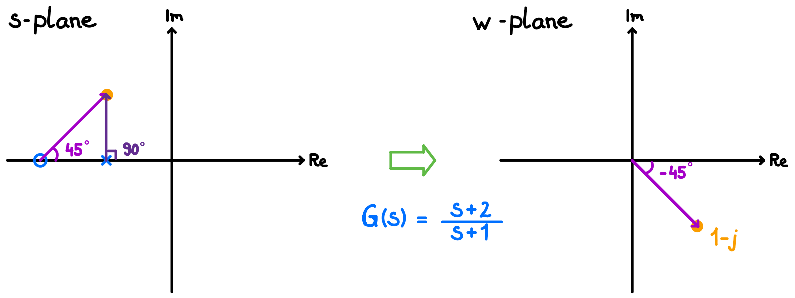
Given a TF G(s), we can plug in different values of s and get another complex number. We can plot this new complex number in a plane and call it the w-plane. We can now say that our TF G(s) maps a point from the s-plane to the w-plane. We can extend this mapping and map entire lines from s to w. We can also look at how closed curves, so called contours, map from s to w.



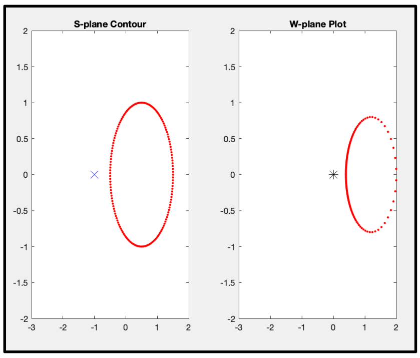
a closed curve in the s-plane will also result in a closed curve in the w-plane. This closed curve in the w-plane now also includes informations about the TF we used to map from s to w. Consider a simple example

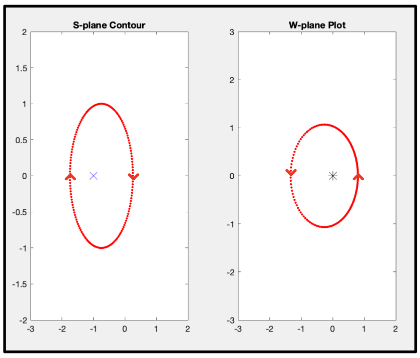


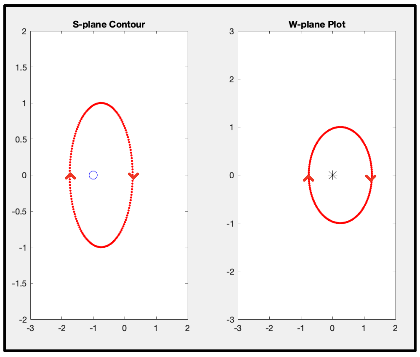
We observe that the phasor of the point in the w-plane is the same as the one from the zero to the original point in the s-plane. Let's add in a pole and see what happens:

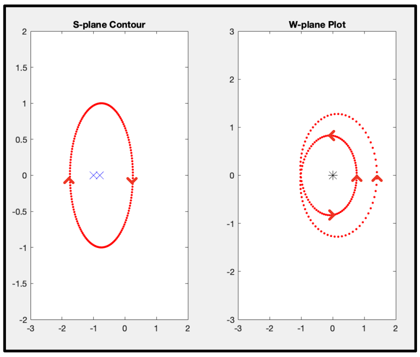


The general rule is: Add phases of zeros subtract phases of poles. Let's look at some different contours.

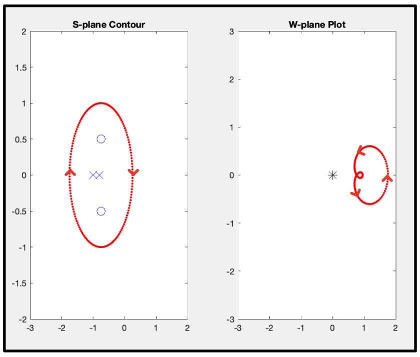


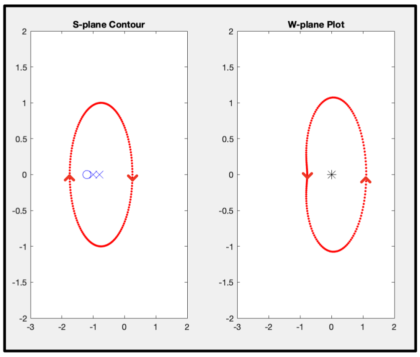






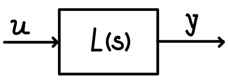
What we can see is that for every time we encircle a pole / zero in the s-plane we also encircle the origin in the w-plane. For each clockwise encirclement of zeros we get one CW encirclement of the origin, and for each CW encirclement of a pole we get one CCW encirclement of the origin. Think of a zero as adding 360° and a pole as subtracting 360°. If we have as many poles as zeros we end up with no encirclements, and if we have one more pole than zeros, we end up with one CCW encirclement of the origin.



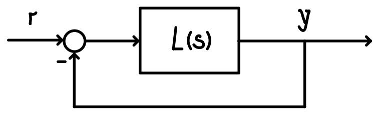


This means that you can tell the relative difference of poles and zeros inside a contour by how many times the plot circles the origin and in which direction.

Let's get back to the Nyquist plot. Recall that for an open loop system,



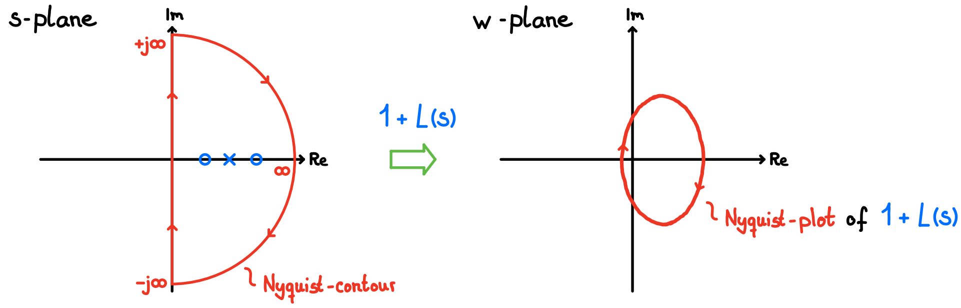
we can check wether the system is stable by looking at the poles of $L(s)$. If there are any poles in the RHP the system is unstable. In closed-loop systems:



we now have to look at the poles of:

$$T(s) = \frac{L(s)}{1 + L(s)}$$

i.e. all points where $1 + L(s) = 0$. To asses the closed loop stability we have to check if any of the the ZEROS of $1 + L(s)$ are in the RHP. To do this we can use the principle of variation of the argument introduced above. We can choose our contour to encircle the entire RHP:



we now know:

$$\boxed{\# \text{Encirclements of } 0 \text{ using } 1 + L(s) = \# \text{Zeros of } 1 + L(s) \text{ in RHP} - \# \text{Poles of } 1 + L(s) \text{ in RHP}}$$

