HOLOMORPHIC ANOMALY EQUATION FOR $(\mathbb{P}^2, E)$ AND THE NEKRASOV-SHATASHVILI LIMIT OF LOCAL $\mathbb{P}^2$

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ABSTRACT. We prove a higher genus version of the genus 0 local-relative correspondence of van Garrel-Graber-Ruddat: for (X, D) a pair with X a smooth projective variety and D a smooth divisor, maximal contact Gromov-Witten theory of (X, D) is related to Gromov-Witten theory of the total space of $\mathcal{O}_X(-D)$ and local Gromov-Witten theory of D.

Specializing to $(X, D) = (S, E)$ for S a del Pezzo surface or a rational elliptic surface and E a smooth anticanonical divisor, we show that maximal contact Gromov-Witten theory of (S, E) is determined by the Gromov-Witten theory of the Calabi-Yau 3-fold $\mathcal{O}_S(-E)$ and the stationary Gromov-Witten theory of the elliptic curve E.

Specializing further to $S = \mathbb{P}^2$, we prove that higher genus generating series of maximal contact Gromov-Witten invariants of $(\mathbb{P}^2, E)$ are quasimodular and satisfy a holomorphic anomaly equation. The proof combines the quasimodularity results and the holomorphic anomay equations previously known for local $\mathbb{P}^2$ and the elliptic curve.

Futhermore, using the connection between maximal contact Gromov-Witten invariants of $(\mathbb{P}^2, E)$ and Betti numbers of moduli spaces of semistable one-dimensional sheaves on $\mathbb{P}^2$, we obtain a proof of the quasimodularity and holomorphic anomaly equation predicted in the physics literature for the refined topological string free energy of local $\mathbb{P}^2$ in the Nekrasov-Shatashvili limit.

CONTENTS

1. Introduction

1.1. **Higher genus local-relative correspondence.** Let X be a smooth projective complex variety and D a smooth effective divisor on X. We assume that D is nef, that is $C \cdot D \geq 0$ for every curve C on X. The main topic of the present paper is the comparison of the relative Gromov-Witten theory of the pair (X, D) and of the local Gromov-Witten theory of the the total space $\mathrm{Tot}(\mathcal{O}_X(-D))$ of the line bundle $\mathcal{O}_X(-D)$.

Let β be a curve class on X such that $\beta \cdot D > 0$. We denote by $\overline{\mathcal{M}}_g(\mathcal{O}_X(-D), \beta)$ the moduli space of genus g stable maps of class β to the total space of $\mathcal{O}_X(-D)$, and by $\overline{\mathcal{M}}_g(X/D, \beta)$ the moduli space of genus g relative stable maps of class β to (X, D) with only one contact condition of maximal tangency along D.

As D is nef and $\beta \cdot D > 0$, $\overline{\mathcal{M}}_g(\mathcal{O}_X(-D), \beta)$ coincides with the moduli space $\overline{\mathcal{M}}_g(X, \beta)$ of genus g stable maps of class β to X. On the other hand, there is a natural morphism $F \colon \overline{\mathcal{M}}_g(X/D, \beta) \to \overline{\mathcal{M}}_g(X, \beta)$ obtained by forgetting the relative marking and stabilizing. Therefore, it makes sense to try to compare the virtual fundamental classes $[\overline{\mathcal{M}}_g(\mathcal{O}_X(-D), \beta)]^{\mathrm{vir}}$ and $F_*[\overline{\mathcal{M}}_g(X/D, \beta)]^{\mathrm{vir}}$ both living on $\overline{\mathcal{M}}_g(X, \beta)$.

In genus 0, van Garrel, Graber and Ruddat [vGR19] proved that

$$[\overline{\mathcal{M}}_0(\mathcal{O}_X(-D),\beta)]^{\mathrm{vir}} = \frac{(-1)^{\beta\cdot D-1}}{\beta\cdot D} F_*[\overline{\mathcal{M}}_0(X/D,\beta)]^{\mathrm{vir}}\,.$$

Our first main result is a generalization of this formula in arbitrary genus.

thm_locrel_intro

Theorem 1.1 (=Theorem 2.5)**.** *For every $g \geq 0$, we have*

$$[\overline{\mathcal{M}}_g(\mathcal{O}_X(-D),\beta)]^{\mathrm{vir}} = \frac{(-1)^{\beta\cdot D-1}}{\beta\cdot D} F_*\left((-1)^g\lambda_g \cap [\overline{\mathcal{M}}_g(X/D,\beta)]^{\mathrm{vir}}\right) + \sum_{\mathcal{G}\in G_{g,\beta}} \frac{1}{|\operatorname{Aut}(\mathcal{G})|}(\tau_{\mathcal{G}})_*\left(C_{\mathcal{G}}\cap[\overline{\mathcal{M}}_{\mathcal{G}}]^{\mathrm{vir}}\right).$$

The right-hand side of the theorem 1.1 is the sum of a leading term and corrections terms. The leading term is obtained by capping $[\overline{\mathcal{M}}_g(X/D,\beta)]^{\mathrm{vir}}$ with the top Chern class λ_g of the Hodge bundle. The corrections terms are explicitly described in Section 2.2 in terms of the classes $(-1)^{g'}\lambda_{g'} \cap [\overline{\mathcal{M}}_{g'}(X/D,\beta')]^{\mathrm{vir}}$ with $g' < g$, $\beta' \leq \beta$ (see Remark 2.6) and the Gromov-Witten theory of the rank 2 vector bundle $\mathcal{O}_D(D)\oplus\mathcal{O}_D(-D)$ over D.

The genus 0 result of [vGR19] is proved by an application of the degeneration formula in Gromov-Witten theory. We prove Theorem 1.1 using the same strategy. The main novelty for $g > 0$ is that the degeneration formula contains new terms which are not present for $g = 0$ and come from the bubble geometry $\mathbb{P}(\mathcal{O}_D \oplus \mathcal{O}_D(D))$. We compute these correction terms using the relative virtual localization formula in Gromov-Witten theory applied to the scaling action of $\mathbb{C}^*$ on the fibers of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_D\oplus\mathcal{O}_D(D))$.

sec:log_K3_intro

1.2. **The case of log Calabi-Yau surfaces with smooth boundary.** We specialize the higher genus local-relative correspondence given by Theorem 1.1 to $(X,D) = (S,E)$, where S is a smooth projective surface over $\mathbb{C}$ and E is a smooth effective anticanonical divisor on S which is nef. By the adjunction formula, E is a genus 1 curve. Examples of such surface S include del Pezzo surfaces and rational elliptic surfaces.

The local geometry $\mathrm{Tot}(\mathcal{O}_S(-E))$ is the total space of the canonical line bundle $\mathcal{O}_S(-E) = K_S$ and is a non-compact Calabi-Yau 3-fold. Let β be a curve class on S such that $\beta\cdot E > 0$. The moduli space $\overline{\mathcal{M}}_g(\mathcal{O}_S(-E),\beta)$ has virtual dimension 0 and we define Gromov-Witten invariants

$$N_{g,\beta}^{K_S} := \int_{[\overline{\mathcal{M}}_g(\mathcal{O}_S(-E),\beta)]^{\mathrm{vir}}} 1\,.$$

The moduli space $\overline{\mathcal{M}}_g(S/E,\beta)$ has virtual dimension g and we define maximal contact relative Gromov-Witten invariants

$$N_{g,\beta}^{S/E} := \int_{[\overline{\mathcal{M}}_g(S/E,d)]^{\mathrm{vir}}} (-1)^g\lambda_g\,.$$

The stationary Gromov-Witten invariants of the elliptic curve E are defined as

$$\langle \omega\psi_1^{a_1}, \dots, \omega\psi_n^{a_n} \rangle^E_{g,n,d} := \int_{[\overline{\mathcal{M}}_{g,n}(E,d)]^{\mathrm{vir}}} \prod_{j=1}^n \psi_j^{a_j} \operatorname{ev}_j^*(\omega) \qquad (1) \quad \boxed{\texttt{eq:gw\_E}}$$

for every $\mathbf{a} = (a_1, \cdots, a_n)$ and $g, d \in \mathbb{Z}_{\geq 0}$, where $\overline{\mathcal{M}}_{g,n}(E,d)$ is the moduli space of n pointed genus g degree d stable maps to E and $\omega \in H^2(E)$ is the (Poincaré dual) class of a point.

We show that the local invariants $N_{g,\beta}^{K_S}$ are explicitly determined by the relative invariants $N_{g,\beta}^{S/E}$ and the stationary theory of E. Moreover, this relation can be inverted: the relative invariants $N_{g,\beta}^{S/E}$ are explicitly determined by the local invariants $N_{g,\beta}^{K_S}$ and the stationary Gromov-Witten theory of E. Okounkov and Pandharipande [OP06a] have completely solved the stationary Gromov-Witten theory of E. Therefore, we establish the complete equivalence of the local theory of K_S and of the relative theory of (S, E) with maximal contact and λ_g insertion.

In order to write down the formula, we introduce some notation. For every effective $\beta \in H_2(S, \mathbb{Z})$, we denote by Q^β the corresponding monomial in the algebra of the monoid of effective curve classes. In particular, we denote by Q^E the monomial Q^β for β the class of E. We form the generating series

$$F_g^{K_S} := \delta_{g,0} F^{K_S}_{\text{classical}} + \delta_{g,1} F^{K_S}_{\text{unstable}} + \sum_{\beta,\, \beta\cdot E>0} N_{g,\beta}^{K_S} Q^\beta\,, \qquad (2) \quad \boxed{\texttt{eq:F\_K\_S}}$$

$$F_g^{S/E} := \delta_{g,0} F^{K_S}_{\text{classical}} + \delta_{g,1} F^{S/E}_{\text{unstable}} + \sum_{\beta,\, \beta\cdot E>0} \frac{(-1)^{\beta\cdot E+g-1}}{\beta\cdot E} N_{g,\beta}^{\mathbb{P}^2/E} Q^\beta\,, \quad (3) \quad \boxed{\texttt{eq:F\_SE}}$$

where

$$F^{K_S}_{\text{classical}} := -\frac{\delta_{(E\cdot E),0}}{3!(E\cdot E)^2} (\log Q^E)^3,$$

$$F^{K_S}_{\text{unstable}} := \left(\frac{\delta_{(E\cdot E),0}}{(E\cdot E)} \frac{\chi(S)}{24} - \frac{1}{24} \right) \log Q^E,$$

$$F^{S/E}_{\text{unstable}} := -\frac{\delta_{(E\cdot E),0}}{(E\cdot E)} \frac{\chi(S)}{24} \log Q^E.$$

Here $\chi(S)$ is the Euler characteristic of S and we adopt the convention that $\delta_{(E,E),0}/(E\cdot E)^2 = 0$ and $\delta_{(E\cdot E),0}/(E\cdot E) = 0$ if $E\cdot E = 0$.

According to the genus 0 result of [vGR19], we have

$$N_{0,\beta}^{S/E} = (-1)^{\beta\cdot E-1} (\beta\cdot E) N_{0,\beta}^{K_S}\,,$$

that is

$$F_0^{S/E} = F_0^{K_S}\,.$$

For every $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{Z}^n_{\geq 0}$, we consider the generating series

$$F^E_{g,\mathbf{a}} := \delta_{g,1}\delta_{n,0} F^E_{\text{unstable}} + \sum_{d\geq 0} \tilde{\mathcal{Q}}^d \, \langle \omega\psi_1^{a_1}, \dots, \omega\psi_n^{a_n} \rangle^E_{g,n,d} \qquad (4) \quad \boxed{\texttt{eq:F\_E}}$$

of stationary Gromov-Witten invariants of E, where

$$F^E_{\text{unstable}} = -\frac{1}{24}\log\left((-1)^{E\cdot E}\tilde{\mathcal{Q}}\right).$$

We view the series $F^E_{g,\mathbf{a}}$ as a function of the variables Q^β through the change of variables

$$\tilde{\mathcal{Q}} =(-1)^{E\cdot E}Q^E \exp\left(\sum_{\beta,\,\beta\cdot E>0}(-1)^{\beta\cdot E}(\beta\cdot E)N^{S/E}_{0,\beta}Q^\beta\right) \quad (5)$$
eq:cQ

$$=(-1)^{E\cdot E}Q^E \exp\left(-\sum_{\beta,\,\beta\cdot E>0}(\beta\cdot E)^2 N^{K_S}_{0,\beta}Q^\beta\right) \quad (6)$$
eq:cQ1

$$=(-1)^{E\cdot E}\exp\left(-D^2F_0^{K_S}\right), \quad (7)$$
eq:cQ2

where D is the differential operator defined by $DQ^\beta = (\beta\cdot E)Q^\beta$.

thm_locrel_SE

Theorem 1.2. *For every $g \geq 0$, we have*

$$F_g^{K_S} = (-1)^g F_g^{S/E} +$$
$$\sum_{n\geq 0}\sum_{\substack{g=h+g_1+\cdots+g_n,\\ \mathbf{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n_{\geq 0}\\ (a_j,g_j)\neq(0,0),\,\sum_{j=1}^n a_j=2h-2}} \frac{(-1)^{h-1}F^E_{h,\mathbf{a}}}{|\operatorname{Aut}(\mathbf{a},\mathbf{g})|}\prod_{j=1}^n(-1)^{g_j-1}D^{a_j+2}F^{S/E}_{g_j}.$$

According to Theorem 1.2, the local series $F_g^{K_S}$ is completely determined by the relative series $F_{g'}^{S/E}$ with $g' \leq g$ and the stationary theory of E. This relation is clearly invertible, that is the relative series $F_g^{S/E}$ is completely determined by the local series $F_{g'}^{K_S}$ with $g' \leq g$ and the stationary theory of E.

Theorem 1.2 is a corollary of the specialization of Theorem 1.1 to $(X, D) = (S, E)$. The non-trivial part of the proof is to express explicitly the correction terms present in Theorem 1.2 in terms of the stationary Gromov-Witten theory of E. This is done using the quantum Riemann-Roch theorem in the form given by Coates and Givental [CG07]. The relatively simple form of Theorem 1.2 relies on several algebraic identities and in particular on the presence of the classical and unstable terms in the definition of $F_g^{K_S}$, $F_g^{S/E}$ and $F^E_{g,\mathbf{a}}$.

localp2generators

1.3. Finite generation for $(\mathbb{P}^2, E)$.

Finite generation, quasimodularity and holomorphic anomaly equation for the local series $F_g^{K_{\mathbb{P}^2}}$ have been recently proven using various techniques by Lho, Pandharipande [LP18], Coates, Iritani [CI18] and Fang, Ruan, Zhang, Zhou [FRZZ19]. We show that similar properties also hold for the relative series $F_g^{\mathbb{P}^2/E}$.

Specializing the formulas (2) and (3) to $\mathbb{P}^2$, we get

$$F_g^{K_{\mathbb{P}^2}} := -\frac{\delta_{g,0}}{18}(\log Q)^3 - \frac{\delta_{g,1}}{12}\log Q + \sum_{d\geq 1} N_{g,d}^{K_{\mathbb{P}^2}} Q^d\,, \tag{8}$$

eq:F_KP2

$$F_g^{\mathbb{P}^2/E} := -\frac{\delta_{g,0}}{18}(\log Q)^3 - \frac{\delta_{g,1}}{24}\log Q + \sum_{d\geq 1} \frac{(-1)^{d+g-1}}{3d} N_{g,d}^{\mathbb{P}^2/E} Q^d\,. \tag{9}$$

eq:F_P2

The I-functions for the Gromov-Witten theory of local $\mathbb{P}^2$ is

$$I^{K_{\mathbb{P}^2}}(z,q) = \sum_{k=0}^{2} I_k(q) H^k z^{1-k} := z\, e^{\frac{H}{z}\log q} \sum_{d\geq 0} q^d \frac{\prod_{k=0}^{3d-1}(-3H-kz)}{\prod_{k=1}^{d}(H+kz)^3}, \tag{10}$$

IfunctionofKP2

where H is the hyperplane class of $\mathbb{P}^2$. In particular, we have $I_0 = 1$, $I_1 = \log q + \bar{I}_1$, $I_2 = \frac{1}{2}(\log q)^2 + \bar{I}_1 \log q + O(q)$, where

$$\bar{I}_1 = 3\sum_{k\geq 1} \frac{(3k-1)!}{(k!)^3}(-1)^k q^k\,.$$

The functions I_0, I_1 and I_2 form a basis of solutions of the linear differential equation

$$\left[\left(q\frac{d}{dq}\right)^3 + 3q\left(q\frac{d}{dq}\right)\left(3\left(q\frac{d}{dq}\right)+1\right)\left(3\left(q\frac{d}{dq}\right)+2\right)\right] I = 0\,. \tag{11}$$

mirror_ODE

The variables q and Q are related by the mirror map

$$Q = e^{I_1}\,. \tag{12}$$

eq:mirror_map

Explicitly, we have

$$Q = q - 6q^2 + 63q^3 - 866q^4 + 13899q^5 - 246366q^6 + \dots$$

and

$$q = Q + 6Q^2 + 9Q^3 + 56Q^4 - 300Q^5 + 3942Q^6 + \dots$$

In particular, we have $\frac{d}{dI_1} = Q\frac{d}{dQ}$. The genus 0 mirror theorem for $K_{\mathbb{P}^2}$ [Giv96, LLY97, CKYZ99] computes the generating series $F_0^{K_{\mathbb{P}^2}}$ in terms of I_1 and I_2:

$$-3Q\frac{dF_0^{K_{\mathbb{P}^2}}}{dQ} = I_2\,. \tag{13}$$

thm_mirror

In order to describe higher genus invariants, we introduce the functions[1]

$$X := (1+27q)^{-1}\,, \tag{14}$$
def_X

$$I_{11} := q\frac{dI_1}{dq}\,, \tag{15}$$
def_I11

$$S := q\frac{d}{dq}\left(\log I_{11} - \frac{1}{3}\log X\right) = q\frac{d}{dq}\log I_{11} - \frac{1}{3}(X-1)\,. \tag{16}$$
def_S

The functions S, X and I_{11} are algebraically independent over $\mathbb{C}$ (see Lemma 4.2). Therefore, the ring of functions generated by S and X is the polynomial ring

$$\mathbf{R} := \mathbb{Q}[X, S]\,.$$

We define a grading on $\mathbf{R}$ by $\deg X = \deg S = 1$ and denote by $\mathbf{R}_k$ the subspace of polynomials with degree no more than k.

The finite generation property for the Gromov-Witten theory of $K_{\mathbb{P}^2}$, proved in [LP18, LP19a, CI18] states that, for every $g \geq 2$, we have

$$F_g^{K_{\mathbb{P}^2}} \in [X^{-(g-1)} \cdot \mathbf{R}_{3g-3}]^{\mathrm{reg}}\,,$$

where $[-]^{\mathrm{reg}}$ (the "orbifold regularity" condition) is defined by

$$[-]^{\mathrm{reg}} := \{f(X,S) : 3\deg_X f + \deg_S f \geq 0\}. \tag{17}$$
orbifoldreg

We prove a finite generation result for the series $F_g^{\mathbb{P}^2/E}$ of relative Gromov-Witten invariants of $(\mathbb{P}^2, E)$.

fgproperty

Theorem 1.3. *For every $g \geq 2$, we have*

$$F_g^{\mathbb{P}^2/E} \in [X^{-(g-1)} \cdot \mathbf{R}_{3g-3}]^{\mathrm{reg}}\,.$$

Moreover, we have $\deg_S F_g^{\mathbb{P}^2/E} \leq 2g-3$.

The bound of the S-degree of $F_g^{\mathbb{P}^2/E}$ by $2g-3$ is specific to the relative theory. In general, the local series $F_g^{K_{\mathbb{P}^2}}$ is of S-degree $3g-3$.

For small genera, the series $F_g^{K_{\mathbb{P}^2}}$ are explicitly known. For example, we have [Hu15]

$$F_1^{K_{\mathbb{P}^2}} = -\frac{1}{12}\log q - \frac{1}{2}\log I_{11} - \frac{1}{12}\log(1+27q)$$

and [LP18]

$$F_2^{K_{\mathbb{P}^2}} = \frac{1}{X}\Big(\frac{5}{8}S^3 + \frac{X}{8}S^2 + \frac{X^2}{96}S + \frac{X^3}{4320} + \frac{X^2}{4320} - \frac{X}{2160}\Big)\,.$$

We prove similar explicit formulas for the series $F_g^{\mathbb{P}^2/E}$ in low genera.

[1]Our S and X are related to the A_2 and L of [LP18] by $A_2 = \frac{3S}{X} + \frac{1}{2}$ and $L = X^{\frac{1}{3}}$. We are defining our generators in the current way in order to have a X-degree bound $3g-3$ for the genus g generating function.

f1formula_intro

Theorem 1.4 (=Theorem 4.11). *We have*

$$F_1^{\mathbb{P}^2/E} = -\frac{1}{24}\log q + \frac{1}{24}\log(1+27q)\,.$$

f2formula

Theorem 1.5. *We have*

$$F_2^{\mathbb{P}^2/E} = \frac{X}{384}S - \frac{X^2}{360} + \frac{X}{240} - \frac{1}{720}\,. \tag{18}$$

eqnF2

The proof of Theorems 1.3, 4.11, 1.5 relies on Theorem 1.2, the corresponding properties for local $\mathbb{P}^2$, and results on the Gromov-Witten theory of the elliptic curve. In particular, the proof of the finite generation in Theorem 1.3 uses the quasimodularity properties of the stationary theory of the elliptic curve [OP06a] and a rewriting of the generators S and X in terms of quasimodular forms discussed in Section 1.4 below. The proof of the S-degree bound in Theorem 1.3 is more difficult and needs the holomorphic anomaly equation described in Section 1.5.

section_quasimod_intro

1.4. **Quasimodularity for** $(\mathbb{P}^2, E)$**.** The mirror geometry of local $\mathbb{P}^2$ is naturally related to modular forms. Indeed, the functions $I_{11} = q\frac{dI_1}{dq}$ and $I_{12} = q\frac{dI_2}{dq}$ are periods of the fibers of the universal family of elliptic curves over the modular curve $Y_1(3) \simeq \{q \in \mathbb{C} | q \neq -\frac{1}{27}, 0\} \cup \{\infty\}$[2]. In the context of mirror symmetry, where $Y_1(3)$ is viewed as the stringy Kähler moduli space of local $\mathbb{P}^2$, the point $q=0$ is the large volume point, $q=-\frac{1}{27}$ is the conifold point and $q=\infty$ is the orbifold point.

The modular curve $Y_1(3)$ is the quotient of the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} | \operatorname{Im} \tau > 0\}$ by the action of the congruence subgroup $\Gamma_1(3)$. The identification between $Y_1(3)$ and $\{q \in \mathbb{C} | q \neq -\frac{1}{27}, 0\} \cup \{\infty\}$ is given by

$$\tau = \frac{1}{2} + \frac{1}{2\pi i}\frac{I_{12}(q)}{I_{11}(q)}\,.$$

We denote $\mathcal{Q} = e^{2\pi i \tau}$. Remark that we have $\mathcal{Q}^3 = \tilde{\mathcal{Q}}$, where $\tilde{\mathcal{Q}} = -\exp(-D^2 F_0^{K_{\mathbb{P}^2}})$ was introduced in (7). Indeed, we have $D = 3Q\frac{d}{dQ} = \frac{3}{I_{11}}q\frac{d}{dq}$ and so $-D^2 F_0^{K_{\mathbb{P}^2}} = 3\frac{I_{12}}{I_{11}}$ follows from the mirror theorem (13).

We define[3]

$$A(\tau) := \left(\frac{\eta(\tau)^9}{\eta(3\tau)^3} + 27\frac{\eta(3\tau)^9}{\eta(\tau)^3}\right)^{\frac{1}{3}}, \quad B(\tau) := \frac{1}{4}\big(E_2(\tau) + 3E_2(3\tau)\big),$$

$$C(\tau) := \frac{\eta(\tau)^9}{\eta(3\tau)^3}\,,$$

where

$$\eta(\tau) := \mathcal{Q}^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-\mathcal{Q}^n),$$

[2]More precisely, it is a description of the coarse moduli space of $Y_1(3)$. As a stack, $Y_1(3)$ has a $\mathbb{Z}/3$-orbifold point at $q=\infty$.

