MAD exercise session 6

Numerical Integration

Pascal Auf der Maur

Motivation

- Calculating integrals numerically is very important in Engineering
  - Domains of integrals can be very difficult. For example a part when calculating its center of mass.
  - Some functions can not be integrated by hand

Discretization

- Numerical integration always depends on discretizing the domain.
- The integral becomes a sum similar to Riemmanns sums.
- These intervals are integrated seperately
- Different approximations for these intervals exist. These approximations are called quadratures.

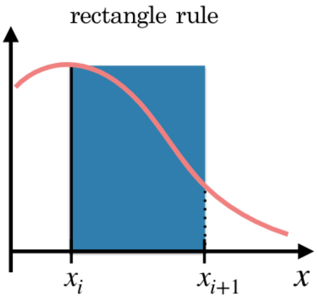
Rectangle Rule

$$I_{R_i} = f(x_i)\Delta_i$$

- Approximates the interval with a rectangle.
- Identical to Riemanns sum, there exists a right and left rectangle rule.
- The method is first order accurate $E_R \approx \mathcal{O}(x_{i+1} - x_1) = \mathcal{O}(h)$
  - Can be derived with Taylor expansion.
  - These derivations are popular questions for the exam

Rectangle Rule

$$I_{R_i} = f(x_i)\Delta_i$$



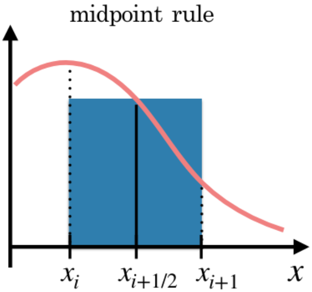
Midpoint Rule

$$I_{M_i} = f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta_i$$

- Evaluates the functionvalue in the middle of the interval
- The method is third order accurate $E_M \approx \mathcal{O}(h^3)$

Midpoint Rule

$$I_{M_i} = f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta_i$$



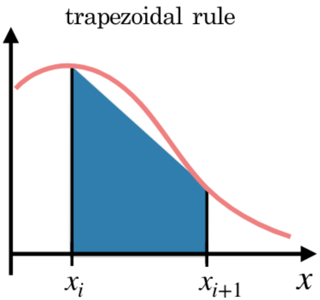
Trapezoidal Rule

$$I_{T_i} = \frac{f(x_i) + f(x_{i+1})}{2} \Delta_i$$

- Mean between left and right limit of interval
- The method is third order accurate $E_M \approx \mathcal{O}(h^3)$

Trapezoidal Rule

$$I_{T_i} = \frac{f(x_i) + f(x_{i+1})}{2}\Delta_i$$



Higher order approximations

- We can perform even better with higher order quadratures.
- These quadratures can be found when we enforce to integrate a higher order polynomial exactly
- For example fitting a parabola and integrating exactly with three evaluated points yields the Simpson’s rule

Higher order approximations

- Constructing these polynomials is done through Lagrange polynomials.
  $p_i = \sum_{k=0}^{M} f(x_k) l_k^M(x)$
- Therefore the quadrature becomes: $I_i \approx \sum_{k=0}^{M} C_k^M f(x_k)$ where $C_k^M = \frac{1}{\Delta_i} \int_{x_i}^{x_{i+1}} l_k^M(x)\,\mathrm{d}x$
- Tips to calculate C_k^M
  - $\sum_{k=0}^{M} C_k^M = 1$
  - If the evaluation points are symmetric: $C_k^M = C_{M-k}^M$

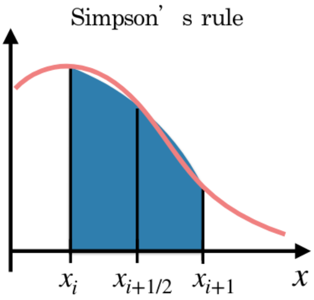
Simpson's Rule

$$I_{S_i} = \frac{f(x_i) + 4f(x_{i+1}) + f(x_{i+2})}{3}\Delta_i$$

- The method is fifth order accurate $E_M \approx \mathcal{O}(h^5)$
- Δ_i refers here to the distance between two datapoints and not the whole interval

Simpson’s Rule

$$I_{S_i} = \frac{f(x_i) + 4f(x_{i+1}) + f(x_{i+2})}{3}\Delta_i$$



Composite rules

- By inspecting the weights of the function evaluations we can formulate a sum for the integral for evenly spaced intervals:
- Rectangle rule: $I \approx \Delta_x \sum_{i=0}^{N-1} f(x_i)$
- Midpoint rule: $I \approx \Delta_x \sum_{i=0}^{N-1} f\left(\frac{x_i + x_{i+1}}{2}\right)$
- Trapezoidal rule: $I \approx \frac{\Delta_x}{2}\left(f(x_0) + 2\sum_{i=1}^{N-1} f(x_i) + f(x_N)\right)$
- Simpson's rule: $I \approx \frac{\Delta_x}{2}\left(f(x_0) + 4\sum_{\substack{i=1\\ i=odd}}^{N-1} f(x_i) + 2\sum_{\substack{i=2\\ i=even}}^{N-2} f(x_i) + f(x_N)\right)$