We can make use of these characteristics:

- Instead of counting the encirclements of $1 + L(s)$ arround zero, we can shift the coordinate system by -1 and count the encirclements of $L(s)$ arround -1.
- If $L(s) = \frac{N(s)}{D(s)}$, then the poles of $L(s)$ are given by D(s). We can also re-write $1 + L(s)$:

$$1 + L(s) = 1 + \frac{N(s)}{D(s)} = \frac{D(s) + N(s)}{D(s)}$$, the poles are also given by D(s)

Thus, $L(s)$ and $1 + L(s)$ have the same poles

We can combine both and get:

$$\boxed{\# \text{Encirclements of } -1 \text{ using } L(s) = \# \text{Zeros of } 1 + L(s) \text{ in RHP} - \# \text{Poles of } L(s) \text{ in RHP}}$$

Which finally yields the Nyquist stability theorem:

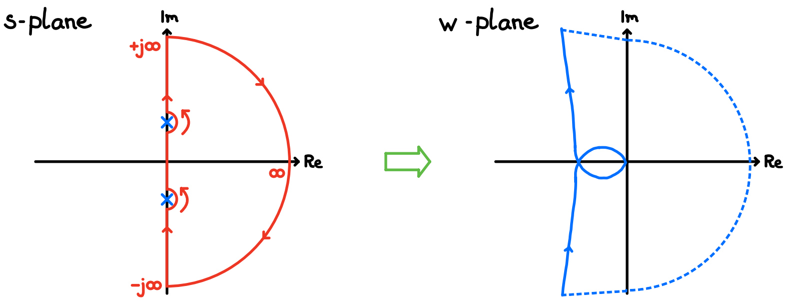
$$Z = N + P$$

- Z: # Zeros of $1 + L(s)$ in RHP = # unstable closed loop poles
- N: # CW Encirclements of -1 using $L(s)$
- P: # Poles of $L(s)$ in RHP = # unstable open loop poles

We can now asses wether a closed loop system is stable, by only looking at the OL poles and the Nyquist-plot

Special case:

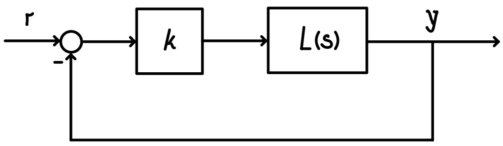
How do we treat poles and zeros of $L(s)$ on the imaginary axis?



We make little indents on the imaginary axis. If you move around the poles CCW, then you have to close the Nyquist-plot CW at infinity.

Note:

If your CL has some gain k:



$$T(s) = \frac{kL(s)}{1 + kL(s)}$$

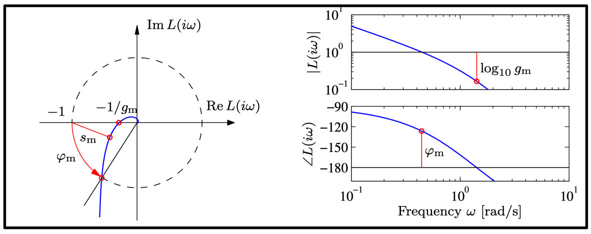
you have to count the #CW Encirclements of $-\frac{1}{k}$.

Stability Margins:

Next to stability of the CL system, the Nyquist-plot can also tell us how far away we are from being unstable.

Assume our OL to be stable, i.e. $P = 0$. For our CL system to be stable we now need $N = 0$. Then $Z = N + P = 0$.

We can now define a phase- and gain margin that tell us how "close" we are to encircling -1.



What does each margin mean?

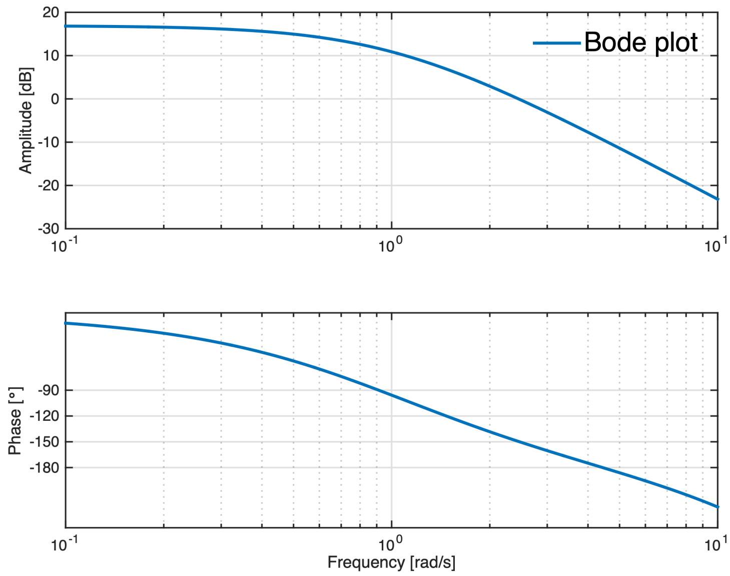
→ Gain Margin g_m: The point at 180°. It tells us how much we can scale until reaching -1.

→ Phase Margin φ_m: Point at magnitude 1. It tells us how much we can change the phase until reaching -1.

9.1 Example Problems

9.1.1 HS 2016, Question 45

Problem: Consider the following Bode plot of a plant:

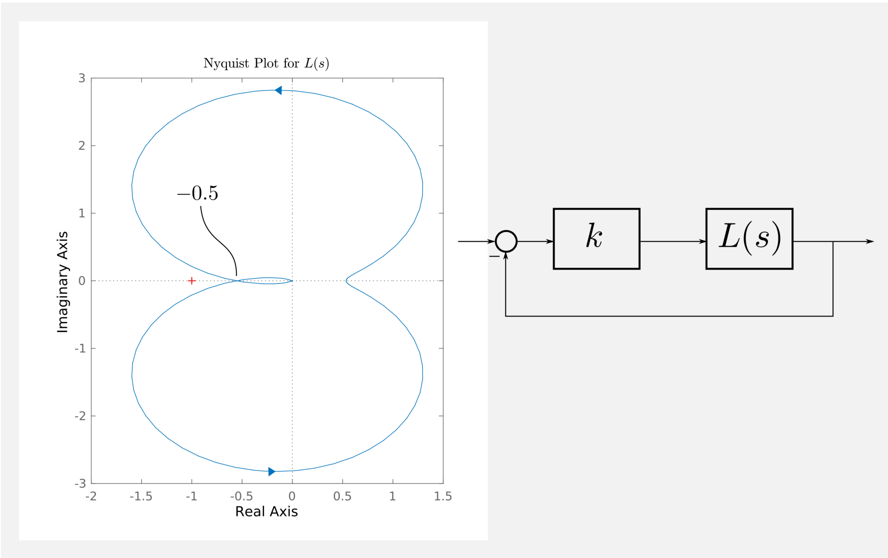


Question *Choose the correct answer. (1 Point)*
Use the bode plot shown above to determine the approximate crossover frequency ω_c, phase margin φ and gain margin γ.