[3]The A, B, C defined here are respectively denoted by A, E, B^3 in [ASYZ14, Zho14].

is the Dedekind eta function and

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \frac{n\mathcal{Q}^n}{1-\mathcal{Q}^n}$$

is the weight 2 Eisenstein series. The functions A, B, and C are quasimodular forms for $\Gamma_1(3)$. More precisely, A and C are modular respectively of weight 1 and 3, and B is quasimodular of weight 2 and depth 1. In fact, A, B, and C freely generate the ring of quasimodular forms of $\Gamma_1(3)$:

$$\mathrm{QMod}(\Gamma_1(3)) = \mathbb{C}[A, B, C]\,. \tag{19}$$

Going from the variable τ to the variable q, we can express the quasimodular forms A, B, C in terms of the functions X, I_{11}, S introduced in Section 1.3 ([ASYZ14, Mai09, Mai11, Zho14]):

$$A = I_{11}, \quad B = \frac{I_{11}^2}{X}(X + 6S), \quad C = \frac{I_{11}^3}{X}.$$

The space $[X^{-(g-1)}\mathbf{R}_{3g-3}]^{\mathrm{reg}}$ of polynomials in S and $X^{\pm}$ introduced in Theorem 1.3 has a very natural interpretation in terms of modular forms. Indeed, we show in Proposition 4.3 that it can be identified with the space of quasimodular forms for $\Gamma_1(3)$ of weight $6g-6$ in the following way:

$$[X^{-(g-1)}\mathbf{R}_{3g-3}]^{\mathrm{reg}} = C^{-(2g-2)} \cdot \mathbb{Q}[A, B, C]_{6g-6}\,. \tag{20}$$

eqn_idgenerators

Therefore, we can rephrase Theorem 1.3 as follows.

fgproperty1

Theorem 1.6. *For every $g \geq 2$, we have*

$$F_g^{\mathbb{P}^2/E} \in C^{-(2g-2)} \cdot \mathbb{Q}[A, B, C]_{6g-6}\,.$$

Moreover, we have $\deg_B F_g^{\mathbb{P}^2/E} \leq 2g-3$.

section_HAE_intro

1.5. Holomorphic anomaly equation for $(\mathbb{P}^2, E)$.

The holomorphic anomaly equation is a remarkable structure conjecturally underlying Gromov-Witten theory of Calabi-Yau varieties [BCOV93, BCOV94]. In the past few years, a series of works has lead to the proof of several instances of the holomorphic anomaly equation: local $\mathbb{P}^2$ [LP18, CI18], $\mathbb{C}^3/\mathbb{Z}^3$ [LP19a, CI18], local $\mathbb{P}^1 \times \mathbb{P}^1$ [Lho18, Wan19a], the formal quintic 3-fold [LP19b], the quintic 3-fold [GJR18, CGLL18], toric Calabi-Yau 3-folds (through the Eynard-Orantin topological recursion)[EMO07, FLZ16, FRZZ19], the formal elliptic curve [Wan19b], elliptic orbifold projective lines [MRS18], elliptic curves [OP18] and elliptic fibrations [OP19]. We combine two a priori distinct directions of this wave of progress, the cases of local $\mathbb{P}^2$ and of the elliptic curve, to formulate and prove a holomorphic equation for the maximal contact Gromov-Witten theory of $(\mathbb{P}^2, E)$.

Let

$$F_{g,n}^{K_{\mathbb{P}^2}} := \left(Q\frac{d}{dQ}\right)^n F_g^{K_{\mathbb{P}^2}}.$$

For $2g-2+n>0$, we have [LP18] $F_{g,n}^{K_{\mathbb{P}^2}} \in \mathbb{Q}[S, X^{\pm}, I_{11}^{-1}]$, homogeneous of degree n with respect to I_{11}^{-1}.

We have the following holomorphic anomaly equation for local $\mathbb{P}^2$, proved in various ways in [LP18], [CI18] and [EMO07, FLZ16, FRZZ19]: for $2g-2+n>0$,

$$\frac{X}{3\,{I_{11}}^2}\frac{\partial}{\partial S}F_{g,n}^{K_{\mathbb{P}^2}} = \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ n_1+n_2=n\\ 2g_i-2+n_i\geq 0}} \binom{n}{n_1} F_{g_1,n_1+1}^{K_{\mathbb{P}^2}} \cdot F_{g_2,n_2+1}^{K_{\mathbb{P}^2}} + \frac{1}{2}F_{g-1,n+2}^{K_{\mathbb{P}^2}}. \quad (21)$$

We prove a holomorphic anomaly equation for the series $F_g^{\mathbb{P}^2/E}$ of relative Gromov-Witten invariants of $(\mathbb{P}^2, E)$.

We denote

$$F_{g,n}^{\mathbb{P}^2/E} := \left(Q\frac{d}{dQ}\right)^n F_g^{\mathbb{P}^2/E}.$$

fgproperty2

Theorem 1.7. *For $2g-2+n>0$,*

$$F_{g,n}^{\mathbb{P}^2/E} \in \mathbb{Q}[S, X^{\pm}, I_{11}^{-1}],$$

homogeneous of degree n with respect to I_{11}^{-1}.

HAEforrelative

Theorem 1.8. *For $2g-2+n>0$, we have the following* holomorphic anomaly equation

$$\frac{X}{3\,{I_{11}}^2}\frac{\partial}{\partial S}F_{g,n}^{\mathbb{P}^2/E} = \frac{1}{2}\sum_{\substack{g_1+g_2=g, n_1+n_2=n\\ 2g_i-2+n_i\geq 0 \text{ for } i=1,2}} \binom{n}{n_1} F_{g_1,n_1+1}^{\mathbb{P}^2/E} \cdot F_{g_2,n_2+1}^{\mathbb{P}^2/E} \quad (22)$$ eq_HAEPE

We note that the holomorphic anomaly equation for $(\mathbb{P}^2, E)$ does not contain a loop term $F_{g-1,n+2}^{\mathbb{P}^2/E}$, unlike what happens for local $\mathbb{P}^2$.

We prove the holomorphic anomaly equation for $(\mathbb{P}^2, E)$ using Theorem 1.2, the holomorphic anomaly equation for local $\mathbb{P}^2$ and the holomorphic anomaly equation for the elliptic curve recently proved by Oberdieck and Pixton [OP18].

We remark that the holomorphic anomaly equation of Theorem 1.8 can be rewritten in terms of quasimodular forms as

$$\frac{\partial}{\partial B}F_{g,n}^{\mathbb{P}^2/E} = \frac{1}{4}\sum_{\substack{g_1+g_2=g, n_1+n_2=n\\ 2g_i-2+n_i\geq 0 \text{ for } i=1,2}} \binom{n}{n_1} F_{g_1,n_1+1}^{\mathbb{P}^2/E} \cdot F_{g_2,n_2+1}^{\mathbb{P}^2/E}. \quad (23)$$ eq_HAE_modular

section:conifold_gap

1.6. **Conifold gap conjecture.** The conifold point is the point $q = -\frac{1}{27}$, that is, the cusp of the modular curve $Y_1(3)$ defined by the $\Gamma_1(3)$-equivalence class of $\tau = 0$. Let $\tau_{\mathrm{con}} := -\frac{1}{3\tau}$ be the modular coordinate in the neighborhood of the conifold point. For every $g \geq 2$, we define $F_{g,\mathrm{con}}^{K_{\mathbb{P}^2}}$ (resp. $F_{g,\mathrm{con}}^{\mathbb{P}^2/E}$), functions of τ_{con}, by replacing in the expression of $F_g^{K_{\mathbb{P}^2}}$ (resp. $F_g^{\mathbb{P}^2/E}$) in terms of A, B, C:

(1) $A(\tau)$ by $A(\tau_{\mathrm{con}})$,

(2) $B(\tau)$ by $B(\tau_{\text{con}})$,
(3) $C(\tau)$ by $A(\tau_{\text{con}})^3 - C(\tau_{\text{con}})$.

Conceptually, $F_{g,\text{con}}^{K_{\mathbb{P}^2}}$ (resp. $F_{g,\text{con}}^{\mathbb{P}^2/E}$) is the holomorphic part near the conifold point of the almost holomorphic modular function on $Y_1(E)$ whose holomorphic part is $F_g^{K_{\mathbb{P}^2}}$ (resp. $F_g^{\mathbb{P}^2/E}$) near the large volume point $q = 0$. We refer to [ASYZ14, Zho14] for details.

Let t_{con} be the flat coordinate near the conifold point defined as the unique solution of (11) such that $t_{\text{con}} = \frac{1}{\sqrt{3}}(1+27q)+O((1+27q)^2)$ near the conifold point. Both t_{con} and $e^{2i\pi\tau_{\text{con}}}$ are local coordinates near the conifold point. In particular, we can view $F_{g,\text{con}}^{K_{\mathbb{P}^2}}$ (resp. $F_{g,\text{con}}^{\mathbb{P}^2/E}$) as functions of t_c. As C has a first order pole and A, B are regular at the conifold point (see [Zho14]), it follows from $F_g^{K_{\mathbb{P}^2}}$ (resp. $F_g^{\mathbb{P}^2/E}$) $\in C^{-(2g-2)} \cdot \mathbb{Q}[A, B, C]_{6g-6}$ that

$$F_{g,\text{con}}^{K_{\mathbb{P}^2}} = O\left(\frac{1}{t_{\text{con}}^{2g-2}}\right), \quad F_{g,\text{con}}^{K_{\mathbb{P}^2}} = O\left(\frac{1}{t_{\text{con}}^{2g-2}}\right) \tag{24}$$

near the conifold point $t_{\text{con}} = 0$.

According to the conifold gap conjecture [HK07, HKQ09, HKR08], we should have, for every $g \geq 2$,

$$F_{g,\text{con}}^{K_{\mathbb{P}^2}} = \frac{B_{2g}}{2g(2g-2)} \frac{1}{t_{\text{con}}^{2g-2}} + O(1) \tag{25}$$ eq:gap_local

near the conifold point $t_{\text{con}} = 0$, where B_{2g} are the Bernoulli numbers. This conjecture can be checked explicitly in low genus (see section 10.8 [CI18]) but is still open in general.

The holomorphic anomaly equation fixes the B-dependence of $F_g^{K_{\mathbb{P}^2}}$ and so determines $F_g^{K_{\mathbb{P}^2}}$ up to some ambiguity living in the $(2g-1)$-dimensional vector space $C^{-(2g-2)} \cdot \mathbb{Q}[A, C]_{6g-6}$. The coefficient of $1/t_{\text{con}}^{2g-2}$ and the $(2g-3)$ vanishings of the coefficients of $1/t_{\text{con}}^j$ for $1 \leq j \leq 2g-3$ predicted by (25), combined with the fact that the Q-expansion of $F_g^{K_{\mathbb{P}^2}}$ has no constant term, uniquely fix this ambiguity and so determine $F_g^{K_{\mathbb{P}^2}}$ completely.

We formulate a version of the conifold gap conjecture for the series $F_g^{\mathbb{P}^2/E}$.

conj_gap **Conjecture 1.9.** *For every $g \geq 2$, we have*

$$F_{g,\text{con}}^{\mathbb{P}^2/E} = -\frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{2g(2g-1)(2g-2)} \frac{1}{t_{\text{con}}^{2g-2}} + O(1) \tag{26}$$ eq:gap_relative

The holomorphic anomaly equation (23) fixes the B-dependence of $F_g^{\mathbb{P}^2/E}$ and so determines $F_g^{\mathbb{P}^2/E}$ up to some ambiguity living in the $(2g-1)$-dimensional vector space $C^{-(2g-2)} \cdot \mathbb{Q}[A, C]_{6g-6}$. The coefficient of $1/t_{\text{con}}^{2g-2}$ and the $(2g-3)$ vanishings of the coefficients of $1/t_{\text{con}}^j$ for $1 \leq j \leq 2g-3$ predicted by (26), combined with the fact that the Q-expansion of $F_g^{\mathbb{P}^2/E}$

has no constant term, uniquely fix this ambiguity and so determine $F_g^{\mathbb{P}^2/E}$ completely.

As reviewed in the next section, $F_{g,\mathrm{con}}^{\mathbb{P}^2/E}$ coincides with the Nekrasov-Shatashvili limit of the refined topological string on local $\mathbb{P}^2$. Therefore, Conjecture 1.9 can be viewed as a special case of the conjectural universal behavior of the refined topological string near a conifold point predicted by physics arguments in [KW11] and [HK12, section 3.2].

It would be interesting to understand how (25) and (26) are related through Theorem 1.2. We leave this question open.

1.7. **Nekrasov-Shatashvili limit of local $\mathbb{P}^2$.** The results described in Sections 1.3–1.6 concern the series $F_g^{\mathbb{P}^2/E}$ of maximal contact relative Gromov-Witten invariants of $(\mathbb{P}^2, E)$ with λ_g-insertion. However, they have exactly the form predicted in the string theory literature [HK12, HKK13] for the Nekrasov-Shatashvili limit of the refined topological string on local $\mathbb{P}^2$. We explain below that this fact is not a coincidence and was one of the main motivation for our work. In combination with [Bou19a], we obtain a proof of a mathematically precise version of these physics conjectures.

According to the Gromov–Witten/stable pairs correspondence [MNOP06], the series $\bar{F}_g^{K_{\mathbb{P}^2}}$ can be described in terms of the stable pairs Pandharipande-Thomas invariants $P_{d,n}$ of local $\mathbb{P}^2$ [PT09]. More precisely, we have

$$1+\sum_{d\geq 1}\sum_{n\in\mathbb{Z}} P_{d,n}(-x)^n Q^d = \exp\left(\sum_{g\geq 0} \bar{F}_g^{K_{\mathbb{P}^2}} u^{2g-2}\right) \tag{27}$$

eq:mnop

where $x=e^{iu}$ and $\bar{F}_g^{K_{\mathbb{P}^2}} := \sum_{d\geq 1} N_{g,d}^{K_{\mathbb{P}^2}} Q^d$.

The stable pairs invariants $P_{d,n}$ are expected to admit a refinement $P_{d,n,j}$, such that $P_{d,n} = \sum_j P_{d,n,j}$. For local $\mathbb{P}^2$, there are several approaches to the definition of $P_{d,n,j}$: cohomological/motivic [JS12, KS08] or K-theoretic [NO16, CKK14], which conjecturally coincide. The refined topological string free energies $F_{g_1,g_2}^{K_{\mathbb{P}^2},\mathrm{ref}}$ are then defined by the expansion

$$1+\sum_{d\geq 1}\sum_{n,j\in\mathbb{Z}} P_{d,n,j} y^j (-x)^n Q^d = \exp\left(\sum_{g\geq 0} \bar{F}_{g_1,g_2}^{K_{\mathbb{P}^2},\mathrm{ref}} (\epsilon_1+\epsilon_2)^{2g_1}(-\epsilon_1\epsilon_2)^{g_2-1}\right) \tag{28}$$

eq:mnop_ref

where $x=e^{i\frac{\epsilon_1-\epsilon_2}{2}}$ and $y=e^{i\frac{\epsilon_1+\epsilon_2}{2}}$. In the unrefined limit $\epsilon_1=-\epsilon_2=u$, $y=1$, (28) reduces to (27) and so

$$\bar{F}_g^{K_{\mathbb{P}^2}} = \bar{F}_{0,g}^{K_{\mathbb{P}^2},\mathrm{ref}}\,.$$

From string theory arguments, the refined series $\bar{F}_{g_1,g_2}^{K_{\mathbb{P}^2},\mathrm{ref}}$ are expected to behave in a way similar to the unrefined series $\bar{F}_g^{K_{\mathbb{P}^2}}$ and in particular should satisfy finite generation, quasimodularity and an appropriate generalization of the holomorphic anomaly equation [KW11,HK12,HKK13]. These conjectures are widely open. The main issue is to get a geometric understanding of the change of variables $x = e^{i\frac{\epsilon_1 - \epsilon_2}{2}}$ and $y = e^{i\frac{\epsilon_1+\epsilon_2}{2}}$. Even in the unrefined case, to prove the properties of the series $F_g^{K_{\mathbb{P}^2}}$ directly from the stable pairs, that is using (27) as a definition and without using the Gromov-Witten interpretation, seems challenging.

However, it is possible to make progress in the Nekrasov-Shatashvili limit $\epsilon_2 \to 0$, that is for the series $\bar{F}_{g,0}^{K_{\mathbb{P}^2},\mathrm{ref}}$, for which we can use an alternative definition. Indeed, in the same way that the genus 0 series $\bar{F}_0^{K_{\mathbb{P}^2}}$ (more precisely, the genus 0 Gopakumar-Vafa invariants $n_{0,d}^{K_{\mathbb{P}^2}}$) can be described in terms of Euler characteristic of moduli spaces of one-dimensional semistable sheaves ([Kat08],[CMT18, Appendix A]), the series $F_{g,0}^{K_{\mathbb{P}^2},\mathrm{ref}}$ are conjecturally described in terms of Betti numbers of moduli spaces of one-dimensional semistable sheaves.

For every $d > 0$ and $\chi \in \mathbb{Z}$, let $M_{d,\chi}$ be the moduli space of one-dimensional Gieseker semistable sheaves on $\mathbb{P}^2$, of degree d and Euler characteristic χ. We denote by $Ib_j(M_{d,\chi})$ the Betti numbers of the intersection cohomology of $M_{d,\chi}$. According to [Bou19b], the odd-degree part of the intersection cohomology of $M_{d,\chi}$ vanishes: $Ib_{2k+1}(M_{d,\chi}) = 0$. For every $d \in \mathbb{Z}_{>0}$ and $\chi \in \mathbb{Z}$, we define

$$\Omega_{d,\chi}^{\mathbb{P}^2}(y^{\frac{1}{2}}) := (-y^{\frac{1}{2}})^{-\dim M_{d,\chi}} \sum_{j=0}^{\dim M_{d,\chi}} Ib_{2j}(M_{d,\chi}) y^j \in \mathbb{Z}[y^{\pm\frac{1}{2}}]\,.$$

It is proved in [Bou19b] that the $\Omega_{d,\chi}^{\mathbb{P}^2}(y^{\frac{1}{2}})$ are the refined Donaldson–Thomas invariants for one-dimensional sheaves on $K_{\mathbb{P}^2}$. For $y^{\frac{1}{2}} = 1$, $\Omega_{d,\chi}^{\mathbb{P}^2}$ coincides with the genus 0 Gopakumar-Vafa invariant $n_{0,d}^{K_{\mathbb{P}^2}}$ ([Kat08],[CMT18, Appendix A]). Therefore, one should view $\Omega_{d,\chi}^{\mathbb{P}^2}(y^{\frac{1}{2}})$ as a refined genus 0 Gopakumar-Vafa invariant of local $\mathbb{P}^2$.

Tensoring by $\mathcal{O}(1)$ gives an isomorphism between $M_{d,\chi}$ and $M_{d,\chi+d}$. Thus, $\Omega_{d,\chi}(y^{\frac{1}{2}})$ can only depend on χ through $\chi \mod d$. We define

$$\Omega_d^{\mathbb{P}^2}(y^{\frac{1}{2}}) := \frac{1}{d} \sum_{\chi \mod d} \Omega_{d,\chi}(y^{\frac{1}{2}})$$

by averaging over the d possible values of $\chi \mod d$. It is conjectured in [Bou19b] that $\Omega_{d,\chi}(y^{\frac{1}{2}})$ is in fact independent of χ. Assuming this conjecture, $\Omega_d^{\mathbb{P}^2}(y^{\frac{1}{2}})$ would be the common value of the $\Omega_{d,\chi}(y^{\frac{1}{2}})$.

Define

$$\bar{F}^{NS} := i \sum_{d \in \mathbb{Z}_{>0}} \sum_{k \in \mathbb{Z}_{>0}} \frac{1}{k} \frac{\Omega_d(y^{\frac{k}{2}})}{y^{\frac{k}{2}} - y^{-\frac{k}{2}}} Q^{kd} \in \mathbb{Q}(y^{\pm\frac{1}{2}})[\![Q]\!]\,. \qquad (29)$$

FNS_def

Using the change of variables $y = e^{i\hbar}$, we define series $\bar{F}_g^{NS} \in \mathbb{Q}[\![Q]\!]$ by the expansion

$$\bar{F}^{NS} = \sum_{g \in \mathbb{Z}_{\geqslant 0}} (-1)^g \bar{F}_g^{NS} \hbar^{2g-1} \,. \tag{30}$$

FgNS_def

Conjecturally, we have $\bar{F}_g^{NS} = \bar{F}_{g,0}^{K_{\mathbb{P}^2},\mathrm{ref}}$. At this point, it is unclear how the definition of $\bar{F}^{NS}$ using one-dimensional sheaves is better than the definition of $\bar{F}_{g,0}^{K_{\mathbb{P}^2},\mathrm{ref}}$ using stable pairs: one still needs to understand geometrically the change of variables $y = e^{i\hbar}$.

It is precisely such understanding which has been obtained in [Bou19b]. More precisely, one of the main results of [Bou19b], building on [Bou18, Bou19c, Bou19a, Gab19], is the equality

$$\bar{F}_g^{NS} = \bar{F}_g^{\mathbb{P}^2/E} \,, \tag{31}$$

eq:NS

for every $g \geq 0$, where $\bar{F}_g^{\mathbb{P}^2/E} := \sum_{d \geq 1} \frac{(-1)^{d+g-1}}{3d} N_{g,d}^{\mathbb{P}^2/E} Q^d$. In other words, the series $\bar{F}_g^{NS}$ have a Gromov-Witten interpretation, not as Gromov-Witten series of local $\mathbb{P}^2$ but as Gromov-Witten series of $(\mathbb{P}^2, E)$! Therefore, all the results of Sections 1.3–1.5 for $\bar{F}_g^{\mathbb{P}^2/E}$ also hold for $\bar{F}_g^{NS}$ and they agree with the predictions of [HK12, HKK13].

thm_NS

Theorem 1.10. *Using (29)-(30) as definition of the Nekrasov-Shatashvili limit of the refined topological string on local $\mathbb{P}^2$, the series $\bar{F}_g^{NS}$ satisfy the finite generation, quasimodularity, holomorphic anomaly equation and low genus explicit formulas predicted by Huang and Klemm [HK12].*

Independently from any motivation from physics, one can view Theorem 1.10 as providing a surprising way to construct quasimodular forms from Betti numbers of moduli spaces of one-dimensional semistable sheaves on $\mathbb{P}^2$.

Finally, we remark that, given (31), we can view Theorem 1.2 as expressing the difference between the unrefined limit and the Nekrasov-Shatashvili limit of the refined topological string in terms of the Gromov-Witten theory of the elliptic curve. Such relation does not seem to have been predicted in the physics literature.

1.8. **Outline of the paper.** In Section 2, we prove Theorem 1.1, that is, the general form of the higher genus local-relative correspondence. In Section 3, we prove Theorem 1.2, that is, the explicit form of the higher genus local-relative correspondence for log Calabi-Yau surfaces. Starting with Section 4, we focus on the case of $(\mathbb{P}^2, E)$ and we prove the finite generation results (Theorems 1.3-1.6) and the low genus explicit formulas (Theorems 1.4-1.5). The holomorphic anomaly equation for $(\mathbb{P}^2, E)$ (Theorem 1.8) and the S-degree bound of Theorem 1.3 are proven in Section 4. Appendix A described a technical result used in Section 2.

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sec:locrel

2. Higher genus local-relative correspondence

2.1. **Relative Gromov-Witten theory.** Foundations of relative Gromov-Witten invariants were made by Li-Ruan [LR01], Ionel-Parker [IP03] and Eliashberg-Givental-Hofer [EGH00] in symplectic geometry and Li [Li01, Li02] in algebraic geometry. Our presentation is based on [Li01, FWY20].

Let X be a smooth projective variety and D a smooth divisor. The intersection number of a curve class β with a divisor D is denoted by $\beta \cdot D$.

A *topological type* Γ is a tuple $(g, n, \beta, \rho, \vec{\mu})$ where g, n are non-negative integers, $\beta \in H_2(X, \mathbb{Z})$ is a curve class and $\vec{\mu} = (\mu_1, \dots, \mu_\rho) \in \mathbb{Z}^\rho_{>0}$ is a partition of the number $\beta \cdot D$.

Let $\overline{\mathcal{M}}_\Gamma(X, D)$ be the moduli of relative stable maps with topological type Γ. There are evaluation maps

$$\begin{aligned} \mathrm{ev}_X &= (\mathrm{ev}_{X,1}, \dots, \mathrm{ev}_{X,n})\colon \overline{\mathcal{M}}_\Gamma(X, D) \to X^n, \\ \mathrm{ev}_D &= (\mathrm{ev}_{D,1}, \dots, \mathrm{ev}_{D,\rho})\colon \overline{\mathcal{M}}_\Gamma(X, D) \to D^\rho. \end{aligned}$$

The *relative Gromov–Witten invariant with topological type* Γ is defined to be

$$\langle \underline{\varepsilon} \mid \underline{\alpha} \rangle_\Gamma^{(X,D)} \coloneqq \int_{[\overline{\mathcal{M}}_\Gamma(X,D)]^{\mathrm{vir}}} \mathrm{ev}_D^* \underline{\varepsilon} \cup \mathrm{ev}_X^* \underline{\alpha},$$

where $\underline{\varepsilon} \in H^*(D)^{\otimes \rho}$, $\underline{\alpha} \in H^*(X)^{\otimes n}$,

$$\mathrm{ev}_D^* \underline{\varepsilon} \coloneqq \prod_{j=1}^{\rho} \mathrm{ev}_{D,j}^* \varepsilon_j, \quad \mathrm{ev}_X^* \underline{\alpha} \coloneqq \prod_{i=1}^{n} \mathrm{ev}_{X,i}^* \alpha_i$$

We also allow disconnected domains. Let $\Gamma = \{\Gamma^\pi\}$ be a set of topological types. The relative invariant with disconnected domain curves is defined by the product rule:

$$\langle \underline{\varepsilon} \mid \underline{\alpha} \rangle_\Gamma^{\bullet(X,D)} \coloneqq \prod_\pi \langle \underline{\varepsilon}^\pi \mid \underline{\alpha}^\pi \rangle_{\Gamma^\pi}^{(X,D)}.$$

Here $\bullet$ means possibly disconnected domains. We will call this Γ a *possibly disconnected topological type*. We now recall the definition of an *admissible graph*.

defn:adm

Definition 2.1 (Definition 4.6, [Li01])**.** *An admissible graph* Γ is a graph without edges plus the following data.

