MAD exercise session 7

Richardson Extrapolation

Rhomberg Integration

Pascal Auf der Maur

Richardson Extrapolation

- Richardson Extrapolation aims to linearly combine lower order approximations to achieve an higher order approximation.
- This approximation can be computed easier than using high order quadratures.

Richardson Extrapolation

- Assume we calculate an approximation of G with a grid-spacing of h. Therefore $G \approx G(h)$
- We can express our approximation with an Taylor expansion where c_i corresponds with the typical constants:

$$G(h) = G(0) + c_1 h + c_2 h^2 + ...$$

- The term $G(0)$ corresponds with the exact value of the quantity

Richardson Extrapolation

- By combining this approximation with another one with half of the intervalsize the order of the error can be reduced.

$$G(h/2) = G(0) + \frac{1}{2}c_1 h + \frac{1}{4}c_2 h^2 + \ldots$$

$$G_1(h) = 2G(h/2) + G(h) = G(0) + c_2' h^2 + c_3' h^3 + \ldots$$

- This procedure can be repeated:

$$G_2(h) = \frac{1}{3}(4G_1(h/2) - G_1(h)) = G(0) + \mathcal{O}(h^3)$$

Richardson Extrapolation

- This results in an recursive formulation:

$$G_n(h) = \frac{1}{2^n - 1}\left(2^n G_{n-1}(h/2) - G_{n-1}(h)\right) = G(0) + \mathcal{O}(h^{n+1})$$

- To predict if the approximation is sufficient, we can approximate the error with:

$$\epsilon(h/2) \approx G(h/2) - G(h)$$

- If the error is not small enough, then the user should keep on subdividing.

Romberg Integration

- Romberg integration is Richardson extrapolation applied to numerical integration.
- We can apply the quadrature to a single interval I_0^1 and then subdivide it and calculate it for two, four, eight, ... intervals (I_0^2, I_0^4, I_0^8, ...).

Romberg Integration

- From the previous exercise session we know that the trapezoidal rule over a whole domain scales with a second order error.

$$I = \int_a^b f(x)\,\mathrm{d}x = \underbrace{\frac{h}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{n-1} f_j\right]}_{I_0^n} + c_1 h^2 + c_2 h^4 + \ldots$$

- Now we apply Richardson extrapolation to our evaluated integrals:

$$I_1^n = \frac{4\,I_0^{2n} - I_0^n}{3} = I + \frac{1}{4} c_2 h^4 + \ldots$$

Romberg Integration

- This results in the following recursive formula:

$$I_k^n = \frac{4^k I_{k-1}^{2n} - I_{k-1}^n}{4^k - 1}$$

- **Attention:** Due to symmetry the trapezoidal rule doesn't have a third order error but goes directly to fourth order. This has to be considered when applying the formula multiple times.