- A $\omega_c \approx 2.5\,\text{rad/s}$, $\varphi \approx 30°$, $\gamma \approx 1$
- B $\omega_c \approx 4.4\,\text{rad/s}$, $\varphi \approx 15°$, $\gamma \approx 3$
- C $\omega_c \approx 2.5\,\text{rad/s}$, $\varphi \approx 30°$, $\gamma \approx 3$
- D $\omega_c \approx 4.4\,\text{rad/s}$, $\varphi \approx 30°$, $\gamma \approx 3$

9.1.2 HS 2016, Question 26

Problem: The Nyquist plot for a loop transfer function $L(s)$ with two unstable poles is shown below on the left. On the right is a feedback configuration for the closed loop system.

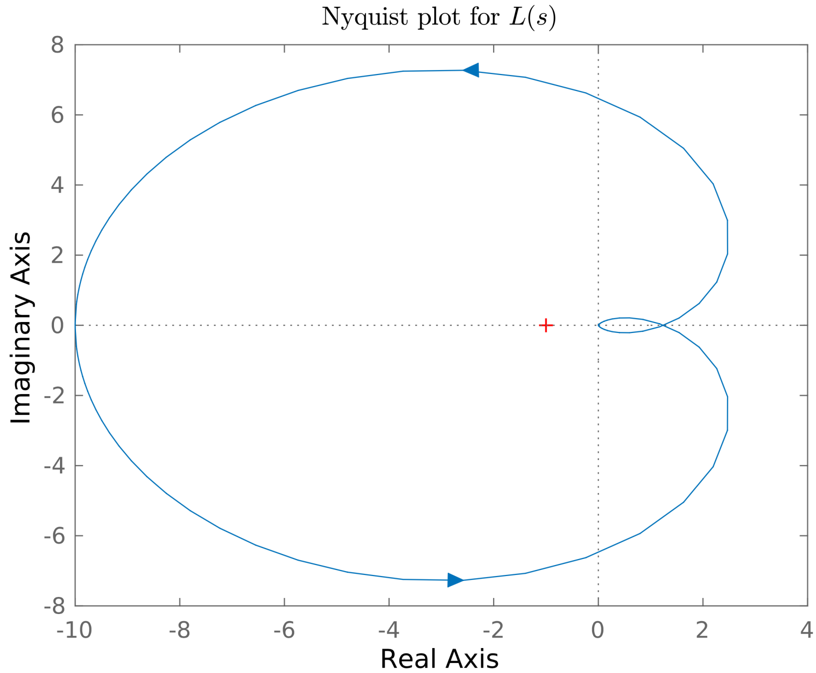


Question *Mark all correct statements. (2 Points)*
Which of the following statements are true about the closed loop system $T(s) = \frac{kL(s)}{1+kL(s)}$?

- A The closed loop system $T(s)$ is stable when $k = 1$.
- B The closed loop system $T(s)$ is unstable when $k > 2$.
- C The closed loop system $T(s)$ is unstable when $k = 1$.
- D The closed loop system $T(s)$ is stable when $k > 2$.

9.1.3 HS 2016, Question 30

Problem: Consider the following Nyquist plot of an open loop gain $L(s)$:



Question *Choose the correct answer. (1 Point)*
Select the transfer function which matches the Nyquist plot.

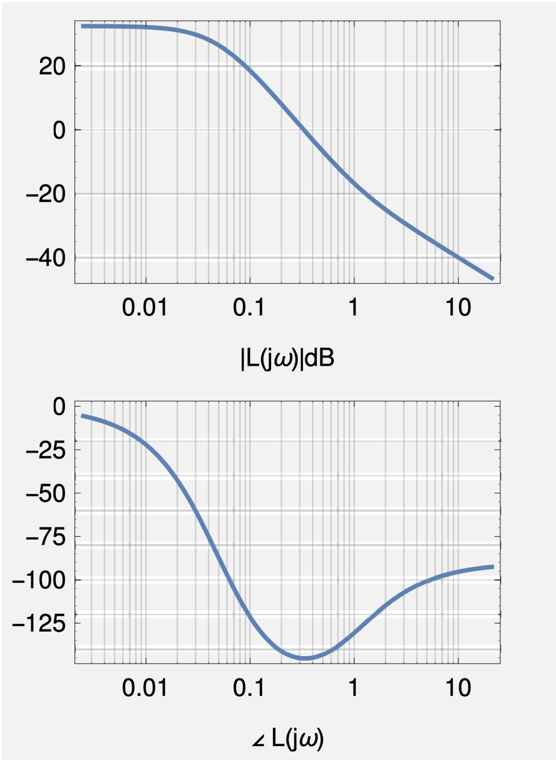
- [A] $L(s) = \frac{10}{(s-1)^3}$
- [B] $L(s) = \frac{10}{(s+1)^3}$
- [C] $L(s) = \frac{-10}{(s+1)^3}$
- [D] $L(s) = \frac{10}{(1-s)^3}$

9.1.4 HS 2017, Question 29

Problem: We consider the following transfer function:

$$L(s) = \frac{1}{10} \cdot \frac{s + \frac{21}{20}}{(s + \frac{1}{20}) \cdot (s + \frac{1}{20})}$$

Furthermore, we consider a Bode plot:

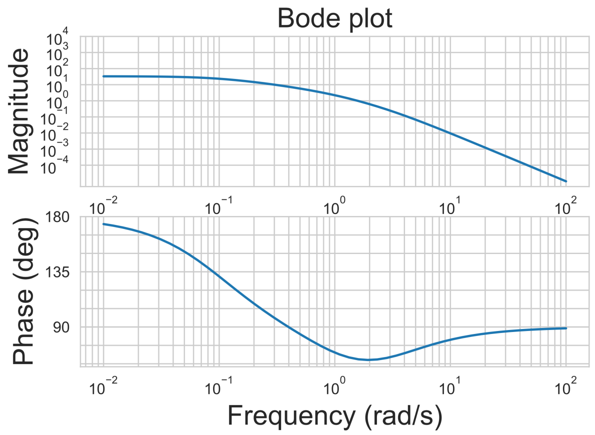


Question *Mark all correct statements. (2 Points)*
Which statements are correct:

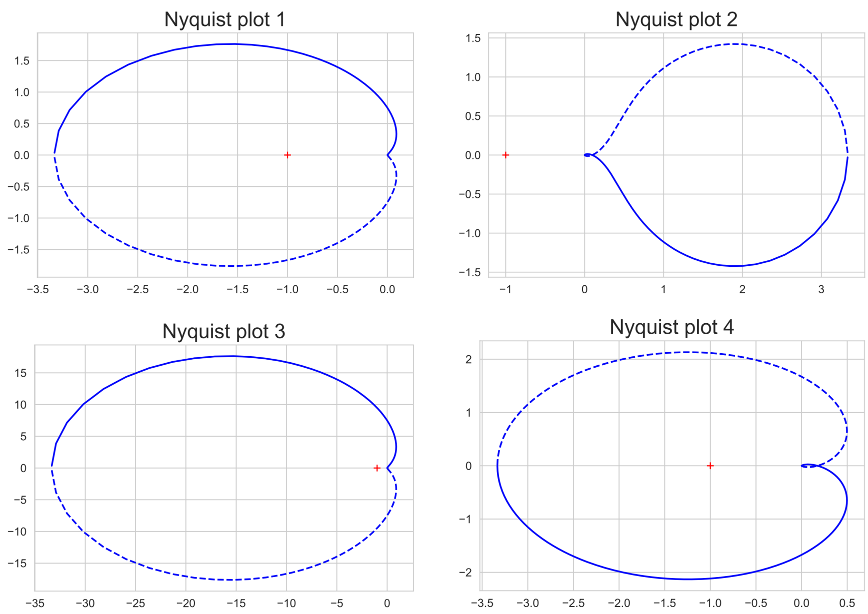
- [A] The phase margin of $L(s)$ is ∞.
- [B] The closed-loop system for $L(s)$ will be able to track a reference signal $r : t \to \sin(t)$ without steady-state error.
- [C] The gain margin of $L(s)$ is ∞.
- [D] At crossover frequency, the system shows a gradient of $-40\,\text{dB/dec}$.
- [E] The displayed Bode plot is the correct Bode plot for the transfer function $L(s)$.

9.1.5 FS 2018, Question 24

Problem: You are given the following Bode plot.



Question *Choose the correct answer. (1 Point)*
What is the associated Nyquist plot to this Bode plot?



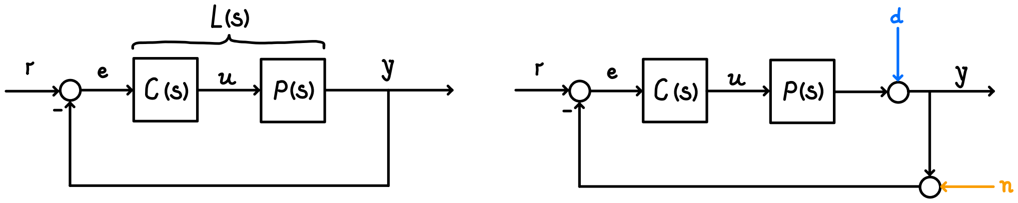
[A] Option 3

[B] Option 4

[C] Option 1

[D] Option 2

Similar to the time-domain specs. that resulted in feasible areas in the s-plane, we can define frequency domain specs, that dictate/shape how our Bode plot should look like. Recall the sensitivity/CL TFs we introduced a while back.



→ Open loop TF:

$L(s) = C(s)P(s)$

→ Complementary Sensitivity:

maps $r \to y$, $n \to y$

$$T(s) = \frac{L(s)}{1+L(s)}$$

→ Sensitivity:

maps $r \to e$, $d \to y$

$$S(s) = \frac{1}{1+L(s)}$$

If we have disturbances d and/or noise n entering our CL system, we can use T and S to map the noise n and disturbances d to the output y. Usually disturbances have low frequencies and noise has high frequencies. The commands we input to our system usually also have a relativley low frequency. Knowing this we can constrain the magnitudes of $S(j\omega)$ and $T(j\omega)$ in the following way:

→ $|S(j\omega)| \ll 1$ at low frequencies for disturbance rejection and good command tracking.

→ $|T(j\omega)| \ll 1$ at high frequencies for noise rejection.

Remember: $S + T = 1 \;\; \forall \omega$. We can't make the sensitivity functions arbitrarily small over all frequencies.

As always, we would like to have these constraints as a function of the OL TF $L(s)$. In this case:

→ $|S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right| \ll 1 \iff$ $L(s)$ has to be large at low frequencies

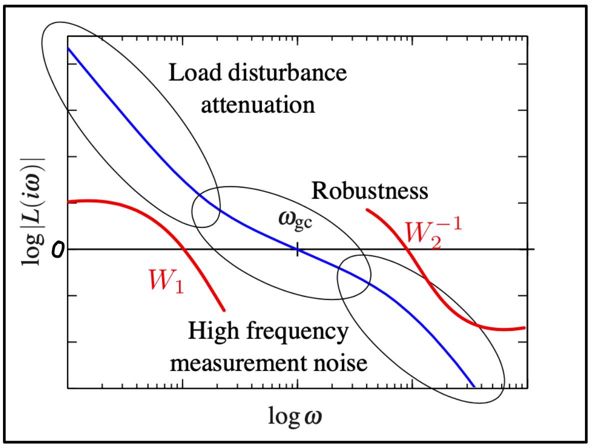
→ $|T(j\omega)| = \left|\frac{L(j\omega)}{1+L(j\omega)}\right| \ll 1 \iff$ $L(s)$ has to be small at high frequencies

We can quantify how large or small we want $L(s)$ to be, with some function $W(j\omega)$. So we can write:

$$|S(j\omega)| \cdot |W_1(j\omega)| < 1 \iff |1+L(j\omega)| > |W_1(j\omega)| \overset{\text{approx.}}{\iff} |L(j\omega)| > |W_1(j\omega)|$$

$$|T(j\omega)| \cdot |W_2(j\omega)| < 1 \underset{T(j\omega)\approx L(j\omega)}{\overset{\text{approx.}}{\iff}} |L(j\omega)| \cdot |W_2(j\omega)| < 1 \iff |L(j\omega)| < |W_2(j\omega)|^{-1}$$

This results in the following "obstacle course" for the Bode plot of $L(j\omega)$.



Next to high- and low-frequency behavior we can also constrain the bandwidth of CL system. The bandwidth tells us the maximum frequency for which the output can track commands within a factor ≈ 0.7. In other words the bandwidth tells us for which max freq. we get satisfactory operation. ($|T(j\omega)| > 1/\sqrt{2}$). We can usually approximate the CL bandwidth with the OL crossover frequency ω_{gc}. But how do we apply these constraints to our $L(j\omega)$?

Loop Shaping

Now we get to design our controller. Usually we approach this problem with some basic building blocks to steer $L(s)$ through the Bode obstacle course. That way we construct a dynamic compensator $C(s)$ that fullfills our requirements.

Proportional compensation:

In this case $C(s) = k$, where k is a simple gain.

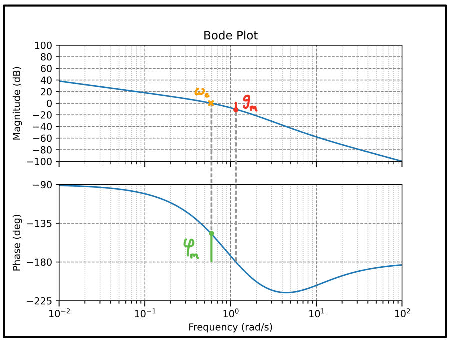
→ Shifts the magnitude, while phase is unaffected.