(1) An ordered collection of legs.
(2) An ordered collection of weighted roots.
(3) A function $\mathrm{g} : V(\Gamma) \to \mathbb{Z}_{\geq 0}$.
(4) A function $b : V(\Gamma) \to H_2(X, \mathbb{Z})$.

Here, $V(\Gamma)$ denotes the set of vertices of Γ. Legs and roots are regarded as half-edges of the graph Γ. A relative stable morphism is associated to an admissible graph in the following way. Vertices in $V(\Gamma)$ correspond to the connected components of the domain curve. Roots and legs correspond to relative markings and interior markings, respectively. Weights on roots correspond to contact orders at the corresponding relative markings.

The functions g, b assign a component to its genus and degree, respectively. We do not spell out the formal definitions in order to avoid heavy notation, but we refer the readers to [Li01, Definition 4.7].

Remark 2.2. A (possibly disconnected) topological type and an admissible graph are equivalent concepts. Different terminologies emphasize different aspects. For example, admissible graphs will be glued at half-edges into actual graphs.

sec:locrel_statement

2.2. **Statement of the local-relative correspondence.** Let X be a smooth projective variety over $\mathbb{C}$ and D be a smooth effective divisor on X. We further assume that D is nef, that is $C \cdot D \geq 0$ for every curve C in X. Let β be a curve class on X such that $\beta \cdot D > 0$. Let $\overline{\mathcal{M}}_g(\mathcal{O}_X(-D), \beta)$ be the moduli space of stable maps to the total space of $\mathcal{O}_X(-D)$. Here the domain curves contain no markings and we omit the number of markings in the notation $\overline{\mathcal{M}}_g(\mathcal{O}_X(-D), \beta)$ for simplicity. The moduli stack $\overline{\mathcal{M}}_g(\mathcal{O}_X(-D), \beta)$ is isomorphic to $\overline{\mathcal{M}}_g(X, \beta)$ thanks to the condition $\beta \cdot D > 0$. In this case, it was proved by van Garrel, Graber and Ruddat [vGR19] that

$$[\overline{\mathcal{M}}_0(\mathcal{O}_X(-D), \beta)]^{\mathrm{vir}} = \frac{(-1)^{(\beta \cdot D)-1}}{\beta \cdot D} F_* \left([\overline{\mathcal{M}}_\Gamma(X, D)]^{\mathrm{vir}} \right)$$

where $\Gamma = (0, 0, \beta, 1, (\beta \cdot D))$ and $F : \overline{\mathcal{M}}_\Gamma(X, D) \to \overline{\mathcal{M}}_{0,0}(X, \beta)$ is the natural stabilization map which also forgets the unique relative marking. Note that the topological type $\Gamma = (0, 0, \beta, 1, (\beta \cdot D))$ corresponds to genus-0 relative stable maps with maximal contact at D. For convenience, whenever $\Gamma = (g, 0, \beta, 1, (\beta \cdot D))$ (genus-g maximal contact), we always denote the above relative moduli space (with only one relative marking) as $\overline{\mathcal{M}}_g(X/D, \beta)$.

We generalize the main result of [vGR19] to higher genera. More precisely, we show that $[\overline{\mathcal{M}}_g(\mathcal{O}_X(-D), \beta)]^{\mathrm{vir}}$ is of the form

$$\frac{(-1)^{\beta \cdot D - 1}}{\beta \cdot D} F_* \left((-1)^g \lambda_g \cap [\overline{\mathcal{M}}_g(X/D, \beta)]^{\mathrm{vir}} \right) + \cdots$$

where λ_g is the g-th Chern class of the Hodge bundle, $\cap$ is the cap product of between cycles and Chow cohomology and "$\cdots$" consists of correction terms which will be made explicit later. Before we explicitly describe the correction terms, we need the following preparation.

defn:star **Definition 2.3.** A *graph of star type* $\mathcal{G}$ is a tuple (V, E, g, b) such that

(1) The set V of vertices admits a disjoint union decomposition $V = \{v\} \coprod V_1$ such that the set E of edges contains exactly one edge between v and v_1 for every $v_1 \in V_1$ and no other edges.
(2) A map $\mathrm{g}\colon V \to \mathbb{Z}_{\geq 0}$.
(3) A map $b\colon \{v\} \cup V_1 \to H_2(D,\mathbb{Z}) \cup H_2(X,\mathbb{Z})$ such that b maps v into $H_2(D,\mathbb{Z})$ and maps V_1 into $H_2(X,\mathbb{Z})$.

The automorphism group of $\mathcal{G}$ consists of automorphisms of the graph (V, E) which commute with g, b. We denote it as $\mathrm{Aut}(\mathcal{G})$.

Definition 2.4. The topological type of a graph of star type is a tuple (g, β) such that

(1) g is the summation of all genera (the values of g).
(2) $\beta = \iota_* b(v) + \sum_{v_i \in V_1} b(v_i) \in H_2(X,\mathbb{Z})$ where $\iota : D \hookrightarrow X$ is the natural inclusion.

We set all the graphs with star type whose topological type is (g, β) to be $G_{g,\beta}$.

For each $\mathcal{G}$, we define

$$\overline{\mathcal{M}}_{\mathcal{G}} \coloneqq \left(\prod_{v_i \in V_1} \overline{\mathcal{M}}_{\mathrm{g}(v_i)}(X/D, b(v_i))\right) \times_{D^{|E|}} \overline{\mathcal{M}}_{\mathrm{g}(v),|E|}(D, b(v)) \qquad (32)$$ eqn:MG

where $\times_{D^{|E|}}$ is the fiber product identifying evaluation maps according to edges. The evaluation map from $\prod_{v_i \in V_1} \overline{\mathcal{M}}_{\mathrm{g}(v_i)}(X/D, b(v_i))$ to $D^{|E|}$ is determined by the relative markings.

The virtual fundamental class $[\overline{\mathcal{M}}_{\mathcal{G}}]^{\mathrm{vir}}$ is given by

$$\Delta^![\prod_{v_i \in V_1} \overline{\mathcal{M}}_{\mathrm{g}(v_i)}(X/D, b(v_i)) \times \overline{\mathcal{M}}_{\mathrm{g}(v),|E|}(D, b(v))]^{\mathrm{vir}}$$

where $\Delta : D^{|E|} \to D^{|E|} \times D^{|E|}$ is the diagonal map and $\Delta^!$ is the Gysin map.

There is a natural stabilization map

$$\mathfrak{s}\colon \prod_{v_i \in V_1} \overline{\mathcal{M}}_{\mathrm{g}(v_i)}(X/D, b(v_i)) \longrightarrow \prod_{v_i \in V_1} \big(\overline{\mathcal{M}}_{\mathrm{g}(v_i),1}(X, b(v_i)) \times_X D\big)$$

where $\times_X$ is the fiber product identifying the unique evaluation map and the inclusion map $D \hookrightarrow X$.

There is also a natural gluing map

$$\left(\prod_{v_i \in V_1} \big(\overline{\mathcal{M}}_{\mathrm{g}(v_i),1}(X, b(v_i)) \times_X D\big)\right) \times_{D^{|E|}} \overline{\mathcal{M}}_{\mathrm{g}(v),|E|}(D, b(v)) \longrightarrow \overline{\mathcal{M}}_g(X, \beta).$$

By composition of the stabilization and gluing maps, we get a map

$$\tau_{\mathcal{G}} \colon \overline{\mathcal{M}}_{\mathcal{G}} \longrightarrow \overline{\mathcal{M}}_g(X,\beta).$$

Let $N_{D/X}$ be the normal bundle of D in X and let $N_{D/X}^\vee$ be its dual. We consider the rank 2 vector bundle over D given by

$$N = N_{D/X} \oplus N_{D/X}^\vee \,. \tag{33}$$ eqn:N

There is a natural *anti-diagonal* scaling action of $\mathbb{C}^*$ on N with weight 1 on $N_{D/X}$ and weight -1 on $N_{D/X}^\vee$.

We have a universal diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & X \\ \big\downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}_{\mathrm{g}(v),|E|}(D,b(v)) & & \end{array}$$

where π is the universal domain curve and f is the universal stable map. We view $-R^\bullet\pi_* f^* N$ as an element of the K-theory of $\overline{\mathcal{M}}_{\mathrm{g}(v),|E|}(D,b(v))$ and we consider its equivariant Euler class

$$e_{\mathbb{C}^*}(-R^\bullet\pi_* f^* N) \in A^*\left(\overline{\mathcal{M}}_{\mathrm{g}(v),|E|}(D,b(v))\right)(t,t^{-1})$$

where t is the equivariant parameter. Now let

$$C_v \coloneqq e_{\mathbb{C}^*}(-R^\bullet\pi_* f^* N) \prod_{v_i\in V_1} \frac{(t+\mathrm{ev}_i^* c_1(N_{D/X}))(-1)^{d_i} d_i}{t+\mathrm{ev}_i^* c_1(N_{D/X}) - d_i\psi_i} \tag{34}$$ eqn:Cv

where

$$\mathrm{ev}_i \colon \overline{\mathcal{M}}_{\mathrm{g}(v),|E|}(D,b(v)) \longrightarrow D \tag{35}$$ eqn:ev_i

is the evaluation map for the i-th marking, $d_i \coloneqq d(v_i)\cdot D$ and ψ_i is the psi-class of the i-th marking. We have

$$C_v \in A^*\left(\overline{\mathcal{M}}_{\mathrm{g}(v),|E|}(D,b(v))\right)(t,t^{-1}).$$

For each $v_i \in V_1$, we define

$$C_{v_i} \coloneqq \frac{t}{t+d_i\bar{\psi}+\bar{\mathrm{ev}}^* c_1(N_{D/X})}(-1)^{g(v_i)}\lambda_{g(v_i)} \tag{36}$$ eqn:Cvi

where

$$\bar{\mathrm{ev}} \colon \overline{\mathcal{M}}_{\mathrm{g}(v_i)}(X/D,b(v_i)) \longrightarrow D \tag{37}$$ eqn:bar_ev

is the evaluation map associated to the unique relative marking, and $\bar{\psi}$ is the psi-class associated to the unique relative marking. We have

$$C_{v_i} \in A^*\left(\overline{\mathcal{M}}_{\mathrm{g}(v_i)}(X/D,b(v_i))\right)(t,t^{-1})\,.$$

Since there is only one relative marking, we have $\bar{\psi} = \mathfrak{s}_i^*\psi_1$ where

$$\mathfrak{s}_i : \overline{\mathcal{M}}_{\mathrm{g}(v_i)}(X/D,b(v_i)) \longrightarrow \overline{\mathcal{M}}_{\mathrm{g}(v_i),1}(X,b(v_i))$$

is the natural stabilization map and ψ_1 is the psi-class associated to the unique marking of $\overline{\mathcal{M}}_{\mathrm{g}(v_i),1}(X, b(v_i))$. So we may also treat C_{v_i} as a pullback class via $\mathfrak{s}_i$. We define

$$C_{\mathcal{G}} := \left[p_v^* C_v \prod_{v_i \in V_1} p_{v_i}^* C_{v_i} \right]_{t^0}$$

where p_v, p_{v_i} are projections from $\overline{\mathcal{M}}_{\mathcal{G}}$ to the corresponding factors and $[\cdots]_{t^0}$ means that we take the constant term.

Now we are ready to state our higher genus local-relative correspondence.

loc2rel

Theorem 2.5 (=Theorem 1.1)**.** *The following relation holds in* $A_*(\overline{\mathcal{M}}_g(X, \beta))$.

$$\begin{aligned}[\overline{\mathcal{M}}_g(\mathcal{O}_X(-D), \beta)]^{\mathrm{vir}} = \quad & \frac{(-1)^{\beta \cdot D - 1}}{\beta \cdot D} F_* \left((-1)^g \lambda_g \cap [\overline{\mathcal{M}}_g(X/D, \beta)]^{\mathrm{vir}} \right) \\ & + \sum_{\mathcal{G} \in G_{g,\beta}} \frac{1}{|\operatorname{Aut}(\mathcal{G})|} (\tau_{\mathcal{G}})_* \left(C_{\mathcal{G}} \cap [\overline{\mathcal{M}}_{\mathcal{G}}]^{\mathrm{vir}} \right).\end{aligned}$$

The proof of Theorem 2.5 is given in Sections 2.3-2.4 and uses the degeneration and localization formulae.

rmk:loc2rel_1

Remark 2.6. For a graph of star type $\mathcal{G}$, if $g(v) = 0$ then it follows from the Riemann-Roch theorem that $e_{\mathbb{C}^*}(-R^\bullet \pi_* f^* N)$ only contains negative powers of t. Therefore, $p_v^* C_v \prod_{v_i \in V_1} p_{v_i}^* C_{v_i}$ must also only contain negative powers of t, and so $C_{\mathcal{G}} = 0$. Thus, a non-vanishing contribution of $\mathcal{G}$ is only possible if $g(v) > 0$, and in particular if $g(v_i) < g$ for each $v_i \in V_1$. That also explains the absence of correction terms if $g = 0$.

sec:deg

2.3. **Degeneration.**

As the first step to prove Theorem 2.5, we apply the degeneration formula to the degeneration to the normal cone of the embedding $D \hookrightarrow X$. This step is identical to the corresponding step in the proof of the main theorem of [vGR19]. More precisely, let $\mathcal{X}$ be the blow-up of $X \times \mathbb{A}^1$ along $D \times \{0\}$. The space $\mathcal{X}$ still admits a projection to $\mathbb{A}^1$ and $\mathcal{X}_0$, the fiber over 0, is a union of the $\mathbb{P}^1$ bundle $\mathbb{P}_D(N_{D/X} \oplus \mathcal{O})$ and X glued along the section $D \cong \mathbb{P}_D(N_{D/X}) \subset \mathbb{P}_D(N_{D/X} \oplus \mathcal{O})$ and the hypersurface $D \subset X$. For convenience, we introduce the following notation.

- Denote the $\mathbb{P}^1$ bundle by P_0 and the component of $\mathcal{X}_0$ isomorphic to X by X_0.
- Denote the section $\mathbb{P}_D(N_{D/X}) \subset \mathbb{P}_D(N_{D/X} \oplus \mathcal{O})$ by D_∞ and the section $\mathbb{P}_D(\mathcal{O}) \subset \mathbb{P}_D(N_{D/X} \oplus \mathcal{O})$ by D_0.

Let $\mathcal{L}$ be the line bundle on $\mathcal{X}$ associated with the divisor of the strict transform of $D \times \mathbb{A}^1 \subset X \times \mathbb{A}^1$. Since we want to relate the local theory with the relative theory, we need to consider the total space $\operatorname{Tot}(\mathcal{L})$ in the degeneration. The general fiber of $\operatorname{Tot}(\mathcal{L})$ over $\mathbb{A}^1$ is isomorphic to the total space of $\mathcal{O}_X(-D)$, which is the target space of our local theory. The special fiber at $0 \in \mathbb{A}^1$ is a union of $\mathcal{L}|_{P_0} = \mathcal{O}_{P_0}(-D_0)$ and $\mathcal{L}|_{X_0} = \mathcal{O}_{X_0}$ glued along the corresponding divisors both of which are isomorphic to $D \times \mathbb{A}^1$. Note that the line bundle $\mathcal{L}|_{X_0}$ is isomorphic to the trivial bundle on X.

The degeneration formula expresses the virtual cycle of $\overline{\mathcal{M}}_g(\mathcal{O}_X(-D), \beta)$ in terms of the ones of $\overline{\mathcal{M}}^\bullet_{\Gamma_1}(\mathcal{L}|_{P_0}, \mathcal{L}|_{D_\infty})$ and $\overline{\mathcal{M}}^\bullet_{\Gamma_2}(X \times \mathbb{A}^1, D \times \mathbb{A}^1)$ summing over all splittings $\mathfrak{i} = (\Gamma_1, \Gamma_2)$ of the genus-g degree-β curve type. The splitting forms a bipartite graph and must have matching contact orders at the corresponding relative markings. The details of the splitting of topological types can be found in [Li02, Definition 4.11]. Since the moduli stack of stable maps to the stack of expanded degeneration admits a morphism to $\overline{\mathcal{M}}_g(X, \beta)$ induced by the projection of the target $\mathcal{X} \to X$, we have the following version of degeneration formula.

$$\begin{aligned} &[\overline{\mathcal{M}}_g(\mathcal{O}_X(-D), \beta)]^{\mathrm{vir}} \\ &= \sum_{\mathfrak{i}=(\Gamma_1,\Gamma_2)} \frac{\prod_{i=1}^{m(\mathfrak{i})} d_i}{\mathrm{Aut}(\mathfrak{i})} \tau_* \Delta^! [\overline{\mathcal{M}}^\bullet_{\Gamma_1}(\mathcal{L}|_{P_0}, \mathcal{L}|_{D_\infty}) \times \overline{\mathcal{M}}^\bullet_{\Gamma_2}(X \times \mathbb{A}^1, D \times \mathbb{A}^1)]^{\mathrm{vir}} \end{aligned} \tag{38}$$

eqn:deg

where there are $m(\mathfrak{i})$ roots (relative markings) on each Γ_1 and Γ_2, d_i are the weights (contact order) of the corresponding roots, $\Delta^!$ is the Gysin pullback induced by the diagonal shown in the following Cartesian diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{\mathfrak{i}} & \longrightarrow & \overline{\mathcal{M}}^\bullet_{\Gamma_1}(\mathcal{L}|_{P_0}, \mathcal{L}|_{D_\infty}) \times \overline{\mathcal{M}}^\bullet_{\Gamma_2}(X \times \mathbb{A}^1, D \times \mathbb{A}^1) \\ \downarrow & & \downarrow \mathrm{ev} \\ (D \times \mathbb{A}^1)^{m(\mathfrak{i})} & \xrightarrow{\Delta} & (D \times \mathbb{A}^1 \times D \times \mathbb{A}^1)^{m(\mathfrak{i})}, \end{array} \tag{39}$$

eq:diag

and τ is the forgetful map from the fiber product $\overline{\mathcal{M}}_{\mathfrak{i}}$ to $\overline{\mathcal{M}}_g(X, \beta)$.

First of all, there is a distinguished term in the degeneration formula where Γ_1 consists of one vertex of genus 0, curve class $\beta \cdot D$ times of fiber class with one root of weight $\beta \cdot D$, and Γ_2 consists of one vertex of genus g, curve class β and a weight-$(\beta \cdot D)$ root. This term can be understood by the following lemma.

Lemma 2.7. *Let Γ_1 be a topological type of genus 0, curve class $\beta \cdot D$ times of fiber class with one root of weight $\beta \cdot D$. Then*

$$\mathrm{ev}_*[\overline{\mathcal{M}}^\bullet_{\Gamma_1}(\mathcal{L}|_{P_0}, \mathcal{L}|_{D_\infty})]^{\mathrm{vir}} = \frac{(-1)^{\beta\cdot D-1}}{(\beta \cdot D)^2}[D],$$

where ev *is the evaluation map of the (unique) relative marking.*

With the help of the known comparison of virtual cycles between moduli of relative stable maps and moduli of log stable maps, this lemma is nothing but [vGR19, Proposition 2.4]. But the localization computation in Section 2.4 also provides a direct proof of this lemma. We do not repeat the proof here.

The lemma implies that this distinguished term equals

$$\frac{(-1)^{\beta\cdot D-1}}{\beta\cdot D}F_*\left((-1)^g\lambda_g\cap[\overline{\mathcal{M}}_g(X/D,\beta)]^{\mathrm{vir}}\right)$$

in Theorem 2.5.

To understand the rest of the terms, we need to state a parallel lemma to [vGR19, Lemma 3.1]. The idea is that in the relative theory of $(X\times\mathbb{A}^1, D\times\mathbb{A}^1)$, we rule out topological types Γ_2 with multiple relative contacts.

lem:root **Lemma 2.8.** *Let $\mathfrak{i}=(\Gamma_1,\Gamma_2)$ be a splitting such that there exists a vertex v in Γ_2 having more than one root (roots corresponds to relative markings). Then*

$$\Delta^![\overline{\mathcal{M}}^\bullet_{\Gamma_1}(\mathcal{L}|_{P_0},\mathcal{L}|_{D_\infty})\times\overline{\mathcal{M}}^\bullet_{\Gamma_2}(X\times\mathbb{A}^1,D\times\mathbb{A}^1)]^{\mathrm{vir}}=0.$$

The proof of [vGR19, Lemma 3.1] can almost apply here word-by-word because the gist of the proof is an intersection theoretic computation on the target spaces and the geometry of moduli spaces plays a minor role. The key step is to observe that if the vertex v has r roots, all the evaluation maps of the corresponding r relative markings must map to the same point on the $\mathbb{A}^1$ factor (here we use the condition that D_0 is a nef divisor in P_0). In other words, the map ev in the diagram (39) factors as follows:

$$\begin{array}{c}\overline{\mathcal{M}}^\bullet_{\Gamma_1}(\mathcal{L}|_{P_0},\mathcal{L}|_{D_\infty})\times\overline{\mathcal{M}}^\bullet_{\Gamma_2}(X\times\mathbb{A}^1,D\times\mathbb{A}^1)\\ \downarrow\\ (D^r\times\mathbb{A}^1\times D^r\times\mathbb{A}^1)\times(D\times\mathbb{A}^1\times D\times\mathbb{A}^1)^{m(\mathfrak{i})-r}\\ \downarrow\\ \left((D\times\mathbb{A}^1)^r\times(D\times\mathbb{A}^1)^r\right)\times(D\times\mathbb{A}^1\times D\times\mathbb{A}^1)^{m(\mathfrak{i})-r},\end{array}$$

where the bottom map sends each of the two D^r in the first bracket factor to the corresponding D^r by identity maps, and each of the two $\mathbb{A}^1$ in the first bracket factor to the corresponding $(\mathbb{A}^1)^r$ by diagonal maps. The bottom map also sends the second bracket to the second bracket by identity. The decomposition of ev induces the decomposition of (39) into two Cartesian squares. It follows that

$$\Delta^![\overline{\mathcal{M}}^\bullet_{\Gamma_1}(\mathcal{L}|_{P_0},\mathcal{L}|_{D_\infty})\times\overline{\mathcal{M}}^\bullet_{\Gamma_2}(X\times\mathbb{A}^1,D\times\mathbb{A}^1)]^{\mathrm{vir}}$$

decomposes into a composition of two Gysin pullbacks. Applying the excess intersection formula, the conclusion is that the Gysin pullback is zero. We refer the readers to the proof of [vGR19, Lemma 3.1] for the details.

A splitting $\mathfrak{i}=(\Gamma_1,\Gamma_2)$ gives rise to a bipartite graph by gluing the corresponding roots. Lemma 2.8 tells us that every vertex of Γ_2 consists of only one root. This suggests that the bipartite graphs are comb-shaped and they match the underlying graphs of Definition 2.3. Thus the degeneration formula is already giving us a general form that looks like Theorem 2.5

except that the pushforward of virtual cycles of $\overline{\mathcal{M}}^\bullet_{\Gamma_1}(\mathcal{L}|_{P_0}, \mathcal{L}|_{D_\infty})$ are not understood in general.

sec:loc

2.4. **Localization.** We use the relative virtual localization formula [GV05] to understand the pushforward of $[\overline{\mathcal{M}}^\bullet_{\Gamma_1}(\mathcal{L}|_{P_0}, \mathcal{L}|_{D_\infty})]^{\mathrm{vir}}$. First recall that P_0 has two sections: D_0 and D_∞. The normal bundle of D_0 is $\mathcal{O}_D(D)$ and the one of D_∞ is $\mathcal{O}_D(-D)$. The line bundle $\mathcal{L}|_{P_0}$ is naturally isomorphic to $\mathcal{O}_{P_0}(-D_0)$. Let $\mathbb{C}^*$ act on $\mathcal{L}|_{P_0}$ as follows:

- $\mathbb{C}^*$ acts fiberwise on P_0 so that the weight of fibers of the normal bundle of D_0 is 1.
- D_0 becomes an invariant divisor under the above $\mathbb{C}^*$-action on P_0 and let $\mathbb{C}^*$ act on $\mathcal{L}|_{P_0}$ in such a way that $\mathcal{L}|_{P_0} \cong \mathcal{O}_{P_0}(-D_0)$ as equivariant line bundles.