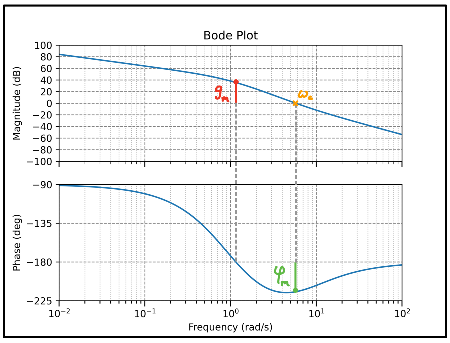
→ For stable OL systems, small k ($k \to 0$) yield stable CLs.

→ Improves command tracking (higher magnitude at low ω) and CL bandwidth (moves crossover freq. to right)

→ Stability can be compromized!!

$L(s)$ | $40 \cdot L(s)$



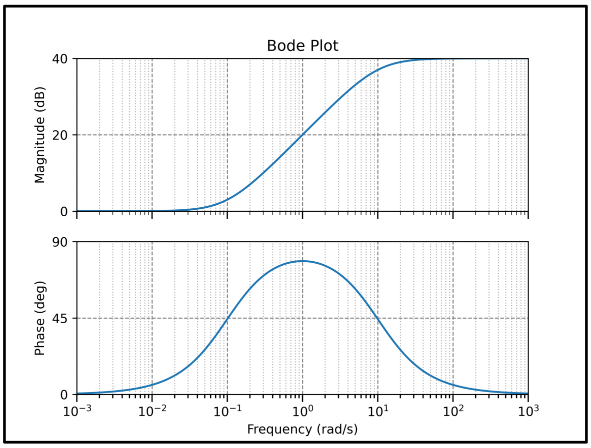


Lead compensator

A lead compensator is a pole zero pair. We can write it in a general form as:

$$C_{lead}(s) = \frac{\frac{s}{a}+1}{\frac{s}{b}+1} = \frac{b}{a}\frac{s+a}{s+b} \qquad 0<a<b$$

In a lead compensator the zero always comes before the pole. Graphically it can look like this:



Example for $a=0.1$, $b=10$, i.e. $C_{lead}(s) = \frac{10s+1}{0.1s+1} = 100\,\frac{s+0.1}{s+10}$

The main effects are:

→ Increase magnitude at high freq. by $\frac{b}{a}$, while low freq. are unafected.

→ Increase slope of magnitude between a and b by $20\,^{dB}/_{dec}$

→ Increase phase around $\sqrt{ab}$ by up to 90°. The max phase increase is of

$$\varphi_{max} = 2\arctan\left[\sqrt{\frac{b}{a}}\right] - 90^\circ$$

Main use: increase phase margin. $C_{lead}(s) = k\,\frac{b}{a}\frac{s+a}{s+b}$ optional to adjust ω_c

i. Pick $\sqrt{ab}$ at desired ω_c

ii. Pick $\frac{b}{a}$ depending on desired phase increase.

iii. Adjust k to put ω_c at desired freq.

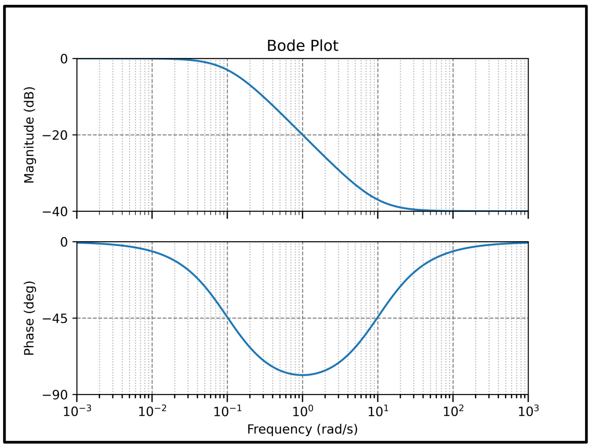
Side effects: increase magnitude at high freq.

Lag compensator:

A lag compensator is also a pole zero pair. We can write it in a general form as:

$$C_{lag}(s) = \frac{\frac{s}{a}+1}{\frac{s}{b}+1} = \frac{b}{a}\frac{s+a}{s+b} \qquad 0 < b < a$$

In a lag compensator the pole always comes before the zero. Graphically it can look like this:



Example for $a = 10$, $b = 0.1$, i.e. $C_{lag}(s) = \frac{0.1s+1}{10s+1} = \frac{1}{100}\frac{s+10}{s+0.1}$

The main effects are:

→ Decrease magnitude at high freq. by $\frac{b}{a}$, while low freq. are unafected.

→ Decrease slope of magnitude between a and b by $20\,{}^{dB}/_{dec}$

→ Decrease phase around $\sqrt{ab}$ by up to $90°$. The max phase increase is of

$$\varphi_{max} = 2\arctan\left[\sqrt{\frac{b}{a}}\right] - 90°$$

Main use: improve command tracking / disturbance rejection. $C_{lag}(s) = k\,\frac{\frac{s}{a}+1}{\frac{s}{b}+1}$

i. Pick $\frac{a}{b}$ as the desired increase in magnitude of low freq.

ii. Multiply k by $\frac{a}{b}$ (high freq. not affected)

iii. Pick a to be sufficiently small not to affect ω_c

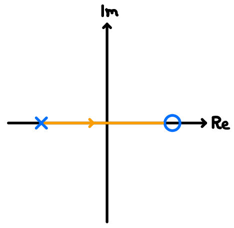
Side effects: reduction of phase margin.

Limitations:

The tools from above work well for stable, minimum phase systems. But what if we have poles and zeros in the RHP?

Loop shaping for non-minimum-phase systems:

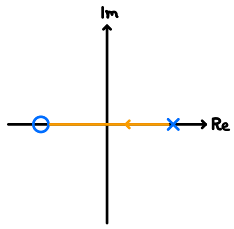
Remember from the root locus that closed loop poles approach open loop zeros. If one of the zeros is non-minimum phase, the closed loop system might become unstable.



This means for large enough gains the closed loop will become unstable. A n.m.p. zero also limits the maximum crossover freq. The consequence is that the CL system becomes slow.

Loop shaping for open-loop unstable systems:

We can look at the root locus again to see what happens when we have a CL pole in the RHP.



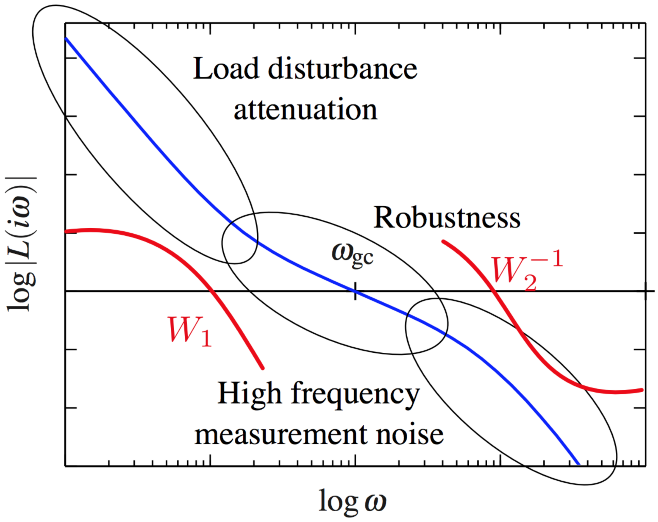
We need a high gain to stabalize the system. In real life a high gain means fast and strong actuators. In this case the crossover freq. becomes large.

This should give you a sense of how, for many systems, there are clear performance limitations. Certain requirements can thus never be satisfied. E.g. have good noise and disturbance rejection whilst also having good command tracking.

10.1 Example Problems

10.1.1 HS 2018, Question 33

Problem: Consider the Bode plot "obstacle course" below.



Question *Mark all correct statements. (2 Points)*
Mark all correct statements:

A Lead elements decrease the magnitude at high frequencies and lag elements increase the magnitude at high frequencies.

B Lead elements are most useful to increase the phase in a certain frequency range.

C A combination of lead/lag elements can re- move the steady-state error to unit steps for any plant $P(s)$.

D It is good practice to start the loop shap- ing design procedure from low frequencies.

10.1.2 HS 2022, Question 37

Problem: Consider a lead compensator

$$C_{\text{lead}}(s) = \frac{\frac{s}{a}+1}{\frac{s}{b}+1}, \qquad 0 < a < b,$$

used in a standatrd feedback architecture to control a given plant P.

Question *Mark the correct answer for each statement. (0.5 Points)*

Statement	True	False
A lead compensator is typically used to reduce the phase margin.		
A lead compensator can increase the sensitivity to high-frequency noise.		
The slope of the magnitude at frequencies between a and b is approximately +20 dB/decade.		

10.1.3 HS 2022, Question 39

Problem: Consider a lag compensator

$$C_{\text{lead}}(s) = \frac{a}{b} \cdot \frac{\frac{s}{a}+1}{\frac{s}{b}+1}, \qquad 0 < b < a,$$

used in a standatrd feedback architecture to control a given plant P.

Question *Mark the correct answer for each statement. (0.5 Points)*

Statement	True	False
A lag compensator is frquently used to improve command tracking/distrubance rejection.		
A lag compensator cannot cause the system to become unstable		
The slope of the magnitude at frequencies between a and b is approximately −20 dB/decade.		

11 Time Delays

After choosing our controler $C(s)$, we have to implement it. Usually we use computers for this task. Unfortunatley, computers have a finite compute time, which means that the control input to a certain error has some delay. Some physical systems themselves also have delays. An extreme example would be communication with e.g. a mars rover.

In this example the delay can be several minutes. How can we take this into account?

Mathematically we can express a time delay as:

$$y(t) = u(t-T)$$

The time delay is a linear operator that transforms an input $u(t)$ into a delayed output $y(t)$. T is the ammount of delay.

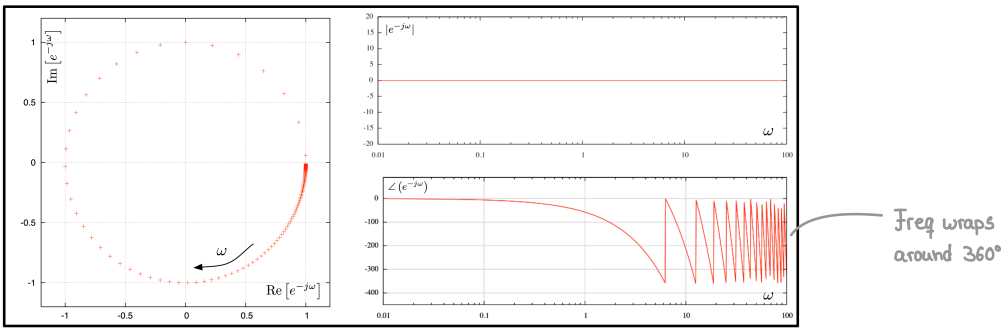
The TF of a time delay is thus given by: e^{-sT}

This not a rational function! We have no poles or zeros! Root locus doesn't work to asses closed loop behavior.

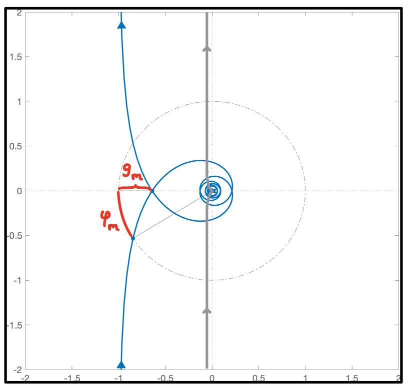
Let's take a closer look at frequency response of a time delay. Remember that for the frequency response we plug in $s = j\omega$ and look at the resulting phase and magnitude. Assume $T = 1$

$$|G(j\omega T)| = |e^{-j\omega T}| = 1$$

$$\angle G(j\omega T) = \angle e^{-j\omega T} = -\omega T$$



Thus we can summarize the effect of a time delay as a phase shift of $-\omega T$.



<u>Gain and phase margins:</u>

To get the gain margin we look at the first time we cross $-180°$, i.e.: $\angle G(j\omega) - \omega T = -180°$

Similarly we can look at the effect on the phase margin and see that: $\varphi_m = \varphi_{m,0} - \omega_c T$

Padé approximation

In the Padé approximation we represent the time delay as a ratio of two polynomials. A first order approximation would look like this:

$$e^{-sT} \approx k\,\frac{s+p}{s+q}$$

To get the values of the coefficients we can compare this term to the taylor series.

$$k\,\frac{s+p}{s+q} = 1 - sT + \frac{1}{2}(sT)^2 - \frac{1}{6}(sT)^3 \ldots$$

And considering therms upto order 2 we get:

$$e^{-sT} \approx \frac{\frac{2}{T} - s}{\frac{2}{T} + s}$$

11.1 Example Problems

11.1.1 FS 2018, Question 34

Question *Choose the correct answer. (1 Point)*
The transfer function of a time delay is...

- A linear and rational.
- B nonlinear and rational
- C nonlinear and not rational
- D linear and not rational

11.1.2 HS 2017, Question 44

Problem: Let $P(s) = \frac{e^{-sT}}{(s+0.1)^2}$.

You are given the frequency $\omega^* = 1$ rad/s at which $\angle P(j\omega^*) = -180°$.

Question *Choose the correct answer. (1 Point)*
What is the time delay T of the system?