Under this construction, the $\mathbb{C}^*$ acts fiberwise on $\mathcal{L}|_{D_0}$ with weight -1 and it acts on $\mathcal{L}|_{D_\infty}$ trivially. The fixed loci of $\mathcal{L}|_{P_0}$ consists of D_0 and the whole total space $\mathcal{L}|_{D_\infty}$. The $\mathbb{C}^*$-action induces an action on $\overline{\mathcal{M}}^\bullet_{\Gamma_1}(\mathcal{L}|_{P_0}, \mathcal{L}|_{D_\infty})$. Invariant curves decompose into components mapping to D_0 and components mapping into rubbers over D_∞. The localization formula can be summarized as follows.

$$\begin{aligned}&[\overline{\mathcal{M}}^\bullet_{\Gamma_1}(\mathcal{L}|_{P_0}, \mathcal{L}|_{D_\infty})]^{\mathrm{eq,vir}} \\ =&\frac{[\overline{\mathcal{M}}^\bullet_{\Gamma_1}(\mathcal{L}|_{P_0}, \mathcal{L}|_{D_\infty})^{\mathrm{simple}}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})} + \sum_{\eta=(\Gamma_1^{(0)},\Gamma_1^{(\infty)})} (\iota_\eta)_* \frac{[\overline{\mathcal{M}}_\eta]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N_\eta^{\mathrm{vir}})}\end{aligned} \tag{40}$$

eqn:virloc

with the notation explained as follows. The first term corresponds to the fixed locus where the target does not degenerate (simple fixed locus according to [GV05]) and the second summand corresponds to the rest of fixed loci. We denote by N^{vir} and N_η^{vir} the corresponding virtual normal bundles. The index of the summation η is a splitting of the topological type Γ_1 into $\Gamma_1^{(0)}$ and $\Gamma_1^{(\infty)}$ with

- $\Gamma_1^{(0)}$ a disjoint union of vertices with a set of ordered half edges (a total of k_η) and the decoration of genus, number of markings and curve class attached to each vertex, and $\overline{\mathcal{M}}^\bullet_{\Gamma_1^{(0)}}(D)$ is the corresponding moduli of stable maps to D with possibly disconnected domain;
- $\Gamma_1^{(\infty)}$ is a possibly disconnected rubber graph (a possibly disconnected graph with assignments of genera, curve classes, inner markings, relative markings to both boundary divisors and their contact orders as detailed in [FWY20, Definition 2.4]) with k_η ordered 0-roots (corresponding to relative markings along the divisor glued to $D_\infty \subset P_0$).

Gluing $\Gamma_1^{(0)}$ and $\Gamma_1^{(\infty)}$ along these ordered k_η half-edges forms a bipartite graph, and $\mathrm{Aut}(\eta)$ is the automorphism group of the resulting graph. $\overline{\mathcal{M}}_\eta$ is the fixed loci corresponds to η. The map ι_η is the inclusion of the fixed loci. It is standard to describe an étale cover of $\overline{\mathcal{M}}_\eta$ as a fiber product of vertex moduli and edge moduli:

$$\overline{\mathcal{M}}^\bullet_{\Gamma_1^{(0)}}(D_0) \times_{D^{k_\eta}} D^{k_\eta} \times_{(D\times\mathbb{A}^1)^{k_\eta}} \overline{\mathcal{M}}^{\bullet\sim}_{\Gamma_1^{(\infty)}}(\mathcal{L}|_{D_\infty}) \to \overline{\mathcal{M}}_\eta,$$

where the fiber products are over evaluation maps and the inclusion of $D^{k_\eta} \to (D \times \mathbb{A}^1)^{k_\eta}$ sends D^{k_η} to $D^{k_\eta} \times \{0\}$.

The combinatorics of graph splittings and the precise formulas of $e_{\mathbb{C}^*}(N_\eta^{\text{vir}})$ are a priori very complicated. But similar to Lemma 2.8, we have the following result to cut down the number of graph types on $\Gamma_1^{(\infty)}$.

Lemma 2.9. *If any vertex of $\Gamma_1^{(\infty)}$ has more than one 0-roots (corresponding to relative markings along the divisor glued to D_∞), we have*

$$[\overline{\mathcal{M}}_\eta]^{\text{vir}} = 0$$

Note that $[\overline{\mathcal{M}}_\eta]^{\text{vir}}$ is a multiple of the pushforward of

$$[\overline{\mathcal{M}}^\bullet_{\Gamma_1^{(0)}}(D_0) \times_{D^{k_\eta}} D^{k_\eta} \times_{(D\times\mathbb{A}^1)^{k_\eta}} \overline{\mathcal{M}}^{\bullet\sim}_{\Gamma_1^{(\infty)}}(\mathcal{L}|_{D_\infty})]^{\text{vir}}.$$

The argument is the same as Lemma 2.8 and we omit the details. The conclusion of this lemma is that we can assume that each $\Gamma_1^{(0)}$ in the summation only consists of one single vertex. Thus, there is a unique edge between the vertex in $\Gamma_1^{(0)}$ and a vertex in $\Gamma_1^{(\infty)}$. Comparing with Definition 2.3 and Equation (32), we see that this shares the shape of graphs of star type, except that the vertices in V_1 represent relative stable maps to X instead of relative stable maps to rubbers over D which appears in our localization formula (40). But combining the virtual localization formula (40) and the degeneration formula (38) altogether, we will be able to match the graph sum with Theorem 2.5.

Let us recap the whole process. Starting with the topological type Γ, we split it into three parts $\Gamma_1^{(0)}, \Gamma_1^{(\infty)}, \Gamma_2$ and sum over cycles

$$\begin{aligned}[\overline{\mathcal{M}}^\bullet_{\Gamma_1^{(0)}}(D_0) \times_{D^{k_\eta}} D^{k_\eta} \times_{(D\times\mathbb{A}^1)^{k_\eta}} &\overline{\mathcal{M}}^{\bullet\sim}_{\Gamma_1^{(\infty)}}(\mathcal{L}|_{D_\infty}) \\ &\times_{(D\times\mathbb{A}^1)^{m(\mathrm{i})}} \overline{\mathcal{M}}^\bullet_{\Gamma_2}(X\times\mathbb{A}^1, D\times\mathbb{A}^1)]^{\text{vir}}\end{aligned}$$

capped with different cohomology classes. To simplify the situation, notice that the sum over $\Gamma_1^{(\infty)}$ and Γ_2 can be combined into a divisor in

$$\overline{\mathcal{M}}^\bullet_{\Gamma_2'}(X\times\mathbb{A}^1, D\times\mathbb{A}^1) = \overline{\mathcal{M}}^\bullet_{\Gamma_2'}(X, D) \times \mathbb{A}^1$$

corresponding to the locus where the target degenerates at least once (denoted by δ). Here Γ_2' is the gluing of $\Gamma_1^{(\infty)}$ and Γ_2 along the corresponding roots. In fact, δ can be written as another divisor that commonly appears in the virtual localization formula as follows.

According to [GV05, Section 2.5], $\overline{\mathcal{M}}^\bullet_{\Gamma_2'}(X, D)$ admits a map to $\mathcal{T}$(or [Li01, Chapter 4] as $\mathfrak{X}^{rel}$), the Artin stack of expanded degenerations of (X, D). Also, by [GV05, Section 2.5], $\mathcal{T}$ is isomorphic to an open substack of $\mathfrak{M}_{0,3}$ (the Artin stack of prestable 3-pointed rational curve) consisting of chains of curves that separate ∞ from $0,1$ (in [GV05]'s notation). There is a divisor corresponding to cotangent lines at ∞, and we denote by Ψ its pullback to $\overline{\mathcal{M}}^{\bullet}_{\Gamma_2'}(X,D)$.

lem:psi

Lemma 2.10. *Ψ is linearly equivalent to δ.*

Proof. When identifying $\mathcal{T}$ as an open substack of $\mathfrak{M}_{0,3}$, the relation $\Psi=\delta$ is simply the pullback of the relation $\psi_1 = D_{1|23}$ on $A^2(\mathfrak{M}_{0,3})$. □

Now that Ψ represents the decomposition of Γ_2' into different $\Gamma_1^{(\infty)}$ and Γ_2, we can simplify the localization formula combined with the degeneration formula into the following form:

$$\begin{aligned}
&[\overline{\mathcal{M}}_g(\mathcal{O}_X(-D),\beta)]^{\mathrm{vir}} \\
=&\Bigg[\sum_{\mathfrak{i}=(\Gamma_1',\Gamma_2')} \frac{1}{\mathrm{Aut}(\mathfrak{i})}(\tau_{\mathfrak{i}})_*\Bigg([\overline{\mathcal{M}}_{\Gamma_1'}(D_0)\times_{D^{k_{\mathfrak{i}}}} \overline{\mathcal{M}}^{\bullet}_{\Gamma_2'}(X,D)]^{\mathrm{vir}}\cap \ \mathrm{Edge}(\mathfrak{i}) \\
&p_1^*\left(\frac{e_{\mathbb{C}^*}(-R^\bullet(\pi_1)_*f_1^*N)}{\prod_{v_i\in V(\Gamma_1')}\left(\frac{t+\mathrm{ev}_i^*c_1(N_{D/X})}{d_i}-\psi_i\right)}\right) p_2^*\left(\left(1+\frac{\Psi}{-t-\Psi}\right)\cdot e(R^1(\pi_2)_*f_2^*\mathcal{O})\right)\Bigg)\Bigg]_{t^0},
\end{aligned} \tag{41}$$

eqn:combine

where

- $\mathfrak{i}$ is a splitting such that each vertex of Γ_2' has one unique 0-root; Γ_1' is allowed to be a degree-zero genus-zero unstable vertex;
- $\tau_{\mathfrak{i}}$ is the stabilization map

$$\tau_{\mathfrak{i}}:\overline{\mathcal{M}}_{\Gamma_1'}(D_0)\times_{D^{k_{\mathfrak{i}}}}\overline{\mathcal{M}}^{\bullet}_{\Gamma_2'}(X,D)\to\overline{\mathcal{M}}_g(X,\beta)=\overline{\mathcal{M}}_g(\mathcal{O}_X(-D),\beta);$$

- $\mathrm{Edge}(\mathfrak{i})$ is certain cohomology class depending on edges in the bipartite graph splitting $\mathfrak{i}$ (corresponding to the edge contribution in the localization);
- p_1 and p_2 are the projections from $\overline{\mathcal{M}}_{\Gamma_1'}(D_0)\times_D\overline{\mathcal{M}}^{\bullet}_{\Gamma_2'}(X,D)$ to the first and second factors, respectively;
- $V(\Gamma_1')$ refers to the set of vertices of Γ_1', $k_{\mathfrak{i}}$ is the number of 0-roots as before;
- N and ev_i are defined by (33) and (35) in Section 2.2;
- π_1,π_2 and f_1,f_2 are the universal curves and the universal maps for $\overline{\mathcal{M}}_{\Gamma_1'}(D_0)$ and $\overline{\mathcal{M}}^{\bullet}_{\Gamma_2'}(X,D)$, respectively (similar to π and f used in Section 2.2).

We make the convention that $\overline{\mathcal{M}}_{\Gamma_1'}(D_0)=D$ when Γ_1' consists of an unstable vertex. In this case, the whole $p_1^*(\dots)$ factor degenerates into 1. We highlight a few key points regarding how to deduce this formula:

(1) Previously, combining the degeneration formula and the localization formula, we have three levels $\Gamma_1^{(0)},\Gamma_1^{(\infty)}$ and Γ_2. To deduce this formula, we turn $\Gamma_1^{(0)}$ into Γ_1' and glue $\Gamma_1^{(\infty)},\Gamma_2$ into Γ_2'. As previously

discussed, the sum over different decompositions of Γ_2' corresponds to the divisor Ψ which is written in the numerator of the factor

$$p_2^*\left(\left(1+\frac{\Psi}{-t-\Psi}\right)\cdot e(R^1(\pi_2)_*f_2^*\mathcal{O})\right).$$

Note that the extra summand "1" corresponds to the simple fixed locus in the localization formula.

(2) Recall that $N = N_{D/X} \oplus N_{D/X}^\vee$ under the anti-diagonal scaling action. The factor $e_{\mathbb{C}^*}(-R^\bullet(\pi_1)_*f_1^*N)$ comes from the moving part computation on vertices over D_0. $N_{D/X}$ comes from the log-tangent bundle of P_0, and $N_{D/X}^\vee$ comes from the restriction of $\mathcal{L}$ on D_0.

(3) The denominators

$$\prod_{v_i\in V(\Gamma_1')}\left(\frac{t+\mathrm{ev}_i^*\,c_1(N_{D/X})}{d_i}-\psi_i\right),\quad -t-\Psi$$

correspond respectively to the smoothing of nodes and the smoothing of the target. Note that Ψ in the numerator decomposes Γ_2' into different $\Gamma_1^{(\infty)}, \Gamma_2$, and Ψ in the denominator becomes the corresponding target psi-class on the rubber moduli associated to $\Gamma_1^{(\infty)}$.

(4) In the formula, the edges should contribute an automorphism factor $1/\prod_{i=1}^{k_\eta} d_i$. But the gluing of edges with the relative markings on rubber components also contribute $\prod_{i=1}^{k_\eta} d_i$. These two factors cancel each other. Note that in the case of the simple fixed locus, there are no rubber components, but there is a factor $\prod_{i=1}^{k_\eta} d_i$ coming from the degeneration formula (38).

One notices that the splitting $\mathfrak{i}$ already resembles the graphs of star type defined in Definition 2.3. We are left to identify the formulas. It boils down to two things: matching suitable factors with the C_v term defined in Equation (34), and matching the rest with the C_{v_i} terms defined in Equation (36).

2.4.1. *Matching with C_v.* By explicitly computing Edge($\mathfrak{i}$), one can combine it with

$$p_1^*\left(\frac{e_{\mathbb{C}^*}(-R^\bullet(\pi_1)_*f_1^*N)}{\prod_{v_i\in V(\Gamma_1)}(t+\mathrm{ev}_i^*\,c_1(N_{D/X})-d_i\psi_i)}\right)$$

to obtain C_v. Some details about computing Edge($\mathfrak{i}$) is presented in the following.

Edge($\mathfrak{i}$) encodes the deformation and obstruction contributions of edge components. More precisely, we need to compute the factor in the moving part of the equivariant Euler class of $R^\bullet(\pi_1)_*f_1^*T\mathcal{L}(-\log\mathcal{L}|_{D_\infty})$ that corresponds to edges. Note that the moving part of the equivariant Euler class of $R^\bullet(\pi_1)_*f_1^*T\mathcal{L}(-\log\mathcal{L}|_{D_\infty})$ is a product of the one of $R^\bullet(\pi_1)_*f_1^*TP_0(-\log D_\infty)$

and the one of $R^\bullet(\pi_1)_* f_1^* \mathcal{O}_{P_0}(-D_0)$. The moving part of the equivariant Euler class of $R^\bullet(\pi_1)_* f_1^* TP_0(-\log D_\infty)$ is explicitly computed in [Wu18, Appendix B]. The computation of the moving part of $R^\bullet(\pi_1)_* f_1^* \mathcal{O}_{P_0}(-D_0)$ is similar. We omit the details here.

Given an edge of multiplicity d, firstly let us suppose that the vertex over D_0 is stable. The log-tangent bundle of P_0 contributes the factor

$$\frac{d^d}{d!(t+\mathrm{ev}_i^* c_1(N_{D/X}))^d}.$$

The $\mathcal{O}_{P_0}(-D_0)$ twisting contributes the factor

$$\frac{(d-1)!(-t-\mathrm{ev}_i^* c_1(N_{D/X}))^{d-1}}{d^{d-1}}. \tag{42}$$

eqn:midfac

Finally, there is an extra factor $-(t+\mathrm{ev}_i^* c_1(N_{D/X}))^2$ (the weight of $T\mathcal{L}$ at the node) coming from the gluing of the edge and the component over D_0. Putting them together, we have

$$\mathrm{Edge}(\mathfrak{i}) = (-1)^d (t+\mathrm{ev}_i^* c_1(N_{D/X})).$$

Next, let us suppose that the vertex over D_0 is unstable (for example, the leading term in Theorem 2.5). The valence-2 case (in this case the unstable vertex connects to two edges) will contribute 0 to the localization formula because the corresponding summand on the right-hand side of (41) only consists of negative powers of t by dimension reason (see also [vGR19, Lemma 5.4]). When the vertex over D_0 is of valence 1, there is an extra factor $(t+\mathrm{ev}_i^* c_1(N_{D/X}))/d$ coming from the moving part of the space of infinitesimal automorphisms of domain curves. Cancelling with the contribution of the log-tangent bundle of P_0, we get

$$\frac{d^{d-1}}{d!(t+\mathrm{ev}_i^* c_1(N_{D/X}))^{d-1}}.$$

Multiplying with the factor (42), we have

$$\mathrm{Edge}(\mathfrak{i}) = (-1)^{d-1}/d.$$

This also explains the coefficient $\dfrac{(-1)^{\beta\cdot D-1}}{\beta\cdot D}$ of the leading term in Theorem 2.5.

2.4.2. *Matching with C_{v_i}.* First, each factor $(-1)^{g(v)}\lambda_{g(v)}$ of C_{v_i} in (36) comes from $e(R^1(\pi_2)_* f_2^* \mathcal{O})$. To match the other factor in C_{v_i} requires slightly more effort. For a vertex $v \in V(\Gamma)$, denote by γ_v the graph consisting of a single vertex v plus all the decorations on v. The goal now is to rewrite the pushforward of

$$\left(1+\frac{\Psi}{-t-\Psi}\right) \cap [\overline{\mathcal{M}}^\bullet_{\Gamma_2'}(X,D)]^{\mathrm{vir}}$$

in terms of pushforward classes from the product $\prod_{v\in V(\Gamma_2')}\overline{\mathcal{M}}_{\gamma_v}(X,D)$ and match it with the remaining factor of C_{v_i}. More precisely, we need the following lemma.

lem_split2

Lemma 2.11. *We have the following identity:*

$$\begin{aligned}&\tau_*\left(\frac{t}{t+\Psi}\cap[\overline{\mathcal{M}}^{\bullet}_{\Gamma_2'}(X,D)]^{\mathrm{vir}}\right)\\ =&\tau'_*\left(\prod_{v\in V(\Gamma_2')}p_v^*\left(\frac{t}{t+\Psi}\right)\cap\Big[\prod_{v\in V(\Gamma_2')}\overline{\mathcal{M}}_{\gamma_v}(X,D)\Big]^{\mathrm{vir}}\right),\end{aligned}$$

where p_v is the projection to the factor corresponding to v, and the two Ψ on both sides are the target psi-classes on their corresponding moduli spaces, and τ,τ' are the corresponding stabilization maps to $\prod_{v\in V(\Gamma_2')}\overline{\mathcal{M}}_{g(v),1}(X,b(v))\times_X D$.

Lemma 2.11 is a special case of Lemma A.4. Each factor on the right-hand side of Lemma 2.11 matches with the $\dfrac{t}{t+d_i\bar\psi+\bar{\mathrm{ev}}^*c_1(N_{D/X})}$ part of C_{v_i} because of the following two lemmas.

Lemma 2.12. *In the moduli of relative stable maps $\overline{\mathcal{M}}_\Gamma(X,D)$, suppose e is a root with multiplicity d, we have the following identity:*

$$\Psi=d\psi_e+\mathrm{ev}_e^*c_1(N_{D/X}),$$

where ψ_e is the psi-class of the relative marking e defined using the universal curve over $\overline{\mathcal{M}}_\Gamma(X,D)$.

For a proof, see [Kat05, Theorem 5.13.1].

Lemma 2.13. *If Γ is a connected admissible graph with only one root. Let ψ be the psi-class of the relative marking and recall that $\bar\psi$ is the pullback of the psi-class from the moduli of stable maps to X. We have $\psi=\bar\psi$.*

Proof. Because Γ has only one root, when the target expands, the relative marking never lies on an unstable component (a multiple cover of a fiber of a rubber target with only 2 relative markings). Thus, the stabilization does not contract components containing the relative marking. As a result, the psi-class and the pullback psi-class are the same. □

sec:locrel_log_K3

3. The case of log Calabi-Yau surfaces

Let S be a smooth projective surface over $\mathbb{C}$ and E a smooth effective anticanonical divisor on S which is nef. By the adjunction formula, E is a genus 1 curve. In this section, we prove Theorem 1.2 expressing the local series $F_g^{K_S}$ in terms of the relative series $F_g^{S/E}$ and the stationary series $F^E_{g,\mathbf{a}}$ of the elliptic curve E. We use systematically the notation introduced in Section 1.2.

3.1. **Specializing Theorem 2.5 to the pair** (S, E)**.** Using Theorem 2.5, we obtain a relation between the local invariants $N_{g,\beta}^{K_S}$ and the relative invariants $N_{g,\beta}^{S/E}$.

Let $N_{E/S}$ be the normal bundle to E in S. We consider the rank 2 vector bundle $N := N_{E/S} \oplus N_{E/S}^\vee$ over E and the anti-diagonal scaling action of $\mathbb{C}^*$ on N with weight 1 on $N_{E/S}$ and weight -1 on $N_{E/S}^\vee$. We denote by t the corresponding equivariant parameter. For every $d_E \geq 0$ and $\mathbf{d} = (d_1, \cdots, d_n)$, we set

$$N_{h,d_E}^{E,\mathrm{tw}}(\mathbf{d}) := \int_{[\overline{\mathcal{M}}_{h,n}(E,d_E)]^{\mathrm{vir}}} \left(\prod_{j=1}^n \frac{t\, \mathrm{ev}_j^* \,\omega}{t - d_j \psi_j} \right) e_{\mathbb{C}^*}(-R^\bullet \pi_* f^* N)$$

where ω is the point class for E. It is easy to deduce from the dimension constraint that $N_{h,d_E}^{E,\mathrm{tw}}(\mathbf{d}) \in \mathbb{Q}$, that is, does not depend on t.

Proposition 3.1. *For every* $\beta \in H_2(S, \mathbb{Z})$ *such that* $\beta \cdot E > 0$ *we have*

$$N_{g,\beta}^{K_S} = \frac{(-1)^{\beta\cdot E - 1}}{\beta \cdot E} N_{g,\beta}^{S/E} +$$

$$\sum_{n\geq 0} \sum_{\substack{g = h + g_1 + \cdots + g_n \\ \beta = d_E[E] + \beta_1 + \cdots + \beta_n \\ d_E \geq 0,\, \beta_j \cdot E > 0}} \frac{N_{h,d_E}^{E,\mathrm{tw}}(\boldsymbol{\beta} \cdot E)}{|\operatorname{Aut}(\boldsymbol{\beta}, \mathbf{g})|} \prod_{j=1}^n \left((-1)^{\beta_j \cdot E} (\beta_j \cdot E) N_{g_j,\beta_j}^{S/E} \right)$$

where

$$\boldsymbol{\beta} \cdot E = (\beta_1 \cdot E, \ldots, \beta_n \cdot E)\,,$$

$$\frac{1}{|\operatorname{Aut}(\boldsymbol{\beta}, \mathbf{g})|} := \frac{1}{|\operatorname{Aut}((\beta_1, g_1), \cdots, (\beta_n, g_n))|}\,.$$

Proof. We apply Theorem 2.5. Let $\mathcal{G} \in G_{g,\beta}$ with $|V_1| = n$, $g = h + \sum_{j=1}^n g_j$, $\beta = d_E[E] + \sum_{j=1}^n \beta_j$. We denote $d_j = \beta_j \cdot E$. The contribution of $\mathcal{G}$ is

$$(\tau_{\mathcal{G}})_* \left[p_v^* C_v \prod_{v_j \in V_1} p_{v_j}^* C_{v_j} \right]$$

where $\tau_{\mathcal{G}}$ is the gluing map

$$\left(\prod_{v_j \in V_1} \overline{\mathcal{M}}_{g_j}(S/E, \beta_j) \right) \times_{E^n} \overline{\mathcal{M}}_{h,n}(E, d_E) \longrightarrow \overline{\mathcal{M}}_g(S, \beta).$$

According to (36), we have

$$C_{v_j} = \frac{t}{t + d_j \bar{\psi} + \bar{\mathrm{ev}}^* c_1(N_{E/S})} (-1)^{g_j} \lambda_{g_j}\,.$$

The key point is that $\dim[\overline{\mathcal{M}}_{g_j}(S/E, \beta_j)]^{\mathrm{vir}} = g_j$. Therefore the insertion of $(-1)^{g_j} \lambda_{g_j}$ already eats up all the dimension of $[\overline{\mathcal{M}}_{g_j}(S/E, \beta_j)]^{\mathrm{vir}}$ and so

C_{v_j} reduces to $(-1)^{g_j}\lambda_{g_j}$ and the only possibly non-vanishing contribution of the class of the diagonal $E \hookrightarrow E \times E$ defining the gluing $\tau_{\mathcal{G}}$ is $p_{v_j}^* 1 \times p_v^* \omega$ on $\overline{\mathcal{M}}_{g_j}(S/E, \beta_j) \times \overline{\mathcal{M}}_{h,n}(E, d_E)$, where $1 \in H^0(E)$ and $\omega \in H^2(E)$. Thus, the contribution of $\overline{\mathcal{M}}_{g_j}(S/E, \beta_j)$ is

$$\int_{[\overline{\mathcal{M}}_{g_j}(S/E,\beta_j)]^{\mathrm{vir}}} (-1)^{g_j}\lambda_j = N_{g_j,\beta_j}^{S/E}$$

and the contribution of $\overline{\mathcal{M}}_{h,n}(E, d_E)$ is

$$\int_{[\overline{\mathcal{M}}_{h,n}(E,d_E)]^{\mathrm{vir}}} C_v \prod_{j=1}^n \mathrm{ev}_j^* \omega \,.$$

According to (34), we have

$$C_v = e_{\mathbb{C}^*}(-R^\bullet \pi_* f^* N) \prod_{j=1}^n \frac{(t + \mathrm{ev}_j^* c_1(N_{E/S}))(-1)^{d_j} d_j}{t + \mathrm{ev}_j^* c_1(N_{E/S}) - d_j \psi_j} \,.$$

As $c_1(N_{E/S}) \cup \omega = 0$ in $H^\bullet(E)$,

$$C_v \prod_{j=1}^n \mathrm{ev}_j^* \omega = e_{\mathbb{C}^*}(-R^\bullet \pi_* f^* N) \prod_{j=1}^n \frac{t(-1)^{d_j} d_j}{t - d_j \psi_j} \mathrm{ev}_j^* \omega$$

and so the contribution of $\overline{\mathcal{M}}_{h,n}(E, d_E)$ is indeed $N_{h,d_E}^{E,\mathrm{tw}}(\mathbf{d}) \prod_{j=1}^n (-1)^{d_j} d_j$. □