- A $T \approx 0.2$ s
- B $T \approx 2.9$ s
- C $T \approx 1.7$ s
- D $T \approx 0.1$ s

11.1.3 HS 2017, Question 45

Problem: You are given the Bode plot of a plant however the time delay has not been included in the model. You have a time delay of $T_{\mathrm{d}} = \frac{\pi}{40}$ s.

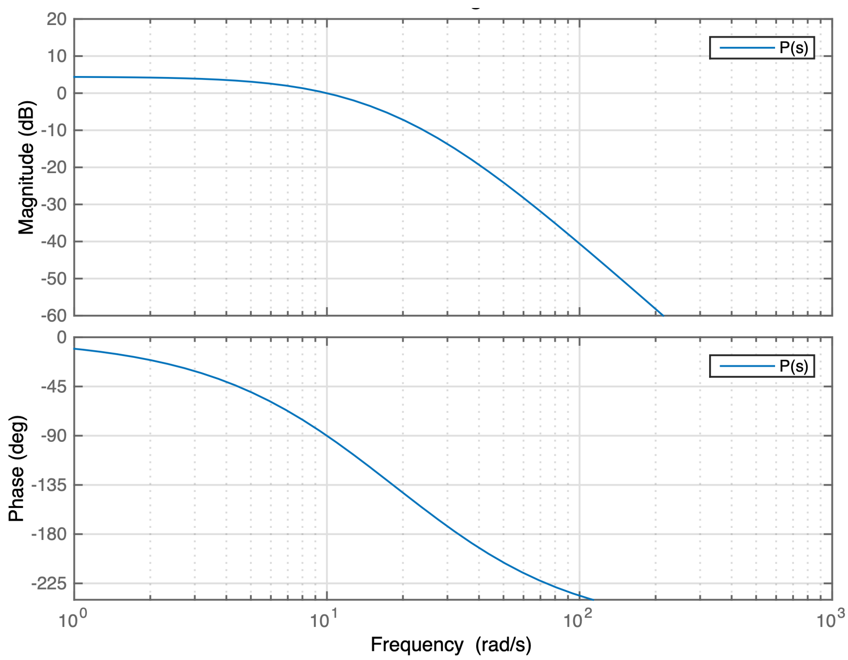


Figure 4: Bode plot of a plant with a time delay.

Question *Choose the correct answer. (1 Point)*
What is the new phase margin of your system?

A $\varphi_{\text{new}} \approx 45°$

B $\varphi_{\text{new}} \approx 30°$

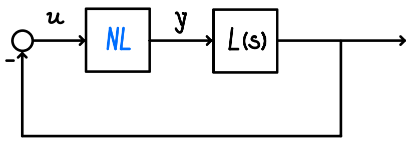
C $\varphi_{\text{new}} \approx 60°$

D $\varphi_{\text{new}} \approx 90°$

12 Nonlinearities and Describing Functions

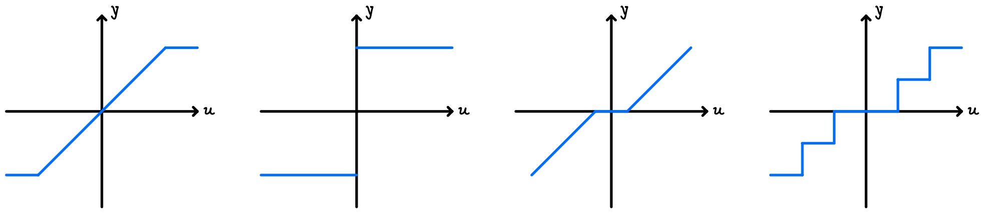
In this section we want to start to understand nonlinear systems. Today we will briefly introduce some concepts that can help us.

We will manliy consider systems like this:



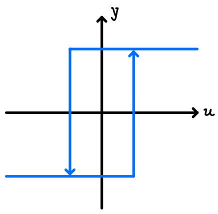
where $L(s)$ is still a linear function, and NL is a non-linear gain. NL can represent some important nonlinearity like:

Static, memoryless:



Saturation | Switch, or Relay | Deadzone | Quantizer

Dynamic, with memory:



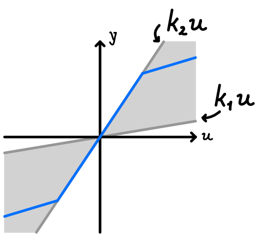
Schmitt Trigger

We can also represent them mathematically. The saturation, e.g.:

$$y = \begin{cases} 1 & \text{if } u \geq 1; \\ u & \text{if } -1 < u < 1; \\ -1 & \text{if } u \geq -1. \end{cases}$$

Stability:

For this we will consider the general case from above where where $L(s)$ is still a linear function, and NL is a non-linear gain. We can now define some boundaries that contain all possible values of NL.



Mathematically:

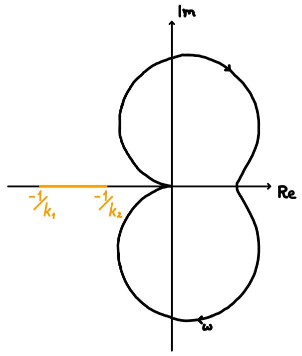
$$k_1 u \leq NL(u) \leq k_2 u$$

$$NL(0) = 0$$

We can check the CL stability of the system with the NL in the Nyquist plot.

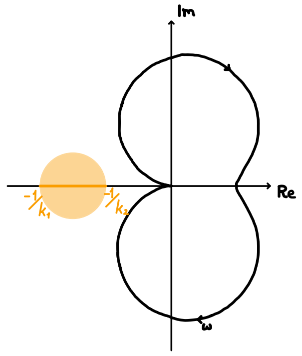
Necessary condition:

The way we defined the NL, it can include all linear gains $k_1 \leq k \leq k_2$. For the Nyquist critereon this means that we have to consider not only the point $-1/k$ but all points in $[-1/k_1, -1/k_2]$. Graphically you have to consider



Sufficient condition:

We go even further and extend the Nyquist condition to count encirclements of a circle with diameter $[-1/k_1, -1/k_2]$.



Describing Functions:

As with the frequency response for linear systems, we will consider what happens when we apply

$$u(t) = A \sin(\omega t).$$

The output will now be of the form

$$y(t) = f(A \sin(\omega t)).$$

That means some periodic function with the same frequency as the input. The output is now also dependend on the input amplitude A.

Since $y(t) = f(A\sin(\omega t))$ is a periodic function we can write the corresponding Fourier series expansion

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)],$$

with the coefficents a_n and b_n given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(t) \cos(n\omega t)\, d(\omega t) \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(t) \sin(n\omega t)\, d(\omega t)$$

for odd functions $a_n = 0 \ \forall n \in \mathbb{N}$. We can use this and approximate the output with the first harmonic, i.e. only consider $n=1$.