Expanding the denominator in the formula for $N_{h,d_E}^{E,\mathrm{tw}}(\mathbf{d})$ and using that $\dim[\overline{\mathcal{M}}_{h,n}(E, d_E)]^{\mathrm{vir}} = 2h - 2 + n$, we get

$$N_{h,d_E}^{E,\mathrm{tw}}(\mathbf{d}) = \sum_{\substack{\mathbf{a}=(a_1,\dots,a_n) \\ a_j \geq 0\,, \sum_j a_j \leq 2h-2}} N_{h,d_E,\mathbf{a}}^{E,\mathrm{tw}} \prod_{j=1}^n d_j^{a_j} \,,$$

where, for every $\mathbf{a} = (a_1, \dots, a_n)$,

$$N_{h,d_E,\mathbf{a}}^{E,\mathrm{tw}} := \frac{1}{t^{\sum_{j=1}^n a_j}} \int_{[\overline{\mathcal{M}}_{h,n}(E,d_E)]^{\mathrm{vir}}} \left(\prod_{j=1}^n \mathrm{ev}_j^*(\omega) \psi_j^{a_j} \right) e_{\mathbb{C}^*}(-R^\bullet \pi_* f^* N) \,.$$

Therefore, we have

$$N_{g,\beta}^{K_S} = \frac{(-1)^{\beta \cdot E - 1}}{\beta \cdot E} N_{g,\beta}^{S/E} +$$

$$\sum_{n \geq 0} \sum_{\substack{g=h+g_1+\cdots+g_n \\ \beta = d_E[E]+\beta_1+\cdots+\beta_n \\ d_E \geq 0,\, \beta_j \cdot E > 0}} \sum_{\substack{\mathbf{a}=(a_1,\dots,a_n) \\ a_j \geq 0\,, \sum_j a_j \leq 2h-2}} \frac{N_{h,d_E,\mathbf{a}}^{E,\mathrm{tw}}}{|\mathrm{Aut}(\boldsymbol{\beta}, \mathbf{g})|} \prod_{j=1}^n \left((-1)^{\beta_j \cdot E} (\beta_j \cdot E)^{a_j+1} N_{g_j,\beta_j}^{S/E} \right)$$

Using generating series, we package the above recursive formula as follows. Let

$$\bar{F}_g^{K_S} := \sum_{\substack{\beta \\ \beta\cdot E>0}} N_{g,\beta}^{K_S} Q^\beta \,, \tag{43}$$ eq:F_bar_K_S

$$\bar{F}_g^{S/E} := \sum_{\substack{\beta \\ \beta\cdot E>0}} \frac{(-1)^{\beta\cdot E+g-1}}{\beta\cdot E} N_{g,\beta}^{S/E} Q^\beta \,, \tag{44}$$ eq:F_bar_SE

$$F_{h,\mathbf{a}}^{E,\mathrm{tw}} := -\frac{\delta_{h,1}\delta_{n,0}}{24}\log((-1)^{E\cdot E}\tilde{\mathcal{Q}}) + \sum_{d_E\geq 0} N_{h,d_E,\mathbf{a}}^{E,\mathrm{tw}} ((-1)^{E\cdot E}\tilde{\mathcal{Q}})^{d_E} \,, \tag{45}$$ eq:F_E_tw

where the variable $\tilde{\mathcal{Q}}$ in $F_{h,\mathbf{a}}^{E,\mathrm{tw}}$ is related to the variable Q in $\bar{F}_g^{S/E}$ and $\bar{F}_g^{K_S}$ by the formula (5).

prop_gen_series_bar **Proposition 3.2.**

$$\bar{F}_g^{K_S} = (-1)^g \bar{F}_g^{S/E} + \frac{\delta_{g,1}}{24}\log Q^E +$$

$$\sum_{n\geq 0} \sum_{\substack{g=h+g_1+\cdots+g_n, \\ \mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{Z}_{\geq 0}^n \\ (a_j,g_j)\neq(0,0),\, \sum_{j=1}^n a_j\leq 2h-2}} \frac{F_{h,\mathbf{a}}^{E,\mathrm{tw}}}{|\operatorname{Aut}(\mathbf{a},\mathbf{g})|} \prod_{j=1}^n (-1)^{g_j-1} D^{a_j+2} \bar{F}_{g_j}^{S/E}$$

Proof. Given $\mathbf{g}=(g_1,\ldots,g_n)$ and $\mathbf{a}=(a_1,\ldots,a_n)$, we have a disjoint sum decomposition

$$\{1,\ldots,n\} = I_{\mathbf{g},\mathbf{a}} \coprod J_{\mathbf{g},\mathbf{a}}$$

where $I_{\mathbf{g},\mathbf{a}}$ is the subset of j such $(g_j,a_j)\neq(0,0)$ and $J_{\mathbf{g},\mathbf{a}}$ is the subset of j such that $(g_j,a_j)=0$. Denote $\mathbf{a}'=(a_j)_{j\in I_{\mathbf{g},\mathbf{a}}}$. If $a_j=0$, then there is no insertion of ψ_j in $N_{h,d_E,\mathbf{a}}^{E,\mathrm{tw}}$, we can remove $\mathrm{ev}_j^*(\omega)$ from the integral using the divisor equation and so we have $N_{h,d_E,\mathbf{a}}^{E,\mathrm{tw}} = (\prod_{j\in J_{\mathbf{g},\mathbf{a}}} d_E) N_{h,d_E,\mathbf{a}'}^{E,\mathrm{tw}}$. There is one exception: we cannot apply the divisor equation if $d_E=0$, $n=1$, $h=0$, $g_1=0$ and $a_1=0$, in which case

$$N_{1,0,(0)}^{E,\mathrm{tw}} = \int_{[\overline{\mathcal{M}}_{1,1}(E,0)]^{\mathrm{vir}}} \mathrm{ev}^*(\omega) = -\frac{1}{24} \,.$$

It follows that the correct general relation is

$$N_{h,d_E,\mathbf{a}}^{E,\mathrm{tw}} = -\frac{\delta_{h,1}\delta_{n,1}\delta_{\mathbf{a},(0)}}{24} + N_{h,d_E,\mathbf{a}'}^{E,\mathrm{tw}} \prod_{j\in J_{\mathbf{g},\mathbf{a}}} d_E \,. \tag{46}$$ eq_relation

It follows that we can replace the sum over $\mathbf{a}$ by a sum over $\mathbf{a}'$.

After summing over β to form generating series, the factors indexed by $j\in J_{g,\mathbf{a}}$ are absorbed in $F_{h,\mathbf{a}}^{E,\mathrm{tw}}$ via the change of variables $Q\mapsto\mathcal{Q}$. Indeed, according to formula 7, we have

$$\tilde{\mathcal{Q}} = (-1)^{E\cdot E}\exp(-D^2 F_0^{S/E}) = (-1)^{E\cdot E} Q^E \exp(-D^2 \bar{F}_0^{S/E})$$

and so

$$((-1)^{E\cdot E}\tilde{\mathcal{Q}})^{d_E} = Q^{d_E E}\sum_{l\geq 0}\frac{1}{l!}(-1)^l (D^2\bar{F}_0^{S/E})^l\,.$$

□

Recall from (2) and (3) that

$$F_g^{K_S} = -\frac{\delta_{(E\cdot E),0}}{3!(E\cdot E)^2}(\log Q^E)^3\delta_{g,0} + \left(\frac{\delta_{(E\cdot E),0}}{(E\cdot E)}\frac{\chi(S)}{24} - \frac{1}{24}\right)(\log Q^E)\delta_{g,1} + \bar{F}_g^{K_S}$$

and

$$F_g^{S/E} = -\frac{\delta_{(E\cdot E),0}}{3!(E\cdot E)^2}(\log Q^E)^3\delta_{g,0} - \frac{\delta_{(E\cdot E),0}}{(E\cdot E)}\frac{\chi(S)}{24}(\log Q^E)\delta_{g,1} + \bar{F}_g^{S/E}\,.$$

prop_gen_series **Proposition 3.3.**

$$F_g^{K_S} = (-1)^g F_g^{S/E} + \sum_{n\geq 0}\sum_{\substack{g=h+g_1+\cdots+g_n,\\ \mathbf{a}=(a_1,\dots,a_n)\in\mathbb{Z}_{\geq 0}^n\\ (a_j,g_j)\neq(0,0),\,\sum_{j=1}^n a_j\leq 2h-2}}\frac{F_{h,\mathbf{a}}^{E,\mathrm{tw}}}{|\operatorname{Aut}(\mathbf{a},\mathbf{g})|}\prod_{j=1}^n(-1)^{g_j-1}\left(D^{a_j+2}F_{g_j}^{S/E} + (E\cdot E)\delta_{g_j,0}\delta_{a_j,1}\right)\,.$$

Proof. We rewrite Proposition 3.2 in terms of the series $F_g^{K_S}$ and $F_g^{S/E}$. One needs to use that $F_1^{K_S} - \bar{F}_1^{K_S} = -(F_1^{S/E} - \bar{F}_1^{S/E}) - \frac{1}{24}\log Q^E$ and $D^{a+2}\bar{F}_0^{S/E} = D^{a+2}F_0^{S/E} + (E\cdot E)\delta_{a,1}$ for $a\geq 1$. □

3.2. **Twisted Gromov-Witten theory of the elliptic curve.** In this section, we compute the twisted Gromov-Witten series $F_{g,\mathbf{a}}^{E,\mathrm{tw}}$ of the elliptic curve in terms of the untwisted Gromov-Witten series $F_{g,\mathbf{a}}^{E}$. Recall from (45) that

$$F_{g,\mathbf{a}}^{E,\mathrm{tw}} = -\frac{\delta_{g,1}\delta_{n,0}}{24}\log((-1)^{E\cdot E}\tilde{\mathcal{Q}}) + \bar{F}_{g,\mathbf{a}}^{E,\mathrm{tw}}$$

where

$$\bar{F}_{g,\mathbf{a}}^{E,\mathrm{tw}} := \sum_{d_E\geq 0}\frac{((-1)^{E\cdot E}\tilde{\mathcal{Q}})^{d_E}}{t^{\sum_{j=1}^n a_j}}\int_{[\overline{\mathcal{M}}_{g,n}(E,d_E)]^{\mathrm{vir}}}\left(\prod_{j=1}^n \mathrm{ev}_j^*(\omega)\psi_j^{a_j}\right)e_{\mathbb{C}^*}(-R^\bullet\pi_* f^* N)\,,$$

and from (4) that

$$F_{g,\mathbf{a}}^{E} = -\frac{\delta_{g,1}\delta_{n,0}}{24}\log((-1)^{E\cdot E}\tilde{\mathcal{Q}}) + \bar{F}_{g,\mathbf{a}}^{E}$$

where

$$\bar{F}_{g,\mathbf{a}}^{E} := \sum_{d_E\geq 0}\tilde{\mathcal{Q}}^{d_E}\int_{[\overline{\mathcal{M}}_{g,n}(E,d_E)]^{\mathrm{vir}}}\prod_{j=1}^n \mathrm{ev}_j^*(\omega)\psi_j^{a_j}\,.$$

prop_localcurve

Proposition 3.4. *For every* $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{Z}^n_{\geq 0}$ *such that* $\sum_{j=1}^n a_j \leq 2g-2$*, the twisted Gromov-Witten theory of the elliptic curve is related with the untwisted one via*

$$\bar{F}^{E,\mathrm{tw}}_{g,\mathbf{a}} = (-1)^{g-1} \frac{(E \cdot E)^m}{m!} \bar{F}^{E}_{g,(\mathbf{a},1^m)}$$

where $m := 2g-2-\sum_{j=1}^n a_j$ *and* $(\mathbf{a}, 1^m) = (a_1, \ldots, a_n, \underbrace{1, \cdots, 1}_{m})$.

The proof of Proposition 3.4 takes the remaining of this section.

Let W be a rank r vector bundle over a projective variety X. Let ρ_i be the Chern roots of W. We fix a fiberwise action of $\mathbb{C}^*$ on W and we denote by λ_i the corresponding equivariant parameters.

Coates and Givental [CG07] have expressed in terms of the Givental's quantization formalism the computation of the W-twisted Gromov-Witten theory in terms of the untwisted one obtained by applying the Grothendiech-Riemann-Roch to the universal curve over the stable maps moduli spaces [Mum83, FP00].

More precisely, according to the main result[4] of [CG07], (the stable part of) the generating function of the W-twisted Gromov-Witten invariants

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{\mathrm{vir}}} \left(\prod_{j=1}^n \mathrm{ev}_j^*(\tau_j) \psi_j^{a_j} \right) e_{\mathbb{C}^*}(-R^\bullet \pi_* f^* W)$$

can be computed via the quantization of the symplectic operator

$$\Delta(z) := \prod_{i=1}^r \sqrt{\lambda_i + \rho_i} \, \exp\Big(- \sum_{m>0} \frac{B_{2m}}{2m(2m-1)} \frac{z^{2m-1}}{(\lambda_i+\rho_i)^{2m-1}} \Big)$$

together with the quantization of the symplectic operator

$$\Psi(z) := \prod_{i=1}^r \exp\Big(- \frac{\rho_i \ln \lambda_i}{z} + \frac{1}{z} \sum_{k>0} \frac{(-1)^{k-1} \rho_i^{k+1}}{k(k+1)\lambda_i^k} \Big)$$

acting on the untwisted Gromov-Witten potential.

For the case relevant to Proposition 3.4, W is the rank 2 vector bundle $N = N_{E/S} \oplus N^\vee_{E/S}$ and we have

$$\lambda_1 = t, \quad \rho_1 = (E \cdot E)\,\omega, \quad \lambda_2 = -t, \quad \rho_2 = -(E \cdot E)\,\omega \,.$$

In this case, the expressions for the symplectic operators become much simpler. Indeed, most of the terms are odd under $\rho_i \mapsto -\rho_i$, $\lambda_i \mapsto -\lambda_i$ and so

[4] Theorem 1 of [CG07] computes the theory twisted by an arbitrary multipicative characteristic class $\exp(s_k \, \mathrm{ch}_k)$ and Corollary 1 of [CG07] describes the twist by $e_{\mathbb{C}^*}(R^\bullet \pi_* f^* N)$. In order to get the twist by $e_{\mathbb{C}^*}(-R^\bullet \pi_* f^* N) = 1/e_{\mathbb{C}^*}(R^\bullet \pi_* f^* N)$, we have to flip the sign of the coefficients s_k.

most of the terms with $i = 1$ cancel pairwise with the terms with $i = 2$. We have

$$\Psi(z) = \exp\left(\frac{\log(-1)(E\cdot E)\,\omega}{z}\right)$$

and $\Delta(z)$ does not depend on z:

$$\Delta(z) = \sqrt{-1}(t + (E\cdot E)\,\omega).$$

Using the divisor equation, we see that the operator Ψ acts like the change of variables (c.f. Remarks under Theorem 1′ in [CG07])

$$\tilde{\mathcal{Q}} \to \tilde{\mathcal{Q}}\, e^{\log(-1)(E\cdot E)\int_E \omega} = (-1)^{(E\cdot E)}\tilde{\mathcal{Q}}\,.$$

The action of $\Delta(z)$ can be viewed as a R-matrix action on the CohFT defined via the Gromov-Witten theory of elliptic curve.

We recall the definition of R-matrix (quantization) action. Suppose we have two symplectic vector spaces V, V', and with units 1, $1'$, and a symplectic transformation $R(z) \in \mathrm{Hom}(V, V')[[z]]$. For each $2g - 2 + n > 0$, let $F_{g,n}(-) = \langle - \rangle_{g,n}$ be the (given) correlation functions with insertions in $V[[z]]$. We define $\hat{R}F_{g,n}$ by the following graph sum

$$\hat{R}F_{g,n}(\tau_1\psi_1^{k_1}, \cdots, \tau_n\psi_n^{k_n}) := \sum_{\Gamma\in G_{g,n}} \frac{1}{\mathrm{Aut}(\Gamma)}\mathrm{Cont}_\Gamma$$

where $G_{g,n}$ is the set of stable graphs with genus g and n legs, and the contributions Cont_Γ are defined via the following construction:

- at each leg l of Γ, we place an insertion
$$R^{-1}(\psi_l)\tau_l\psi_l^{k_l};$$
- at each edge $e = (v_1, v_2)$ of Γ, we place a bi-vector as a two-direction insertion
$$\frac{\sum_\alpha e_\alpha \otimes e^\alpha - \sum_\alpha R^{-1}(\psi_{(e,v_1)})e'_\alpha \otimes R^{-1}(\psi_{(e,v_2)})e'^\alpha}{\psi_{(e,v_1)} + \psi_{(e,v_2)}}$$
where $\{e_\alpha\}, \{e'_\alpha\}$ are arbitrary bases of V, V', with the dual basis $\{e^\alpha\}$ and $\{e'^\alpha\}$, respectively;
- at each vertex v of Γ, we place the map
$$\tau'_1\psi_1^{l_1} \otimes \cdots \otimes \tau'_{n_v}\psi_{n_v}^{l_{n_v}} \mapsto$$
$$\sum_{k\geq 0} \frac{1}{k!} F_{g_v, n_v+k}\big(\tau'_1\psi_1^{l_1}, \cdots, \tau'_{n_v}\psi_{n_v}^{l_{n_v}}, T(\psi_{n_v+1}), \cdots, T(\psi_{n_v+k})\big),$$
where $(\tau'_i, l_i) \in \{(\tau_1, k_1), \cdots, (\tau_n, k_n)\}$, and $T(z) := z\mathbf{1} - z\,R^{-1}(z)\mathbf{1}'$.

In our case, $V = H^\bullet(E)((t))$ with pairing $(\alpha, \beta) = \int_E \alpha \wedge \beta$ and $V' = H^\bullet(E)((t))$ with pairing $(\alpha,\beta)' = \int_E \frac{\alpha}{\sqrt{-1}(t+(E\cdot E)\omega)} \wedge \frac{\beta}{\sqrt{-1}(t+(E\cdot E)\omega)}$. The R-matrix $R = \Delta$ is independent of z and so the edge bi-vector is simply zero.

Hence the contribution is non-vanishing only if the graph is a single vertex. Namely

$$\langle \tau_1\psi_1^{a_1},\cdots,\tau_n\psi_n^{a_n}\rangle_{g,n}^{E,\mathrm{tw}} = \sum_{k\geq 0}\frac{1}{k!}\left\langle R^{-1}(\psi_1)\tau_1\psi_1^{a_1},\cdots,R^{-1}(\psi_n)\tau_n\psi_n^{a_n},T(\psi_{n+1}),\cdots,T(\psi_{n+k})\right\rangle_{g,n+k}^{E}|_{\tilde{\mathcal{Q}}\mapsto(-1)^{(E\cdot E)}\tilde{\mathcal{Q}}}$$

where $\tau_j\in H^\bullet(E)$,

$$\langle \tau_1\psi_1^{a_1},\cdots,\tau_n\psi_n^{a_n}\rangle_{g,n}^{E,\mathrm{tw}} := \sum_{d_E\geq 0}\tilde{\mathcal{Q}}^{d_E}\int_{[\overline{\mathcal{M}}_{g,n}(E,d_E)]^{\mathrm{vir}}}\left(\prod_{j=1}^n\psi_j^{a_j}\,\mathrm{ev}_j^*(\tau_j)\right)e_{\mathbb{C}^*}(-R^\bullet\pi_*f^*N),$$

$$\langle \tau_1\psi_1^{a_1},\cdots,\tau_n\psi_n^{a_n}\rangle_{g,n}^{E} := \sum_{d_E\geq 0}\tilde{\mathcal{Q}}^{d_E}\int_{[\overline{\mathcal{M}}_{g,n}(E,d)]^{\mathrm{vir}}}\prod_{j=1}^n\psi_j^{a_j}\,\mathrm{ev}_i^*(\tau_j)\,.$$

We have

$$R(z)=\Delta(z)=\sqrt{-1}(t+(E\cdot E)\omega)\,,$$
$$R^{-1}(z)=\frac{1}{\sqrt{-1}t}-\frac{(E\cdot E)\omega}{\sqrt{-1}t^2}\,,$$
$$T(z)=z\left(1-\frac{1}{\sqrt{-1}t}+\frac{(E\cdot E)\omega}{\sqrt{-1}t^2}\right)\,.$$

We focus on the case where all the insertions τ_i are point classes ω. We have

$$\langle \omega\psi_1^{a_1},\cdots,\omega\psi_n^{a_n}\rangle_{g,n}^{E,\mathrm{tw}} = \sum_{k\geq 0}\frac{1}{k!}\left\langle R^{-1}(\psi_1)\omega\psi_1^{a_1},\cdots,R^{-1}(\psi_n)\omega\psi_n^{a_n},T(\psi_{n+1}),\cdots,T(\psi_{n+k})\right\rangle_{g,n+k}^{E}|_{\tilde{\mathcal{Q}}\mapsto(-1)^{(E\cdot E)}\tilde{\mathcal{Q}}}\,.$$

For every j, we have $R^{-1}(\psi_j)\omega=\frac{1}{\sqrt{-1}t}\omega$ and so

$$\langle \omega\psi_1^{a_1},\cdots,\omega\psi_n^{a_n}\rangle_{g,n}^{E,\mathrm{tw}} = \frac{1}{(\sqrt{-1}t)^n}\sum_{k\geq 0}\frac{1}{k!}\langle \omega\psi_1^{a_1},\cdots,\omega\psi_n^{a_n},T(\psi_{n+1}),\cdots,T(\psi_{n+k})\rangle_{g,n+k}^{E}|_{\tilde{\mathcal{Q}}\mapsto(-1)^{(E\cdot E)}\tilde{\mathcal{Q}}}\,.$$

Writing

$$T(\psi_j)=\left(1-\frac{1}{\sqrt{-1}t}\right)\psi_j+\frac{(E\cdot E)}{\sqrt{-1}t^2}\psi_j\omega$$

and expanding, we obtain

$$\langle \omega\psi_1^{a_1},\cdots,\omega\psi_n^{a_n}\rangle_{g,n}^{E,\mathrm{tw}} = \frac{1}{(\sqrt{-1}t)^n}\sum_{m,l\geq 0}\frac{1}{m!l!}\left(\frac{(E\cdot E)}{\sqrt{-1}t^2}\right)^m\left(1-\frac{1}{\sqrt{-1}t}\right)^l\times$$
$$\langle \omega\psi_1^{a_1},\cdots,\omega\psi_n^{a_n},\omega\psi_{n+1},\cdots,\omega\psi_{n+m},\psi_{n+m+1},\cdots,\psi_{n+m+l}\rangle_{g,n+m+l}^{E}|_{\tilde{\mathcal{Q}}\mapsto(-1)^{(E\cdot E)}\tilde{\mathcal{Q}}}\,.$$

The sum over l can be evaluated using the dilaton equation and the binomial theorem

$$\begin{aligned}&\sum_{l\geq 0}\frac{1}{l!}\left(1-\frac{1}{\sqrt{-1}t}\right)^{l}\langle -,\psi_{n+m+1},\cdots,\psi_{m+m+l}\rangle^{E}_{g,n+m+l}\\ =&\sum_{l\geq 0}\binom{-(2g-2+n+m)}{l}\left(\frac{1}{\sqrt{-1}t}-1\right)^{l}\langle -\rangle^{E}_{g,n+m}\\ =&\left(\frac{1}{\sqrt{-1}t}\right)^{-(2g-2+n+m)}\langle -\rangle^{E}_{g,n+m}\,.\end{aligned}$$

Hence, collecting the powers of $\sqrt{-1}$ and t, we obtain

$$\langle \omega\psi_1^{a_1},\cdots,\omega\psi_n^{a_n}\rangle^{E,\mathrm{tw}}_{g,n}=(\sqrt{-1})^{2g-2}\times$$
$$\sum_{m\geq 0}\frac{(E\cdot E)^m}{m!}t^{2g-2-m}\langle \omega\psi_1^{a_1},\cdots,\omega\psi_n^{a_n},\omega\psi_{n+1},\cdots,\omega\psi_{n+m}\rangle^{E}_{g,n+m}|_{\tilde{\mathcal{Q}}\mapsto(-1)^{(E\cdot E)}\tilde{\mathcal{Q}}}.$$

By dimension constraint, the correlator in the sum is non-vanishing only if $m=2g-2-\sum_{j=1}^n a_i$. Therefore, we have

$$\begin{aligned}\langle \omega\psi_1^{a_1},\cdots,\omega\psi_n^{a_n}\rangle^{E,\mathrm{tw}}_{g,n}=(-1)^{g-1}t^{\sum_{j=1}^n a_j}\frac{(E\cdot E)^m}{m!}\times\\ \langle \omega\psi_1^{a_1},\cdots,\omega\psi_n^{a_n},\omega\psi_{n+1},\cdots,\omega\psi_{n+m}\rangle^{E}_{g,n+m}|_{\tilde{\mathcal{Q}}\mapsto(-1)^{(E\cdot E)}\tilde{\mathcal{Q}}}\end{aligned}$$

where $m=2g-2-\sum_{j=1}^n a_i$. This concludes the proof of Proposition 3.4.

3.3. **Conclusion of the proof of Theorem 1.2.** By Proposition 3.3, we have

$$F_g^{K_S}=(-1)^g F_g^{S/E}+$$
$$\sum_{n\geq 0}\sum_{\substack{g=h+g_1+\cdots+g_n,\\ \mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 0}\\ (a_j,g_j)\neq(0,0),\,\sum_{j=1}^n a_j\leq 2h-2}}\frac{1}{|\operatorname{Aut}(\mathbf{a},\mathbf{g})|}F^{E,\mathrm{tw}}_{h,\mathbf{a}}\prod_{j=1}^n(-1)^{g_j-1}\left(D^{a_j+2}\bar{F}^{S/E}_{g_j}+(E\cdot E)\delta_{g_j,0}\delta_{a_j,1}\right).$$

For every $\mathbf{g}=(g_j)_j$ and $\mathbf{a}=(a_j)_j$ such that $(a_j,g_j)\neq(0,0)$ for every j and $\sum_j a_j\leq 2h-2$, we denote $m=2h-2-\sum_{j=1}^n a_j$ and we define

$$\tilde{\mathbf{g}}\coloneqq(g_1,\cdots,g_n,\underbrace{0,\cdots,0}_{m})$$

and

$$\tilde{\mathbf{a}}\coloneqq(a_1,\cdots,a_n,\underbrace{1,\cdots,1}_{m})\,.$$

We have $(\tilde{g}_j,\tilde{a}_j)\neq(0,0)$ for every j and $\sum_j\tilde{a}_j=2h-2$.

According to Proposition 3.4, we have

$$F^{E,\mathrm{tw}}_{h,\mathbf{a}}=(-1)^{h-1}\frac{(E\cdot E)^m}{m!}F^{E}_{h,\tilde{\mathbf{a}}}\,.$$

Therefore, we can rewrite the sum over $\mathbf{a} = (a_j)_j$ with $\sum_j a_j \leq 2h-2$ as a sum over $\mathbf{a} = (\bar{a}_j)_j$ with $\sum_j \bar{a}_j = 2h-2$. More precisely, let $\bar{g} = (\bar{g}_j)_j$ and $\bar{a} = (\bar{a}_j)$ be of the form

$$\bar{\mathbf{g}} = (\bar{g}_1, \cdots, \bar{g}_k, \underbrace{0, \cdots, 0}_{l})$$

$$\bar{\mathbf{a}} = (\bar{a}_1, \ldots, \bar{a}_k, \underbrace{1, \cdots, 1}_{l})$$

with $(\bar{a}_j, \bar{g}_j) \neq (0,0), (1,0)$ for $1 \leq j \leq k$, $g = h + \sum_{j=1}^k \bar{g}_j$ and $\sum_{j=1}^{k+l} \bar{a}_j = 2h-2$. The total contribution of $(\bar{\mathbf{a}}, \bar{\mathbf{g}})$ in the rewritten formula is the sum of contributions of $(\mathbf{a}, \mathbf{g})$ with $(\tilde{\mathbf{a}}, \tilde{\mathbf{g}}) = (\bar{\mathbf{a}}, \bar{\mathbf{g}})$. Such $(\mathbf{a}, \mathbf{g})$ is of the form

$$\mathbf{g} = (\bar{g}_1, \cdots, \bar{g}_k, \underbrace{0, \cdots, 0}_{l-m})$$

$$\mathbf{a} = (\bar{a}_1, \ldots, \bar{a}_k, \underbrace{1, \cdots, 1}_{l-m})$$

with $0 \leq m \leq l$. Using that

$$|\operatorname{Aut}(\mathbf{a}, \mathbf{g})| = |\operatorname{Aut}(\{(\bar{a}_1, \bar{g}_1), \cdots, (\bar{a}_k, \bar{g}_k)\})|(l-m)!\,,$$

we get that the total contribution of $(\bar{\mathbf{a}}, \bar{\mathbf{g}})$ is given by

$$\frac{(-1)^{h-1} F_{h,\bar{a}}^E}{|\operatorname{Aut}(\{(\bar{a}_1, \bar{g}_1), \cdots, (\bar{a}_k, \bar{g}_k)\})|} \left(\prod_{j=1}^k (-1)^{\bar{g}_j - 1} D^{\bar{a}_j + 1} F_{\bar{g}_j}^{S/E} \right) \times$$
$$\sum_{m=0}^{l} (-1)^{l-m} \frac{(E \cdot E)^m (D^3 F_0^{S/E} + (E \cdot E))^{l-m}}{m!(l-m)!}.$$

Using the binomial theorem, this can be rewritten as

$$\frac{(-1)^{h-1} F_{h,\bar{a}}^E}{|\operatorname{Aut}(\{(\bar{a}_1, \bar{g}_1), \cdots, (\bar{a}_k, \bar{g}_k)\})|} \left(\prod_{j=1}^k (-1)^{\bar{g}_j - 1} D^{\bar{a}_j + 1} F_{\bar{g}_j}^{S/E} \right) (-1)^l \frac{(D^3 F_0^{S/E})^l}{l!}\,.$$

As $|\operatorname{Aut}(\bar{\mathbf{a}}, \bar{\mathbf{g}})| = |\operatorname{Aut}(\{(\bar{a}_1, \bar{g}_1), \cdots, (\bar{a}_k, \bar{g}_k)\})| l!$, we finally obtain

$$\frac{(-1)^{h-1} F_{h,\bar{a}}^E}{|\operatorname{Aut}(\bar{\mathbf{a}}, \bar{\mathrm{g}})|} \left(\prod_{j=1}^{k+l} (-1)^{\bar{g}_j - 1} D^{\bar{a}_j + 1} F_{\bar{g}_j}^{S/E} \right)\,.$$

This ends the proof of Theorem 1.2.

sec:gw_elliptic_curve

3.4. **Stationary Gromov-Witten theory of the elliptic curve.** According to Theorem 1.2, the local series $F_g^{K_S}$ and the relative series $F_g^{S/E}$ determine each other through the stationary series $F_{g,\mathbf{a}}^E$ of the elliptic curve. In this section, we review the computation by Okounkov and Pandharipande [OP06a, §5] of the stationary Gromov-Witten theory of the elliptic curve.

Recall that, for every $(a_1, \dots, a_n) \in \mathbb{Z}_{\geq 0}^n$, we denote by

$$\bar{F}_{g,\mathbf{a}}^E := \sum_{d_E \geq 0} \tilde{\mathcal{Q}}^{d_E} \int_{[\overline{\mathcal{M}}_{g,n}(E,d_E)]^{\mathrm{vir}}} \prod_{j=1}^n \mathrm{ev}_j^*(\omega) \psi_j^{a_j}$$

the generating series of stationary Gromov-Witten invariants of the elliptic curve E.

For every $k \geq 1$, we consider the Eisenstein series

$$E_{2k}(\tilde{\tau}) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \frac{n^{2k-1} \tilde{\mathcal{Q}}^n}{1 - \tilde{\mathcal{Q}}^n}.$$

where $\tilde{\mathcal{Q}} = e^{2i\pi\tilde{\tau}}$ and the Benoulli numbers B_{2k} are defined by $\frac{t}{e^t - 1} = \sum_{n \geq 0} B_n \frac{t^n}{n!}$. As functions on the upper half-plane $\{\tilde{\tau} \in \mathbb{C} | \operatorname{Im} \tilde{\tau} > 0\}$, E_{2k} is modular of weight $2k$ for $SL(2,\mathbb{Z})$ for every $k \geq 2$, and E_2 is quasimodular of weight 2 for $SL(2,\mathbb{Z})$. The ring $\mathbb{C}[E_2, E_4, E_6]$, graded by the weight, is exactly the graded ring $\mathrm{QMod}(SL(2,\mathbb{Z}))$ of quasimodular forms for $SL(2,\mathbb{Z})$ [Zag08]. For every $k \geq 0$, we denote by $\mathbb{C}[E_2, E_4, E_6]_k$ the weight k subspace of $\mathbb{C}[E_2, E_4, E_6]$.

quasimod_eliptic_curve

Theorem 3.5 (Okounkov-Pandharipande [OP06a])**.** *For every $g \geq 0$, $n \geq 1$ and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}_{\geq 0}^n$, we have*

$$\bar{F}_{g,\mathbf{a}}^E \in \mathbb{Q}[E_2, E_4, E_6]_{\sum_{j=1}^n (a_j+2)} .$$

In fact, Okounkov and Pandharipandes give an explicit formula computing $\bar{F}_{g,\mathbf{a}}^E$ as a polynomial in E_2, E_4, E_6.

For every $n \geq 1$, let

$$F(z_1, \cdots, z_n) := \delta_{1,n} z_1^{-1} + \sum_{\substack{\mathbf{a}=(a_1,\cdots,a_n) \in \mathbb{Z}_{\geq 0}^n, \\ 2g-2 = \sum_{i=j}^n a_j}} \bar{F}_{g,\mathbf{a}}^E \cdot z^{a_1+1} \cdots z^{a_n+1}$$

and let $F^\bullet(z_1, \cdots, z_n)$ be the disconnected generating function defined by [5]

$$F^\bullet(z_1, \cdots, z_n) := \sum_{k>0} \sum_{\substack{I_1 \sqcup \cdots \sqcup I_k = \{1,\cdots,n\} \\ I_j \neq \emptyset}} \frac{1}{|\operatorname{Aut}(I_1, \cdots, I_k)|} \prod_{j=1}^k F(z_{I_j}) .$$

[5]The condition $I_j \neq \emptyset$ excludes genus-1 unmarked connected components and so only series $F(z_1, \dots, z_n)$ with $n \geq 1$ enter the formula. For $n = 0$, the series $-\frac{1}{24} + \bar{F}_{1,\emptyset}^E = -\log \eta$ is not quasimodular.

This relation can be inverted to compute the connected series $F(z_1, \dots, z_n)$ in terms of the disconnected ones.

We introduce the odd theta function

$$\vartheta(\tilde{\tau}, z) := \vartheta_{\frac{1}{2},\frac{1}{2}}(\tau, z) = \sum_{k=-\infty}^{\infty} (-1)^k e^{(k+\frac{1}{2})z} e^{\pi i \tilde{\tau}(k+\frac{1}{2})^2}$$

and following Bloch and Okounkov [BO00] we denote

$$\begin{aligned}\Theta(z) &:= \eta(\tilde{\tau})^{-3}\vartheta(\tilde{\tau}, z) \\ &= z \exp\left(\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} E_{2k}(\tilde{\tau}) z^{2k}\right)\end{aligned}$$

thm_ellipticGW **Theorem 3.6** (Okounkov-Pandharipande [OP06a]). *For every $n \geq 1$, we have*

$$F^\bullet(z_1, \cdots, z_n) = \sum_{\substack{\sigma \text{ permutation} \\ \text{of } z_1 \cdots, z_n}} \frac{\det\left[\frac{\Theta^{(j-i+1)}(z_{\sigma(1)}+\cdots+z_{\sigma(n-j)})}{(j-i+1)!}\right]_{i,j=1}^{n}}{\Theta(z_{\sigma(1)})\Theta(z_{\sigma(1)}+z_{\sigma(2)})\cdots\Theta(z_{\sigma(1)}+\cdots+z_{\sigma(n)})}$$

where $\Theta^{(k)}$ is the k-th derivative of Θ with respect to z and $\frac{\Theta^{(j-i+1)}}{(j-i+1)!}$ is interpreted as 0 if $j-i+1<0$.

For example, using that $\Theta'(0) = 1$ and $\Theta''(0) = 0$, we get

$$\begin{aligned}F(z) = F^\bullet(z) = \frac{1}{\Theta(z)} &= z^{-1} - \frac{E_2}{24} z + \left(\frac{E_4}{2880} + \frac{{E_2}^2}{1152}\right) z^3 \\ &\quad - \left(\frac{E_6}{181440} + \frac{E_2 E_4}{69120} + \frac{{E_2}^3}{82944}\right) z^5 + \cdots \\ F(z_1, z_2) &= F^\bullet(z_1, z_2) - F^\bullet(z_1) F^\bullet(z_2) \\ &= \frac{1}{\Theta(z_1+z_2)}\left(\frac{\Theta'(z_1)}{\Theta(z_1)} + \frac{\Theta'(z_2)}{\Theta(z_2)}\right) - \frac{1}{\Theta(z_1)\Theta(z_2)} \\ &= -\frac{(E_2^2 - E_4) z_1 z_2}{288} + \frac{\left(5\,{E_2}^3 - 3\,E_2E_4 - 2\,E_6\right) z_1^2 z_2^2}{25920} \\ &\quad + \frac{\left(5\,{E_2}^3 - E_2E_4 - 4\,E_6\right)(z_1 z_2^3 + z_1^3 z_2)}{34560} + \cdots\end{aligned}$$

Hence, we obtain

$$F^E_{1,(0)} = -\frac{E_2}{24}, \tag{47}$$ eq_F10

$$F_{1,(0^2)} = -\frac{1}{12 \cdot 24}(E_2^2 - E_4), \tag{48}$$ eq_F100

$$F^E_{2,(1^2)} = -\frac{1}{9^2 \cdot 320}(2E_6 + 3E_2E_4 - 5E_2^3), \tag{49}$$ eq_F211

$$F^E_{2,(2)} = \frac{1}{9 \cdot 640}(2E_4 + 5E_2^2). \tag{50}$$ eq_F22

4. Finite generation and quasimodularity for $(\mathbb{P}^2, E)$

ssetifig:quasimod

4.1. Quasimodular forms for $\Gamma_1(3)$. We refer to [DS05, Zag08] for basics on modular and quasimodular forms. For every Γ congruence subgroup of $SL(2,\mathbb{Z})$, we denote by $\mathrm{Mod}(\Gamma) = \bigoplus_{k\geq 0} \mathrm{Mod}(\Gamma)_k$ the ring of modular forms for Γ, graded by the weight, and by $\mathrm{QMod}(\Gamma) = \bigoplus_{k\geq 0} \mathrm{QMod}(\Gamma)_k$ the ring of quasimodular forms for Γ, also graded by the weight.

We focus on the congruence subgroups $\Gamma_1(3)$ and $\Gamma_0(3)$ defined by

$$\Gamma_1(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod 3 \right\},$$

$$\Gamma_0(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod 3 \right\}.$$

We have $\Gamma_1(3) \subset \Gamma_0(3)$, subgroup of index 2, with $-I \in \Gamma_0(3)$ and $-I \notin \Gamma_1(3)$. It follows that the modular (resp. quasimodular) forms for $\Gamma_0(3)$ are exactly the modular (resp. quasimodular) forms for $\Gamma_1(3)$ of even weight.

Let

$$A(\tau) := \left(\frac{\eta(\tau)^9}{\eta(3\tau)^3} + 27 \frac{\eta(3\tau)^9}{\eta(\tau)^3} \right)^{\frac{1}{3}} = 1 + 6\mathcal{Q} + 6\mathcal{Q}^3 + 6\mathcal{Q}^4 + 12\mathcal{Q}^7 + 6\mathcal{Q}^9 + \dots$$

$$C(\tau) := \frac{\eta(\tau)^9}{\eta(3\tau)^3} = 1 - 9\mathcal{Q} + 27\mathcal{Q}^2 - 9\mathcal{Q}^3 - 117\mathcal{Q}^4 + 216\mathcal{Q}^5 + 27\mathcal{Q}^6 - 450\mathcal{Q}^7 + \dots$$

where $\eta(\tau) := \mathcal{Q}^{\frac{1}{24}} \prod_{n=1}^\infty (1-\mathcal{Q}^n)$ is the Dedekind eta function and $\mathcal{Q} = e^{2i\pi\tau}$. It follows from the known modular properties of the eta function that A and C are modular forms for $\Gamma_1(3)$ respectively of weight 1 and 3. We refer to section 3 of [Mai11] for details[6]. The combination

$$\frac{A^3 - C}{27} = \frac{\eta(3\tau)^9}{\eta(\tau)^3} = \mathcal{Q} + 3\mathcal{Q}^2 + 9\mathcal{Q}^3 + 13\mathcal{Q}^4 + 24\mathcal{Q}^5 + 27\mathcal{Q}^6 + 50\mathcal{Q}^7 + \dots$$

is a cusp form of weight 3 for $\Gamma_1(3)$.

lem_alg_ind_A_C

Lemma 4.1. *The functions A and C are algebraically independent over $\mathbb{C}$.*

Proof. If there exists a non-trivial polynomial relation between A and C, then there exists a non-trivial polynomial relation between A and C which is homogeneous for the weight and so of the form $\sum_{n+3m=k} a_{nm} A^n C^m$ for some k. If such non-trivial relation existed, then, dividing by A^k, we would get that C/A^3 is solution of a non-trivial polynomial equation with complex coefficients and so C/A^3 would be constant, which is not the case as $C/A^3 = 1 - 27\mathcal{Q} + O(\mathcal{Q}^2)$. □

[6] In terms of the notation $\mathcal{A}_3$ and $\mathcal{B}_3$ of [Mai11], we have $A = \mathcal{A}_3$ and $C = \mathcal{B}_3^3$.

According to [DS05, Figure 3.4][7], the dimension of $\mathrm{Mod}(\Gamma_1(3))_k$ is $\lfloor \frac{k}{3} \rfloor + 1$. Therefore, Lemma 4.1 implies[8] that

$$\mathrm{Mod}(\Gamma_1(3)) = \mathbb{C}[A, C]\,.$$

By Proposition 20 of [Zag08], the Eisenstein series $E_2(\tau)$ for $SL(2,\mathbb{Z})$ is algebraically independent of $\mathrm{Mod}(\Gamma_1(3))$ over $\mathbb{C}$ and

$$\mathrm{QMod}(\Gamma_1(3)) = \mathrm{Mod}(\Gamma_1(3))[E_2]\,.$$

The difference $3E_2(3\tau) - E_2(\tau)$ is modular for $\Gamma_1(3)$. Using that the space of weight 2 modular forms for $\Gamma_1(3)$ is of dimension 1, we find that $3E_2(3\tau) - E_2(\tau) = 2A^2$. Therefore, we can use

$$B(\tau) \coloneqq \frac{1}{4}\big(E_2(\tau) + 3E_2(3\tau)\big) = 1 - 6\mathcal{Q} - 18\mathcal{Q}^2 - 42\mathcal{Q}^3 - 42\mathcal{Q}^2 + \cdots$$

instead of E_2 as depth 1 weight 2 generator and we have

$$\mathrm{QMod}(\Gamma_1(3)) = \mathbb{C}[A, B, C]\,.$$

The ring $\mathbb{Q}[A, B, C]$ is closed under the differential operator

$$\partial_\tau \coloneqq \tfrac{1}{2\pi i}\tfrac{d}{d\tau} = \mathcal{Q}\tfrac{d}{d\mathcal{Q}}.$$

To be explicit, ∂_τ maps the quasimodular forms of weight k to quasimodular forms of weight $k+2$ via the following Ramanujan type identities

$$\begin{gathered} \partial_\tau A = \frac{1}{6}A(B + A^2) - \frac{C}{3}, \\ \partial_\tau B = \frac{1}{6}(B^2 - A^4), \qquad \partial_\tau C = \frac{1}{2}C(B - A^2). \end{gathered} \tag{51}$$

eqn_Ramid

Each of these identities is easy to prove: as we know the modular properties of both sides, it is enough to identify finitely many terms of the $\mathcal{Q}$-expansions.

4.2. **Mirror of local $\mathbb{P}^2$ and quasimodular forms.** We review the relation between the mirror family of local $\mathbb{P}^2$ and modular forms, following [ASYZ14, Zho14, CI18].

Let $\mathbb{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}\,\tau > 0\}$ be the upper half-plane. We consider the modular curve $Y_1(3) = [\mathbb{H}/\Gamma_1(3)]$. It is a smooth orbifold, whose coarse moduli space can be identified with $\{q \in \mathbb{C} | q \neq -\frac{1}{27}, 0\} \cup \{\infty\}$, and with a single $\mathbb{Z}/3$-orbifold point at $q = \infty$. The modular curve $Y_1(3)$ has two cusps, given by $q = 0$ and $q = -\frac{1}{27}$ (corresponding respectively to the $\Gamma_1(3)$-equivalence classes of $\tau = i\infty$ and $\tau = 0$). In the context of mirror

[7] In [DS05], the formula is a priori valid only for $k \geq 2$. We get that $\dim \mathrm{Mod}(\Gamma_1(3))_1 = 1$ because it is > 0 by the existence of A, and ≤ 1 as $\dim \mathrm{Mod}(\Gamma_1(3))_2 = 1$.

[8] Therefore, A is the unique weight 1 modular form for $\Gamma_1(3)$ with constant term 1, and so can be described as the weight 1 Eisenstein series $E_1^{\psi,1} = 1 + 6\sum_{n\geq 1}\sum_{d|n}\psi(d)\mathcal{Q}^n$, where ψ is the non-trivial character of $(\mathbb{Z}/3)^* \simeq \{\pm 1\}$ (extended by $\psi(d) = 0$ if $3|d$), or as the theta series of the hexagonal lattice $\mathbb{Z}[\sqrt{-3}]$. In particular, the coefficient of $\mathcal{Q}^n$ in A is the number of $(x, y) \in \mathbb{Z}^2$ such that $|x + e^{\frac{2i\pi}{3}}y|^2 = x^2 - xy + y^2 = n$ (see [DS05, Exercise 4.11.5]).

symmetry, where $Y_1(3)$ is viewed as the stringy Kähler moduli space of local $\mathbb{P}^2$, the point $q=0$ is the large volume point, $q=-\frac{1}{27}$ is the conifold point and $q=\infty$ is the orbifold point.

The coordinate q on $Y_1(3)$ is expressed in terms of τ [Mai09][9] by

$$\frac{1}{1+27q} = 1 + 27\frac{\eta(3\tau)^{12}}{\eta(\tau)^{12}}\,, \tag{52}$$

eq:hauptmodul

that is, denoting $X \coloneqq (1+27q)^{-1}$, by

$$X = \frac{A^3}{C}\,. \tag{53}$$

The periods of the universal family of elliptic curves with $\Gamma_1(3)$-level structure are solution of the Picard-Fuchs equation [Mai11]

$$\left[\left(q\frac{d}{dq}\right)^2 - 3q\left(3\left(q\frac{d}{dq}\right)+1\right)\left(3\left(q\frac{d}{dq}\right)+2\right)\right]\Pi = 0\,. \tag{54}$$

PF_ODE

In Section 1.3, we defined the functions S, $I_{11} = q\frac{dI_1}{dq}$, $I_{12} = q\frac{dI_2}{dq}$ in terms of solutions I_1 and I_2 of the differential equation (11) describing genus 0 mirror symmetry for local $\mathbb{P}^2$. As the differential equation (11) is obtained from (54) by applying $q\frac{d}{dq}$ and flipping the sign of q, we deduce that τ, viewed as a multivalued function of q, is given by

$$\tau = \frac{1}{2} + \frac{1}{2\pi i}\frac{I_{12}(q)}{I_{11}(q)}\,, \tag{55}$$

that is,

$$\mathcal{Q} = e^{2\pi i\tau} = -\exp\left(\frac{I_{12}(q)}{I_{11}(q)}\right)\,. \tag{56}$$

eq:cQ_formula

One should not confuse the variables q, $\mathcal{Q}$, Q:

- q is such that $(1+27q)^{-1}$ is a globally defined coordinate on $Y_1(3)\,$,
- $\mathcal{Q} = e^{2i\pi\tau}$ is the flat modular coordinate for the family of elliptic curves parametrized by $Y_1(3)\,$,
- Q is the flat coordinate determined by the mirror of local $\mathbb{P}^2$.

The variables q and $\mathcal{Q}$ are related by (52)-(56). The variables q and Q are related by the mirror transformation (12).

According to [ASYZ14, Mai09, Mai11, Zho14], the functions X, I_{11}, S and the quasimodular forms A, B, C determine each other through the identities

$$A = I_{11}, \quad B = \frac{I_{11}^2}{X}(X+6S), \quad C = \frac{I_{11}^3}{X}\,, \tag{57}$$

eq:ABS_SIX

of inverse

$$X = \frac{A^3}{C}, \quad I_{11} = A, \quad S = \frac{1}{6}\frac{AB-A^3}{C}\,. \tag{58}$$

eq:SIX_ABC

[9]The description of the Hauptmodul of the genus 0 modular curve $Y_1(3)$ in terms of eta functions goes back to Klein and Fricke at the end of the 19th century.

In particular, viewed as functions of τ, X is a modular function of weight 0 for $\Gamma_1(3)$, A is a modular form of weight 1 for $\Gamma_1(3)$ and S is a quasimodular function of weight 0 for $\Gamma_1(3)$.

lem_alg_ind_S_X

Lemma 4.2. [10] *The functions S, X, I_{11} are algebraically independent over $\mathbb{C}$.*

Proof. It is a direct corollary of the algebraic independence of the quasimodular forms A, B, C reviewed in Section 4.1. □

By Lemma 4.2, the ring of functions generated by S and X is the polynomial ring

$$\mathbf{R} \coloneqq \mathbb{Q}[X,S]\,.$$

We define a grading on $\mathbf{R}$ by $\deg X = \deg S = 1$ and denote by $\mathbf{R}_{\leq k}$ the subspace of polynomials with degree no more than k.