For an odd NL this would mean

$$y(t) \approx b_1 \sin(\omega t).$$

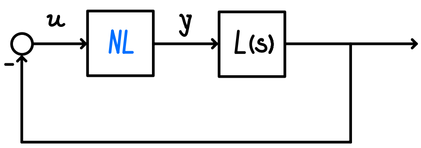
This approximation works since physical systems usually attenuate high frequencies, acting as a low pass filter. The value of b_1 is a function of A, the input amplitude. We now define the describing function to be

$$N(A) = \frac{b_1(A)}{A} = \frac{1}{\pi A}\int_{-\pi}^{\pi} y(t)\sin(\omega t)\,d(\omega t),$$

the ratio $\frac{b_1}{A}$. We can use this function to approximate NL as an amplitude-dependend gain. The rusulting describing function can also be complex, if the NL introduces a phase shift.

How is this useful?

Consider again the starting point:

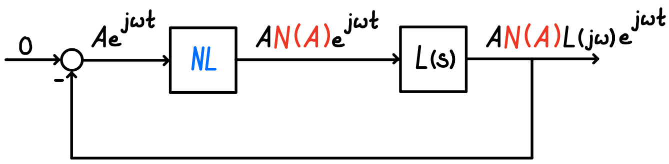


We can now say for some $u(t) = A\sin(\omega t)$ we can approximate $y(t) \approx b_1\sin(\omega t)$. Since $N(A) = \frac{b_1}{A}$ we can also say

$$y(t) \approx A\,N(A)\sin(\omega t)$$

Limit Cycles:

Consider the case of the input to your NL to be of the form $Ae^{j\omega t}$. Let's see how this signal propagates.



we can observe that if

$$A = -A\,N(A)L(j\omega),$$

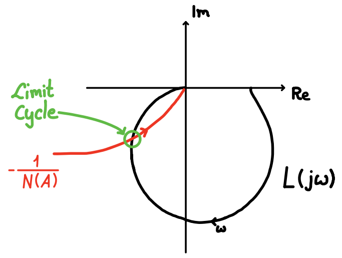
then the feedback loop is self-sustaining. The input $Ae^{j\omega t}$ produces an output of $-Ae^{j\omega t}$ which is negativley fed-back. This causes the system to oscilate indefinetley, causing a so called limit cycle. A condition for the exsistance of limit cycles can be derived from above:

$$-\frac{1}{N(A)} = L(j\omega)$$

Checking for limit cycles:

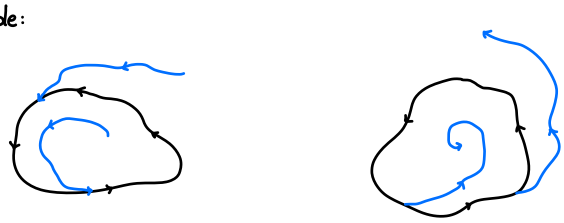
There is a nice way to verify graphically wether there are limit cycles present:

- Sketch polar plot of $L(j\omega)$
- Sketch polar plot of $-\frac{1}{N(A)}$
- Check for intersections.



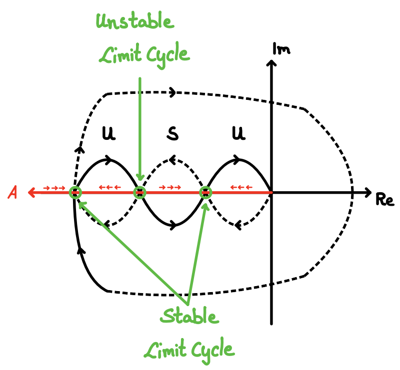
Stability of limit cycles:

Limit cycles can be stable or unstable:



How does this look on the Nyquist plot? (Assume OL stability).

If a point $-\frac{1}{N(A)}$ is in an unstable part of the Nyquist plot (U), the amplitude of oscilations will increase. If a point $-\frac{1}{N(A)}$ is in a stable part of the Nyquist plot (S), the amplitude of oscilations will decrease.



12.1 Example Problems

12.1.1 HS 2023, Questions 40 and 41

Problem: Consider the closed-loop system T shown in the Figure 5, where L is a linear time-invariant system and NL is a saturation non-linearity.

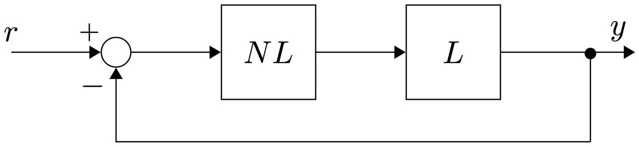


Figure 5: Standard feedback architecture. System L and saturation non-linearity denoted by NL.

The describing function $N(A)$ of the saturation non-linearity is given by,

$$N(A) = \frac{2}{\pi}\left[\arcsin\left(\frac{1}{A}\right) + \frac{1}{A}\sqrt{1-\left(\frac{1}{A}\right)^2}\right].$$

It is known that $L(s)$ has a pole at the origin, i.e. at $s = 0$, and that $L(s)$ has one unstable pole. Figure 6 shows the Nyquist plot of $L(s)$ (solid line) together with $\frac{-1}{N(A)}$ (dashed line). Further, assume that all assumptions required for a describing function analysis are met.

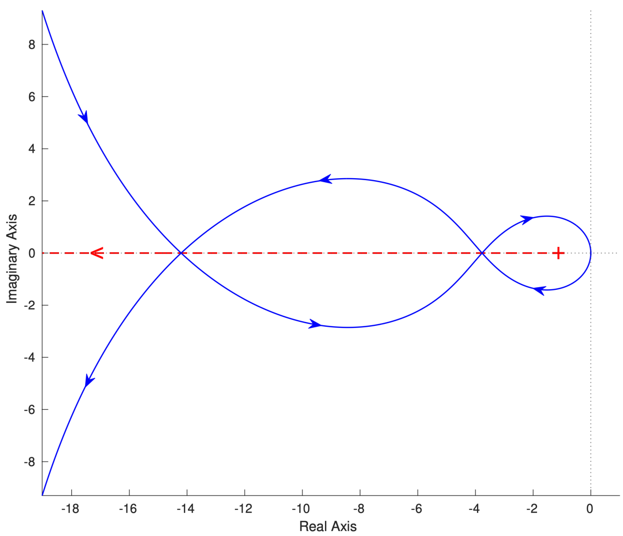


Figure 6: [Solid line] Nyquist plot of $L(s)$. [Dashed line] Plot of $\frac{-1}{N(A)}$, where the arrow indicated the direction of $\frac{-1}{N(A)}$ as A increases.

Question *Mark the correct answer. (0.5 Points)*
First, without doing any stability analysis, what is the maximum number M of potential limit cycles that the closed-loop system T could support?

- A $M = 0$
- B $M = 2$
- C $M \to \infty$
- D $M = 1$

Question *Mark the correct answer. (1 Point)*
What is the number M_s of stable limit cycles that exsist in the closed-loop system T?

- A $M_s = 0$
- B $M_s = 2$
- C $M_s \to \infty$
- D $M_s = 1$