We consider the vector space $[X^{-(g-1)} \cdot \mathbf{R}_{\leq 3g-3}]^{\mathrm{reg}}$, where $[-]^{\mathrm{reg}}$ (the "orbifold regularity" condition) is defined by

$$[-]^{\mathrm{reg}} := \{f(X,S) : 3\deg_X f + \deg_S f \geq 0\}. \tag{59}$$

prop_spaces_equality

Proposition 4.3. *The expression of S and X in terms of the quasimodular forms A, B, C induces an identification*

$$[X^{-(g-1)} \cdot \mathbf{R}_{\leq 3g-3}]^{\mathrm{reg}} = C^{-(2g-2)} \cdot \mathbb{Q}[A,B,C]_{6g-6}$$

for every $g \geq 2$.

Proof. We first prove that $[X^{-(g-1)} \cdot \mathbf{R}_{\leq 3g-3}]^{\mathrm{reg}} \subset C^{-(2g-2)} \cdot \mathbb{Q}[A,B,C]_{6g-6},$. Let $X^{-(g-1)} \cdot X^j S^k \in [X^{-(g-1)} \cdot \mathbf{R}_{\leq 3g-3}]^{\mathrm{reg}}$. According to (58), the C-degree of $X^{-(g-1)} \cdot X^j S^k$ is $-j-k+(g-1) \geq -(2g-2)$, using that $j+k \leq 3g-3$ by definition of $\mathbf{R}_{\leq 3g-3}$. The B-degree is $k \geq 0$. The A-degree of each term is $\geq -(3g-3)+3j+k$, which is ≥ 0 by definition of $[-]^{\mathrm{reg}}$. Therefore, $X^{-(g-1)} \cdot X^j S^k = C^{-(2g-2)} f(A,B,C)$ for some $f \in \mathbb{Q}[A,B,C]$. As $X^{-(g-1)} \cdot X^j S^k$ is of weight 0 and C of weigth 3, f is of weight $6g-6$.

We prove that conversely $C^{-(2g-2)} \cdot \mathbb{Q}[A,B,C]_{6g-6} \subset [X^{-(g-1)} \cdot \mathbf{R}_{\leq 3g-3}]^{\mathrm{reg}}$. Let $C^{-(2g-2)} \cdot A^m B^n C^l \in C^{-(2g-2)} \cdot \mathbb{Q}[A,B,C]_{6g-6}$. According to (57), the power of I_{11} is equal to the weight. As $C^{-(2g-2)} \cdot A^m B^n C^l$ is of weight 0, $C^{-(2g-2)} \cdot A^m B^n C^l$ is independent of I_{11} and is only a function of S and X. The S-degree of $C^{-(2g-2)} \cdot A^m B^n C^l$ is $n \geq 0$. The X-degree of each term in $C^{-(2g-2)} \cdot A^m B^n C^l$ is $\geq (2g-2)-n-l$ and $\leq (2g-2)-l$. Therefore, $\deg_X + \deg_S \leq (2g-2)-l \leq 2g-2$. On the other hand, $3\deg_X + \deg_S \geq (6g-6)-3n-3l+n = (6g-6)-2n-3l$, which is ≥ 0 as $A^m B^n C^l$ is of weight $6g-6$, B is of weight 2 and C is of weight 3. □

[10]In [LP18], the algebraic independence of S, X, I_{11} is not considered as known, and explicit lifts of functions to the ring $\mathbb{Q}[S,X]$ are constructed. This extra work is not necessary by Lemma 4.2.

Remark 4.4. It follows from the proof of Proposition 4.3 that the constraint defined by $[-]^{\mathrm{reg}}$ is equivalent to the absence of negative powers of A. According to [Mai11], A has a zero at the orbifold point $q = \infty$. Therefore, $[-]^{\mathrm{reg}}$ is indeed the condition imposed by the regularity at the orbifold point.

We consider the differential operator

$$D = 3Q\frac{d}{dQ}\,.$$

Lemma 4.5.

$$D = 3I_{11}^{-1} \cdot q\frac{d}{dq} \tag{60}$$ D_Q_q

$$= 3C^{-1} \cdot \mathcal{Q}\frac{d}{d\mathcal{Q}}\,. \tag{61}$$ D_q_cQ

Proof. The equality (60) is clear as $Q = e^{I_{11}(q)}$ by (12).

In order to prove (61), we use (56):

$$\mathcal{Q}\frac{d}{d\mathcal{Q}} = I_{22}^{-1}q\frac{d}{dq}$$

where

$$I_{22} \coloneqq q\frac{d}{dq}\left(\frac{I_{12}(q)}{I_{11}(q)}\right)\,.$$

Theorem 2 in [ZZ08] shows that $I_{22} = \frac{X}{I_{11}^2}$. We conclude using that $C = \frac{I_{11}^3}{X}$ according to (57). □

lem_D_weight **Lemma 4.6.** *For every $n, k \geq 0$, we have*

$$D(C^{-n} \cdot \mathbb{Q}[A,B,C]_k) \subset C^{-(n+1)} \cdot \mathbb{Q}[A,B,C]_{k+2}\,.$$

Proof. This follows from (61) and (51). The important point is that $\partial_\tau C$ is divisible by C. □

lem_SX_derivative **Lemma 4.7.** *We have*

$$q\tfrac{d}{dq}S = -S^2 + \tfrac{X-1}{3}\,S - \tfrac{X(X-1)}{9}\,, \quad \tfrac{d}{dq}X = X(X-1)\,. \tag{62}$$ eq:SX_derivative

Proof. The formula for X is clear as $X = (1+27q)^{-1}$ by definition.

The formula for S follows from the differential equation (11). Alternatively, one can use the expression (58) in terms of quasimodular forms and (51)-(61). □

4.3. **Quasimodular forms: from $SL(2,\mathbb{Z})$ to $\Gamma_1(3)$.** As reviewed in Section 3.4, the ring of quasimodular forms for $SL(2,\mathbb{Z})$ is generated by the Eisenstein series $E_2(\tilde{\tau})$, $E_4(\tilde{\tau})$, $E_6(\tilde{\tau})$. On the other hand, we have seen in Section 4.1 that the ring of quasimodular forms for $\Gamma_1(3)$ is generated by the functions $A(\tau)$, $B(\tau)$, $C(\tau)$.

The following result shows that after the change of variables $\tilde{\tau} = 3\tau$ the ring of quasimodular forms for $SL(2,\mathbb{Z})$ embeds in the ring of quasimodular form for $\Gamma_1(3)$.

prop_from_sl2_to_gamma **Proposition 4.8.** *We have the embedding of graded rings*

$$\mathbb{Q}[E_2(3\tau), E_4(3\tau), E_6(3\tau)] \subset \mathbb{Q}[A(\tau), B(\tau), C(\tau)]\,.$$

Explicitly, we have the identities

$$\begin{aligned} 3\,E_2(3\tau) &= 2B(\tau) + A(\tau)^2 = \frac{3\,{I_{11}}^2}{X}(X+4S), \\ 9\,E_4(3\tau) &= A(\tau)^4 + 8\,A(\tau)C(\tau) = \frac{{I_{11}}^4}{X}(X+8), \\ 27\,E_6(3\tau) &= -A(\tau)^6 + 20\,A(\tau)^3C(\tau) + 8\,C(\tau)^2 = \frac{-{I_{11}}^6}{X^2}(X^2-20X-8). \end{aligned} \tag{63}$$

modularformtogenerator

Proof. For every $n \geq 1$, if $f(\tau)$ is a modular (resp. quasimodular) form for $SL(2,\mathbb{Z})$ of weight k then $f(n\tau)$ is a modular (resp. quasimodular) form for $\Gamma_0(n)$ of weight k.

Once we know the modularity properties of each side, the identities (63) are easy to prove: it is enough to match finitely many terms of the $\mathcal{Q}$-expansions. Expressions in terms of X, I_{11} and S follow from (57). □

4.4. **Genus 0 invariants of $(\mathbb{P}^2, E)$.** Applying Theorem 1.2 for $g = 0$ (which reduces in this case to the genus 0 local-relative correspondence of [vGR19]), we get

$$F_0^{\mathbb{P}^2/E} = F_0^{K_{\mathbb{P}^2}}\,.$$

Lemma 4.9.

$$DF_0^{K_{\mathbb{P}^2}} = -I_2 \tag{64}$$

eq:D_F0

$$D^2F_0^{K_{\mathbb{P}^2}} = -3\frac{I_{12}}{I_{11}} \tag{65}$$

eq:D2_F0

$$D^3F_0^{K_{\mathbb{P}^2}} = -9C^{-1} = -\frac{9X}{I_{11}^3}\,. \tag{66}$$

eq:D3_F0

Proof. The formula (64) is the genus 0 mirror theorem for $K_{\mathbb{P}^2}$ [Giv96, LLY97, CKYZ99]. Formula (65) follows directly from (64) and (60).

Taking the derivative of (65), we obtain $D^3F_0^{K_{\mathbb{P}^2}} = -9I_{22}$ where

$$I_{22} := q\frac{d}{dq}\left(\frac{I_{12}(q)}{I_{11}(q)}\right)\,.$$

Theorem 2 in [ZZ08] shows that $I_{22} = \frac{X}{I_{11}^2}$. We end the proof of (66) using that $C = \frac{I_{11}^3}{X}$ according to (57). □

prop_Dn_F0 **Proposition 4.10.** *For every $n \geq 1$, we have*

$$D^{n+2}F_0^{K_{\mathbb{P}^2}} \in C^{-n}\mathbb{Q}[A,B,C]_{2n-2}\,.$$

Proof. The case $n = 1$ is clear by (66). The general case follows by induction on n from Lemma 4.6. □

For example, using (60)-(62) or (61)-(51), we get

$$D^4 F_0^{K_{\mathbb{P}_2}} = \frac{81SX}{I_{11}^4} = \frac{27}{2} C^{-2}(B - A^2)\,. \quad (67)$$

eq:D4_F0

4.5. Genus 1 invariants of $(\mathbb{P}^2, E)$.

f1formula

Theorem 4.11 (=Theorem 1.4)**.** *We have*

$$\begin{aligned} F_1^{\mathbb{P}^2/E} &= -\frac{1}{24}\log(-\mathcal{Q}) + \frac{1}{2}\sum_{n\geq 1}\log(1-\mathcal{Q}^n) - \frac{1}{2}\sum_{n\geq 1}\log(1-\mathcal{Q}^{3n}) \\ &= -\frac{1}{24}\log q + \frac{1}{24}\log(1+27q)\,. \end{aligned}$$

Proof. According to Theorem 1.2, we have

$$F_1^{K_{\mathbb{P}^2}} = -F_1^{\mathbb{P}^2/E} + F_{1,\emptyset}^E\,.$$

By formulas (A.3) and (A.15) of [Hu15], we have

$$\begin{aligned} F_1^{K_{\mathbb{P}^2}} &= -\frac{1}{12}\log q - \frac{1}{2}\log I_{11} - \frac{1}{12}\log(1+27q) \\ &= -\frac{1}{12}\log(-\mathcal{Q}) - \frac{1}{2}\sum_{n\geq 1}\log(1-\mathcal{Q}^{3n}) - \frac{1}{2}\sum_{n\geq 1}\log(1-\mathcal{Q}^n)\,. \end{aligned}$$

On the other hand, by (4) we have

$$F_{1,\emptyset}^E = -\frac{1}{24}(-\tilde{\mathcal{Q}}) + \bar{F}_{1,\emptyset}^E$$

and it is well-known [Dij95] that

$$\bar{F}_{1,\emptyset}^E = -\sum_{n\geq 1}\log(1-\tilde{\mathcal{Q}}^n)\,.$$

As $\tilde{\mathcal{Q}} = \mathcal{Q}^3$, we get

$$F_{1,\emptyset}^E = -\frac{1}{8}\log(-\mathcal{Q}) - \sum_{n\geq 1}\log(1-\mathcal{Q}^{3n})\,,$$

and so

$$\begin{aligned} F_1^{\mathbb{P}^2/E} =& F_{1,\emptyset}^E - F_1^{K_{\mathbb{P}^2}} \\ =& -\frac{1}{8}\log(-\mathcal{Q}) - \sum_{n\geq 1}\log(1-\mathcal{Q}^{3n}) \\ &+ \frac{1}{12}\log(-\mathcal{Q}) + \frac{1}{2}\sum_{n\geq 1}\log(1-\mathcal{Q}^{3n}) + \frac{1}{2}\sum_{n\geq 1}\log(1-\mathcal{Q}^n) \\ =& -\frac{1}{24}\log(-\mathcal{Q}) + \frac{1}{2}\sum_{n\geq 1}\log(1-\mathcal{Q}^n) - \frac{1}{2}\sum_{n\geq 1}\log(1-\mathcal{Q}^{3n})\,, \end{aligned}$$

which equals to

$$-\frac{1}{24}\log q + \frac{1}{24}\log(1+27q)$$

by formula (A.14) of [Hu15]. □

Remark 4.12. Expanding the right-hand side of Theorem 4.11, we get

$$F_1^{\mathbb{P}^2/E} = -\frac{1}{24}\log Q + \frac{7Q}{8} - \frac{129Q^2}{16} + \frac{589Q^3}{6} - \frac{43009Q^4}{32} + \frac{392691Q^5}{20} + \dots$$

Lemma 4.13.

$$DF_1^{\mathbb{P}^2/E} = -\frac{1}{8}\frac{X}{I_{11}} = -\frac{1}{8}\frac{A^2}{C}\,. \tag{68}$$

eq:D_F1

Proof. Using Theorem 4.11, we obtain

$$\begin{aligned} Q\frac{dF_1^{\mathbb{P}^2/E}}{dQ} &= -\frac{1}{24}\frac{1}{I_{11}}q\frac{d}{dq}(\log q - \log(1+27q)) = -\frac{1}{24}\frac{1}{I_{11}}\left(1-\frac{27q}{1+27q}\right) \\ &= -\frac{1}{24}\frac{(1+27q)^{-1}}{I_{11}} = -\frac{1}{24}\frac{X}{I_{11}}\,. \end{aligned}$$

We get the expression in terms of quasimodular forms using (58). □

prop_Dn_F1

Proposition 4.14. *For every $n \geq 1$, we have*

$$D^n F_1^{\mathbb{P}^2/E} \in C^{-n}\mathbb{Q}[A,B,C]_{2n}\,.$$

Proof. The case $n=1$ is clear by (68). The general case follows by induction on n from Lemma 4.6. □

For example, using (60)-(62) or (51)-(61), we get

$$D^2F_1^{\mathbb{P}^2/E} = \frac{3X}{8I_{11}^2}\left(S - \frac{2}{3}(X-1)\right) = \frac{A}{16C^2}(-5A^3+AB+4C)\,. \tag{69}$$

eq:D2_F1

Remark 4.15. We have $\frac{\partial}{\partial S}(DF_1^{\mathbb{P}^2/E}) = 0$, as predicted by the holomorphic anomaly equation (22).

4.6. **Proof of finite generation and quasimodularity.** We prove the finite generation statements of Theorems 1.3 and 1.6. By Proposition 4.3, it is enough to prove the finite generation part of Theorem 1.3.

The finite generation property for local $\mathbb{P}^2$ is known by [LP18, CI18]: we have

$$F_g^{K_{\mathbb{P}^2}} \in C^{-(2g-2)}\cdot\mathbb{Q}[A,B,C]_{6g-6} \tag{70}$$

eq:fg_local_P2

for every $g \geq 2$. More precisely, the result proved in [LP18] is slightly weaker, using the generator $L = X^{1/3}$ instead of X, not getting the optimal degree bound on X, and not mentioning the "orbifold regularity". However, using the R-matrix techniques used in [LP19a] and its appendix, it is possible to prove that $F_g^{K_{\mathbb{P}^2}} \in [X^{-(g-1)}\cdot\mathbf{R}_{\leq 3g-3}]^{\mathrm{reg}}$ for every $g \geq 2$. Such refinement of the R-matrix is described in [GJR18] for the proof of a "graded finite generation" for the quintic 3-fold. Once we know that $F_g^{K_{\mathbb{P}^2}} \in [X^{-(g-1)}\cdot$

$\mathbf{R}_{\leq 3g-3}]^{\mathrm{reg}}$, we get that $F_g^{K_{\mathbb{P}^2}} \in C^{-(2g-2)} \cdot \mathbb{Q}[A,B,C]_{6g-6}$ by Proposition 4.3. Alternatively, one can use [CI18] which proves directly the result in terms of quasimodular forms.

We show by induction on g that, for every $g \geq 2$, we have

$$F_g^{\mathbb{P}^2/E} \in C^{-(2g-2)} \cdot \mathbb{Q}[A,B,C]_{6g-6}\,.$$

Let $g \geq 2$. According to Theorem 1.2, we have

$$(-1)^{g-1} F_g^{\mathbb{P}^2/E} = -F_g^{K_{\mathbb{P}^2}} +$$

$$\sum_{n\geq 0} \sum_{\substack{g=h+g_1+\cdots+g_n,\\ \mathbf{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n_{\geq 0}\\ (a_j,g_j)\neq(0,0),\, \sum_{j=1}^n a_j=2h-2}} \frac{(-1)^{h-1}F^E_{h,\mathbf{a}}}{|\operatorname{Aut}(\mathbf{a},\mathbf{g})|}\prod_{j=1}^n (-1)^{g_j-1} D^{a_j+2} F_{g_j}^{\mathbb{P}^2/E}\,,$$

where the variable $\tilde{\mathcal{Q}}$ in the definition (4) of $F^E_{h,\mathbf{a}}$ is expressed in terms of the variable Q in the definitions (8) and (9) of $F_g^{K_{\mathbb{P}^2}}$ and $F_g^{\mathbb{P}^2/E}$ by (7):

$$\tilde{\mathcal{Q}} = \exp\left(-D^2 F_0^{K_{\mathbb{P}^2}}\right)\,.$$

By (70), we know that $F_g^{K_{\mathbb{P}^2}} \in C^{-(2g-2)} \cdot \mathbb{Q}[A,B,C]_{6g-6}$. Therefore, it remains to show that each summand

$$\frac{(-1)^{h-1}F^E_{h,\mathbf{a}}}{|\operatorname{Aut}(\mathbf{a},\mathbf{g})|}\prod_{j=1}^n (-1)^{g_j-1} D^{a_j+2} F_{g_j}^{\mathbb{P}^2/E} \tag{71}$$

belongs to $C^{-(2g-2)} \cdot \mathbb{Q}[A,B,C]_{6g-6}$.

Terms with $n=0$ only arises for $h=1$ and so $g=1$. Thus, for $g \geq 2$, only the terms with $n \geq 1$ contribute and by Theorem 3.5 the series $F^E_{h,\mathbf{a}}$ are quasimodular as functions of $\tilde{\tau}$, where $\tilde{\mathcal{Q}} = e^{2i\pi\tilde{\tau}}$. More precisely, we have

$$F^E_{h,\mathbf{a}} \in \mathbb{Q}[E_2(\tilde{\tau}), E_4(\tilde{\tau}), E_6(\tilde{\tau})]_{\sum_{j=1}^n (a_j+2)}\,.$$

Our aim is to show quasimodularity as functions of τ, that is given by (56) as

$$\mathcal{Q} = e^{2i\pi\tau} = -\exp\left(\frac{I_{12}}{I_{11}}\right)\,.$$

By (65), we have $D^2 F_0^{K_{\mathbb{P}^2}} = -3\frac{I_{12}}{I_{11}}$, so $\tilde{\tau} = 3\tau$ and $\tilde{\mathcal{Q}} = \mathcal{Q}^3$. Using Proposition 4.8, we deduce that

$$F^E_{h,\mathbf{a}} \in \mathbb{Q}[A,B,C]_{\sum_{j=1}^n (a_j+2)}\,.$$

By induction on the genus, we know that for every $g_j \geq 2$

$$F_{g_j}^{\mathbb{P}^2/E} \in C^{-(2g_j-2)} \cdot \mathbb{Q}[A,B,C]_{6g_j-6}\,,$$

and so

$$D^{a_j+2} F_{g_j}^{\mathbb{P}^2/E} \in C^{-(2g_j-2+a_j+2)} \cdot \mathbb{Q}[A,B,C]_{6g_j-6+2a_j+4}$$

using Lemma 4.6. By Propositions 4.10 and 4.14, this result also holds for $g_j = 0$ (in this case, $a_j \geq 1$ so $a_j + 2 \geq 3$) and $g_j = 1$.

Therefore,

$$\frac{(-1)^{h-1} F^E_{h,\mathbf{a}}}{|\operatorname{Aut}(\mathbf{a},\mathbf{g})|} \prod_{j=1}^{n} (-1)^{g_j-1} D^{a_j+2} F^{\mathbb{P}^2/E}_{g_j} \in C^{-(2g-2)} \cdot \mathbb{Q}[A,B,C]_{6g-6}$$

follows from

$$h + \sum_{j=1}^{n} g_j = g, \quad \sum_{j=1}^{n} a_j = 2h-2.$$

sec:genus_2

4.7. Genus 2 invariants of $(\mathbb{P}^2, E)$. We prove Theorem 1.5.

By Theorem 1.2, we have

$$F_2^{K_{\mathbb{P}^2}} = F_2^{\mathbb{P}^2/E} + D^2 F_1^{\mathbb{P}^2/E} \cdot F^E_{1,(0)} - \frac{1}{2}\left(D^3 F_0^{\mathbb{P}^2/E}\right)^2 \cdot F^E_{2,(1,1)} + D^4 F_0^{\mathbb{P}^2/E} \cdot F^E_{2,(2)}\,.$$

By [LP18], we have

$$F_2^{K_{\mathbb{P}^2}} = \frac{5}{8}\frac{S^3}{X} + \frac{1}{8}S^2 + \frac{1}{96}SX + \frac{X^2}{4320} + \frac{X}{4320} - \frac{1}{2160}\,.$$

Using (69)-(47)-(63), we get

$$D^2 F_1^{\mathbb{P}^2/E} \cdot F^E_{1,(0)} = -\frac{S^2}{16} + \frac{5SX}{192} - \frac{S}{24} + \frac{X^2}{96} - \frac{X}{96}\,.$$

Using (66)-(49)-(63), we obtain

$$-\frac{1}{2}(D^3 F_0^{\mathbb{P}^2/E})^2 \cdot F^E_{2,(1,1)} = -\frac{S^3}{2X} - \frac{3S^2}{8} - \frac{11SX}{120} + \frac{S}{60} - \frac{X^2}{135} + \frac{7X}{1080} + \frac{1}{1080}\,.$$

Using (67)-(50)-(63), we have

$$D^4 F_0^{\mathbb{P}^2/E} \cdot F^E_{2,(2)} = \frac{9S^3}{8X} + \frac{9S^2}{16} + \frac{47SX}{640} + \frac{S}{40}\,.$$

Therefore, we find that in $F_2^{\mathbb{P}^2/E}$ the coefficient of $\frac{S^3}{X}$ is

$$\frac{5}{8} - \left(-\frac{1}{2} + \frac{9}{8}\right) = 0\,,$$

the coefficient of S^2 is

$$\frac{1}{8} - \left(-\frac{1}{16} - \frac{3}{8} + \frac{9}{16}\right) = 0\,,$$

the coefficient of SX is

$$\frac{1}{96} - \left(\frac{5}{192} - \frac{11}{120} + \frac{47}{640}\right) = \frac{1}{384}\,,$$

the coefficient of X^2 is

$$\frac{1}{4320} - \left(\frac{1}{96} - \frac{1}{135}\right) = -\frac{1}{360}\,,$$

the coefficient of X is

$$\frac{1}{4320} - \left(-\frac{1}{96} + \frac{7}{1080}\right) = \frac{1}{240}\,,$$

and the constant term is

$$-\frac{1}{2160} - \frac{1}{1080} = -\frac{1}{720}\,.$$

This concludes the proof of Theorem 1.5.

Remark 4.16. Expanding the right-hand side of Theorem 1.5, we get

$$F_2^{\mathbb{P}^2/E} = \frac{29Q}{640} - \frac{207Q^2}{64} + \frac{18447Q^3}{160} - \frac{526859Q^4}{160} + \frac{5385429Q^5}{64} + \dots$$

Using (58), we can rewrite Theorem 1.5 as

$$F_2^{\mathbb{P}^2/E} = \frac{1}{11520C^2}(-37A^6 + 5A^4B + 48A^3C - 16C^2)\,. \tag{72}$$

Taking the S-derivative of Theorem 1.5, we obtain

$$\frac{\partial}{\partial S}F_2^{\mathbb{P}^2/E} = \frac{X}{384}\,,$$

and so, using (68),

$$\frac{3X}{I_{11}^2}\frac{\partial}{\partial S}F_2^{\mathbb{P}^2/E} = \frac{1}{2}(DF_1^{\mathbb{P}^2/E})^2\,,$$

as predicted by the holomorphic anomaly equation (22).

sec:HAE

5. Holomorphic anomaly equation for ($\mathbb{P}^2, E$)

In this section, we prove Theorem 1.8, that is the holomorphic anomaly equation for the series $F_{g,n}^{\mathbb{P}^2/E}$.

We will use the following definitions.

Definition 5.1. A partition $\mathbf{a}$ of length n is an ordered set $(a_1, \cdots, a_n)$ such that

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$$

Note we allow the entries a_i to be zero, which is different from the ordinary definition of a partition.

defncuppartition

Definition 5.2. For any two partitions $\mathbf{a}$ and $\mathbf{b}$ of of length n_1 and n_2 respectively. We define

$$\mathbf{a} \cup \mathbf{b}$$

to be the partition of length n_1+n_2 with entries which are exactly the entries of $\mathbf{a}$ and $\mathbf{b}$ with decreasing ordering.

5.1. **Holomorphic anomaly equation for the elliptic curve.** We first review some known results for the elliptic curve. Recall that we denote

$$\left\langle \tau_1\psi_1^{k_1}, \cdots, \tau_n\psi_n^{k_n} \right\rangle_{g,n}^E := \sum_{d\geq 0} \tilde{\mathcal{Q}}^d \int_{[\overline{M}_{g,n}(E,d)]^{\mathrm{vir}}} \prod_{i=1}^n \psi_i^{k_i} \operatorname{ev}_i^* \tau_i$$

the generating series of Gromov-Witten invariants of the elliptic curve E, with $\tau_i \in H^\bullet(E)$. Recall also that we denote by $\omega \in H^2(E)$ the (Poincaré dual) class of a point and

$$F_{g,\mathbf{a}}^E := -\frac{\delta_{g,1}\delta_{n,0}}{24} \log\left((-1)^{E\cdot E}\tilde{\mathcal{Q}}\right) + \sum_{d\geq 0} \tilde{\mathcal{Q}}^d \left\langle \omega\psi_1^{a_1}, \ldots, \omega\psi_n^{a_n}\right\rangle_{g,n,d}^E$$

Using the polynomiality of the double ramification cycle in the parts of the ramification profiles, Oberdieck and Pixton [OP18] proved the following holomorphic anomaly equation for the Gromov-Witten theory of the elliptic curve: for $2g-2+n>0$, we have

$$\begin{aligned} &-24\,\tfrac{\partial}{\partial E_2} \left\langle \omega\psi_1^{a_1}, \cdots, \omega\psi_n^{a_n}\right\rangle_{g,n}^E = \\ &\sum_{\substack{g_1+g_2=g,\\ \mathbf{a}'\cup\mathbf{a}''=\mathbf{a}}} \left\langle \omega\psi_1^{a_1'}, \cdots, \omega\psi_s^{a_s'}, 1\right\rangle_{g_1,s+1}^E \left\langle \omega\psi_1^{a_1''}, \cdots, \omega\psi_{n-s}^{a_{n-s}''}, 1\right\rangle_{g_2,n-s+1}^E \\ &+ \left\langle \omega\psi_1^{a_1}, \cdots, \omega\psi_n^{a_n}, 1, 1\right\rangle_{g-1,n+2}^E - 2\sum_{j=1}^n \left\langle \omega\psi_1^{a_1}, \cdots, \psi_j^{a_j+1}, \cdots, \omega\psi_n^{a_n}\right\rangle_{g,n}^E. \end{aligned}$$

By the Virasoro constraints proved by Okounkov and Pandharipande [OP06b], we have

$$\begin{aligned} &\sum_{j=1}^n \left\langle \omega\psi_1^{a_1}, \cdots, \psi_j^{a_j+1}, \cdots, \omega\psi_n^{a_n}\right\rangle_{g,n}^E \\ &= \sum_{1\leq i\neq j\leq n} \binom{a_i+a_j+1}{a_i} \left\langle \omega\psi_1^{a_1}, \cdots, \widehat{\omega\psi_i^{a_i}}, \cdots, \widehat{\omega\psi_j^{a_j}}, \cdots, \omega\psi_n^{a_n}, \omega\psi_i^{a_i+a_j}\right\rangle_{g,n-1}^E \end{aligned}$$

Together with the string equation, we obtain the following form of the holomorphic anomaly equation for the series $F_{g,\mathbf{a}}^E$. Let $\mathbf{a}$ be a partition of $2h-2$, i.e. $\sum_{i=1}^n a_i = 2h-2$, then, for $2g-2+n>0$, we have

$$\begin{aligned} -24\,\tfrac{\partial}{\partial E_2} F_{h,\mathbf{a}}^E = &\sum_{1\leq i,j\leq n} F_{h-1,\mathbf{a}-\vec{e}_i-\vec{e}_j}^E + \sum_{\substack{h_1+h_2=h\\ \mathbf{a}'\cup\mathbf{a}''=\mathbf{a}}} \sum_{\substack{1\leq i\leq l(\mathbf{a}')\\ 1\leq j\leq l(\mathbf{a}'')}} F_{h_1,\mathbf{a}'-\vec{e}_i}^E F_{h_2,\mathbf{a}''-\vec{e}_j}^E \\ &- 2\sum_{1\leq i\neq j\leq n} \binom{a_i+a_j+1}{a_i} F_{h,\,\mathcal{G}_{ij}(\mathbf{a})}^E \end{aligned} \tag{73}$$

HAEforE

Here for a partition $\mathbf{a}=(a_1,\cdots,a_n)$, we define the gluing operation by

$$\mathcal{G}_{ij}(\mathbf{a}) = (a_1, \cdots \hat{a}_i, \cdots \hat{a}_j, \cdots, a_n) \cup (a_i+a_j),$$

and for the vectors $\vec{e}_i$ $(i = 1, \cdots, n)$:

$$\vec{e}_i = (0, \cdots, 1, \cdots, 0) \ \text{ with 1 lies in the } i\text{-th component},$$

we define (c.f. Definition 5.2)

$$\mathbf{a} - \vec{e}_i := (a_1, \cdots, \widehat{a}_i, \cdots, a_n) \cup (a_i - 1),$$

$$\mathbf{a}-\vec{e}_i-\vec{e}_j := \begin{cases} (a_1, \cdots, \widehat{a}_i, \cdots, a_n) \cup (a_i - 2), & \text{if } i = j, \\ (a_1, \cdots, \widehat{a}_i, \cdots, \widehat{a}_j, \cdots, a_n) \cup (a_i - 1, a_j - 1), & \text{otherwise.} \end{cases}.$$

Example 5.3. Using (47)–(50), one can check directly that

$$-12 \frac{\partial}{\partial E_2} F^E_{2,(1^2)} = F^E_{1,(0^2)} + \left(F^E_{1,(0)}\right)^2 - 6F^E_{2,(2)}\,,$$

$$-24 \frac{\partial}{\partial E_2} F^E_{2,(2)} = F^E_{1,(0)}\,.$$

5.2. **Holomorphic anomaly equation for local $\mathbb{P}^2$.** We denote by $F^{K_{\mathbb{P}^2}}_{g,n}$ the generating function for the local $\mathbb{P}^2$ theory with n insertions of the hyperplane classes. By the divisor equation,

$$F^{K_{\mathbb{P}^2}}_{g,n} = \left(Q\frac{d}{dQ}\right)^n F^{K_{\mathbb{P}^2}}_g.$$

We have the following holomorphic anomaly equation which was proved using various techniques in [LP18], [CI18] and [EMO07, FLZ16, FRZZ19]:

$$\frac{X}{3\,{I_{11}}^2} \cdot \frac{\partial}{\partial S} F^{K_{\mathbb{P}^2}}_{g,n} = \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n \\ 2g_i-2+n_i\geq 0}} \binom{n}{n_1} F^{K_{\mathbb{P}^2}}_{g_1,n_1+1} \cdot F^{K_{\mathbb{P}^2}}_{g_2,n_2+1} + \frac{1}{2} F^{K_{\mathbb{P}^2}}_{g-1,n+2}. \tag{74}$$

HAEforKP2

5.3. **Proof of the holomorphic anomaly equation for $(\mathbb{P}^2, E)$.** We prove the holomorphic anomaly equation (22) for $F^{\mathbb{P}^2/E}_{g,n}$ (Theorem 1.8) by induction on the genus g. We have $F^{\mathbb{P}^2/E}_{0,n} = F^{K_{\mathbb{P}^2}}_{0,n}$ and so (22) holds for $g = 0$ by (74).

Let g and n such that $2g - 2 + n > 0$. By taking derivatives of both sides of Theorem 1.2, we obtain

$$\begin{aligned} F^{K_{\mathbb{P}^2}}_{g,n} &= (-1)^g F^{\mathbb{P}^2/E}_{g,n} + \sum_{\substack{0<h\leq g, \\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g,n}(h)}} (-1)^{h-1} F^E_{h,\mathbf{a}} \cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})} \\ &= (-1)^g F^{\mathbb{P}^2/E}_{g,n} + 9\,F^E_{1,(0)} \cdot (-1)^{g-2} F^{\mathbb{P}^2/E}_{g-1,n+2} + \sum_{\substack{0<h\leq g, \\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}^+_{g,n}(h)}} (-1)^{h-1} F^E_{h,\mathbf{a}} \cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})} \end{aligned} \tag{75}$$

PErecursioninsertion

where we define

$$\mathcal{P}_{g,n}(h) := \left\{ (\mathbf{g}, \mathbf{a}, \mathbf{B}) = \{(g_i, a_i, B_i)\}^l_{i=1} \middle| \substack{h+\sum_i g_i=g,\ \sum_i a_i=2h-2, \\ \sum_i |B_i|=n,\ 2g_i-2+2a_i+2|B_i|\geq 0, \\ g_i,\ a_i\in\mathbb{Z}_{\geq 0},\ \coprod_i B_i=\{1,2,\cdots,n\}} \right\},$$

$$\mathcal{P}^+_{g,n}(h) := \mathcal{P}_{g,n}(h) \Big\backslash \Big\{ \big(g-1, 0, \{1, 2, \cdots, n\}\big) \Big\},$$

$$\mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})} := \frac{1}{|\operatorname{Aut}(\mathbf{g},\mathbf{a},\mathbf{B})|}\prod_i (-1)^{g_i-1}3^{a_i+2}F^{\mathbb{P}^2/E}_{g_i,a_i+b_i+2}.$$

Note that $\mathbf{B}$ corresponds to the assignment of insertions, and we allow B_i to be empty in the definition of $\mathcal{P}_{g,n}(h)$. Given an element $\{(g_i,a_i,B_i)\}_{i=1}^l \in \mathcal{P}_{g,n}(h)$, we set $b_i = |B_i|$ to be the number of elements in B_i, and set $\mathbf{g} = \{g_1,\cdots,g_{l(\mathbf{g})}\}$, $\mathbf{a} = \{a_1,\cdots,a_{l(\mathbf{a})}\}$, $\mathbf{B} = \{B_1,\cdots,B_{l(\mathbf{B})}\}$ where $l(\mathbf{g})$, $l(\mathbf{a})$, $l(\mathbf{B})$ are lengths of the corresponding partitions (both of them equal to l). Finally, $\operatorname{Aut}(\mathbf{g},\mathbf{a},\mathbf{B})$ is the symmetry group consists of permutation symmetries of $(\mathbf{g},\mathbf{a},\mathbf{B})$.

We denote $\partial_S := \frac{X}{3\,I_{11}{}^2}\cdot\frac{\partial}{\partial S}$. By (63) we have

$$\partial_S \;=\; \tfrac{1}{18}\cdot 24\tfrac{\partial}{\partial_{E_2}}\,.$$

By applying the operator ∂_S on (75) and using (47), we obtain

$$\begin{aligned}\partial_S F^{K_{\mathbb{P}^2}}_{g,n} &= (-1)^g\partial_S F^{\mathbb{P}^2/E}_{g,n} + \frac{(-1)^{g-1}}{2}F^{\mathbb{P}^2/E}_{g-1,n+2} + \sum_{\substack{0<h\le g,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g,n}(h)}}(-1)^{h-1}F^E_{h,\mathbf{a}}\cdot\partial_S\mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})}\\ &\quad+\sum_{\substack{0<h\le g,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}^+_{g,n}(h)}}(-1)^{h-1}\partial_S F^E_{h,\mathbf{a}}\cdot\mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})} \qquad (76)\\ &= (-1)^g\partial_S F^{\mathbb{P}^2/E}_{g,n} + \frac{1}{2}(-1)^{g-1}F^{\mathbb{P}^2/E}_{g-1,n+2} + \mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3+\mathcal{C}_4\end{aligned}$$

where $\mathcal{C}_1$ is the contribution of the last term in the first line, and $\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4$ are the three contributions obtained by applying the holomorphic anomaly equation (73) to the term in the second line:

$$\mathcal{C}_1 := \sum_{\substack{0<h\le g,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g,n}(h)}}(-1)^{h-1}F^E_{h,\mathbf{a}}\cdot\partial_S\mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})},$$

$$\mathcal{C}_2 := \frac{1}{18}\sum_{\substack{0<h\le g,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}^+_{g,n}(h)}}\;\sum_{1\le i,j\le l(\mathbf{a})}(-1)^h F^E_{h-1,\mathbf{a}-\vec{e}_i-\vec{e}_j}\cdot\mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})},$$

$$\mathcal{C}_3 := \frac{1}{18}\sum_{\substack{0<h\le g,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}^+_{g,n}(h)}}\;\sum_{\substack{h_1+h_2=h\\ \mathbf{a}'\sqcup\mathbf{a}''=\mathbf{a}}}\;\sum_{\substack{1\le i\le l(\mathbf{a}')\\ 1\le j\le l(\mathbf{a}'')}}(-1)^{h_1-1}F^E_{h_1,\mathbf{a}'-\vec{e}_i}(-1)^{h_2-1}F^E_{h_2,\mathbf{a}''-\vec{e}_j}\cdot\mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})},$$

$$\mathcal{C}_4 := \frac{1}{18}\sum_{\substack{0<h\le g,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}^+_{g,n}(h)}}\;\sum_{1\le i\ne j\le l(\mathbf{a})}2\binom{a_i+a_j+1}{a_i}(-1)^{h-1}F^E_{h,\,\mathcal{G}_{ij}(\mathbf{a})}\cdot\mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})}.$$

On the other hand, we can first apply the holomorphic anomaly equation (74) for local $\mathbb{P}^2$. We then apply equation (75) to the right-hand side of

(74). We have

$$\partial_S F^{K_{\mathbb{P}^2}}_{g,n} = \frac{1}{2}\Big((-1)^{g-1}F^{\mathbb{P}^2/E}_{g-1,n+2} + \sum_{\substack{0<h\le g-1,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g-1,n+2}(h)}} (-1)^{h-1}F^E_{h,\mathbf{a}}\cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})}\Big)$$
$$+\frac{1}{2}\sum_{\substack{g_1+g_2=g\\ n_1+n_2=n\\ 2g_i-2+n_i\ge 0}}\binom{n}{n_1}\prod_{i=1}^{2}\Big((-1)^{g_i}F^{\mathbb{P}^2/E}_{g_i,n_i+1} + \sum_{\substack{0<h\le g_i,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g_i,n_i+1}(h)}} (-1)^{h-1}F^E_{h,\mathbf{a}}\cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})}\Big)$$
$$= \frac{(-1)^{g-1}}{2}F^{\mathbb{P}^2/E}_{g-1,n+2} + \mathcal{C}_2' + \frac{(-1)^g}{2}\sum_{\substack{g_1+g_2=g\\ n_1+n_2=n\\ 2g_i-2+n_i\ge 0}}\binom{n}{n_1}(-1)^{g_1-1}F^{\mathbb{P}^2/E}_{g_1,n_1+1}(-1)^{g_2-1}F^{\mathbb{P}^2/E}_{g_2,n_2+1}$$
$$+\mathcal{C}_1' + \mathcal{C}_3',$$

where $\mathcal{C}_2'$ is the contribution of the last term in the first line, and $\mathcal{C}_1', \mathcal{C}_3'$ are the two types of contributions of the terms in the second line:

$$\mathcal{C}_2' = \frac{1}{2}\sum_{\substack{0<h\le g-1,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g-1,n+2}(h)}} (-1)^{h-1}F^E_{h,\mathbf{a}}\cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})},$$
$$\mathcal{C}_1' = \sum_{\substack{g_1+g_2=g\\ n_1+n_2=n\\ 2g_i-2+n_i\ge 0}}\binom{n}{n_1}(-1)^{g_1}F^{\mathbb{P}^2/E}_{g_1,n_1+1}\sum_{\substack{0<h\le g_2,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g_2,n_2+1}(h)}} (-1)^{h-1}F^E_{h,\mathbf{a}}\cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})},$$
$$\mathcal{C}_3' = \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ n_1+n_2=n\\ 2g_i-2+n_i\ge 0}}\binom{n}{n_1}\sum_{\substack{0<h\le g_1,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g_1,n_1+1}(h)}} (-1)^{h-1}F^E_{h,\mathbf{a}}\cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})}\sum_{\substack{0<h\le g_2,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g_2,n_2+1}(h)}} (-1)^{h-1}F^E_{h,\mathbf{a}}\cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})}.$$

Suppose that the holomorphic anomaly equation for $(\mathbb{P}^2, E)$ (Theorem 1.8) holds for $g' < g$. Namely

$$\partial_S F^{\mathbb{P}^2/E}_{g',n} = \frac{1}{2}\sum_{\substack{g_1+g_2=g'\\ n_1+n_2=n\\ 2g_i-2+n_i\ge 0}}\binom{n}{n_1}F^{\mathbb{P}^2/E}_{g_1,n_1+1}\cdot F^{\mathbb{P}^2/E}_{g_2,n_2+1} \quad \text{for } g'<g\ . \tag{77}$$

HAEforrelativeg

We prove the holomorphic anomaly equation for genus g case by showing that

$$\mathcal{C}_1 + \mathcal{C}_4 = \mathcal{C}_1', \qquad \mathcal{C}_2 = \mathcal{C}_2', \qquad \mathcal{C}_3 = \mathcal{C}_3'.$$

These three identities follow from the following three lemmas.

Lemma 5.4. *Suppose that the holomorphic anomaly equation for* $(\mathbb{P}^2, E)$ *holds for all* $g' < g$. *Then we have*

$$\mathcal{C}_1 + \mathcal{C}_4 = \mathcal{C}_1'\,.$$

Proof. By using the fact that

$$\binom{a_i+a_j+1}{a_i}+\binom{a_i+a_j+1}{a_j}=\binom{a_i+a_j+2}{a_i+1},$$

we may write $\mathcal{C}_4$ as

$$\frac{1}{18}\sum_{\substack{0<h\leq g,\\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g,n}(h)}}\ \sum_{1\leq i\neq j\leq l(\mathbf{a})}\binom{a_i+a_j+2}{a_i+1}(-1)^{h-1}F^E_{h,\ \mathcal{G}_{ij}(\mathbf{a})}\cdot\mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})}. \quad (78) \quad \text{pflm1-1}$$

In the summation, we can replace $\mathcal{P}^+_{g,n}(h)$ by $\mathcal{P}_{g,n}(h)$ since $l(\mathbf{a})\geq 2$.

Let us fix a $(\mathbf{g},\mathbf{a},\mathbf{B})=\{(g_i,a_i,B_i)\}_{i=1}^l\in\mathcal{P}_{g,n}(h)$ and $1\leq s\neq t\leq l$. If we sum over those $i\neq j$ in (78) such that

$$(g_i,a_i,B_i)=(g_s,a_s,B_s),\quad (g_j,a_j,B_j)=(g_t,a_t,B_t),$$

then the total contribution can be written as

$$\begin{aligned}&(-1)^{h-1}F^E_{h,\ \mathcal{G}_{ij}(\mathbf{a})}\frac{1}{|\operatorname{Aut}(P_1)|}\mathrm{Cont}^{\mathbb{P}^2/E}_{P_1}\cdot\\ &\left(-\frac{1}{2}(-1)^{g_s+g_t-1}3^{a_s+a_t+2}\binom{a_s+a_t+2}{a_s+1}F^{\mathbb{P}^2/E}_{g_s,a_s+b_s+2}F^{\mathbb{P}^2/E}_{g_t,a_t+b_t+2}\right)\end{aligned} \quad (79) \quad \text{pflm1-2}$$

where $P_1=\{(g_k,a_k,B_k)\in(\mathbf{g},\mathbf{a},\mathbf{B})\mid k\neq s,t\}$ and

$$\mathrm{Cont}^{\mathbb{P}^2/E}_{P_1}=\prod_{k\neq s,t}(-1)^{g_k-1}3^{a_k+2}F^{\mathbb{P}^2/E}_{a_k+b_k+2}.$$

Now let us vary $(\mathbf{g},\mathbf{a},\mathbf{B})\in P_{g,n}(h)$. We sum over those (79) with

$$(\mathbf{g},\mathbf{a},\mathbf{B})=P_1\cup\{(g_s,a'_s,B'_s)\}\cup\{(g_t,a'_t,B'_t)\}$$

such that $a'_s+b'_s=a_s+b_s$ and $a'_t+b'_t=a_t+b_t$. Since $\sum a'_i=2h-2$ is fixed, we also deduce that $a'_s+a'_t=a_s+a_t$ and $b'_s+b'_t=b_s+b_t$. Then we get

$$\begin{aligned}(-1)^{h-1}F^E_{h,\ \mathcal{G}_{ij}(\mathbf{a})}\frac{\mathrm{Cont}^{\mathbb{P}^2/E}_{P_1}}{|\operatorname{Aut}(P_1)|}\cdot&\left(-\frac{1}{2}(-1)^{g_s+g_t-1}3^{a_s+a_t+2}F^{\mathbb{P}^2/E}_{g_s,a_s+b_s+2}F^{\mathbb{P}^2/E}_{g_t,a_t+b_t+2}\right)\\ &\sum_{\substack{a'_s+b'_s=a_s+b_s\\ 0\leq a'_s\leq a_s+a_t\\ 0\leq b'_s\leq b_s+b_t}}\binom{a_s+a_t+2}{a'_s+1}\binom{b_s+b_t}{b'_s}.\end{aligned}$$

Using the Vandermonde's identity

$$\begin{aligned}&\sum_{\substack{a'_s+b'_s=a_s+b_s\\ 0\leq a'_s\leq a_s+a_t\\ 0\leq b'_s\leq b_s+b_t}}\binom{a_s+a_t+2}{a'_s+1}\binom{b_s+b_t}{b'_s}=\\ &\qquad\binom{a_s+b_s+a_t+b_t+2}{a_s+b_s+1}-\binom{b_s+b_t}{a_s+b_s+1}-\binom{b_s+b_t}{a_t+b_t+1},\end{aligned}$$

the above equation can be further written as

$$T_1 + T_2$$

where

$$\begin{aligned} T_1 =& (-1)^{h-1} F^E_{h,\ \mathcal{G}_{ij}(\mathbf{a})} \frac{\mathrm{Cont}^{\mathbb{P}^2/E}_{P_1}}{|\operatorname{Aut}(P_1)|} \cdot \\ & \left(-\frac{1}{2}(-1)^{g_s+g_t-1} 3^{a_s+a_t+2} \binom{a_s+b_s+a_t+b_t+2}{a_s+b_s+1} F^{\mathbb{P}^2/E}_{g_s,a_s+b_s+2} F^{\mathbb{P}^2/E}_{g_t,a_t+b_t+2}\right), \\ T_2 =& (-1)^{h-1} F^E_{h,\ \mathcal{G}_{ij}(\mathbf{a})} \frac{\mathrm{Cont}^{\mathbb{P}^2/E}_{P_1}}{|\operatorname{Aut}(P_1)|} \cdot \left(\binom{b_s+b_t}{a_s+b_s+1} + \binom{b_s+b_t}{a_t+b_t+1} \right) \cdot \\ & \frac{1}{2}(-1)^{g_s+g_t-1} 3^{a_s+a_t+2} F^{\mathbb{P}^2/E}_{g_s,a_s+b_s+2} F^{\mathbb{P}^2/E}_{g_t,a_t+b_t+2}. \end{aligned}$$

Using the holomorphic anomaly equation (77) for $g' < g$, it is easy to see that the total contribution of those T_1 when we vary $(\mathbf{g}, \mathbf{a}, \mathbf{B})$ and s, t is

$$\begin{aligned} -\mathcal{C}_1 + \sum_{\substack{0<h\leq g, \\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g,n}(h)}} (-1)^{h-1} F^E_{h,\mathbf{a}} \frac{1}{\operatorname{Aut}(\mathbf{g},\mathbf{a},\mathbf{B})} \sum_{1\leq i\leq l(\mathbf{a})} \prod_{j\neq i} (-1)^{g_j-1} 3^{a_j+2} F^{\mathbb{P}^2/E}_{g_j,a_j+b_j+2} \cdot \\ \left(\sum_{g_1'+g_2'=g_i} (-1)^{g_1'} F^{\mathbb{P}^2/E}_{g_1',1} (-1)^{g_2'-1} 3^{a_2+2} F^{\mathbb{P}^2/E}_{g_2',a_i+b_i+3} \right) \end{aligned}$$

It is easy to check that

$$\begin{aligned} \sum_{\substack{0<h\leq g, \\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g,n}(h)}} (-1)^{h-1} F^E_{h,\mathbf{a}} \frac{1}{\operatorname{Aut}(\mathbf{g},\mathbf{a},\mathbf{B})} \sum_{1\leq i\leq l(\mathbf{a})} \prod_{j\neq i} (-1)^{g_j-1} 3^{a_j+2} F^{\mathbb{P}^2/E}_{g_j,a_j+b_j+2} \cdot \\ \left(\sum_{g_1'+g_2'=g_i} (-1)^{g_1'} F^{\mathbb{P}^2/E}_{g_1',1} (-1)^{g_2'-1} 3^{a_2+2} F^{\mathbb{P}^2/E}_{g_2',a_i+b_i+3} \right) \end{aligned}$$

equals to

$$\sum_{g_1+g_2=g} (-1)^{g_1} F^{\mathbb{P}^2/E}_{g_1,1} \cdot \sum_{\substack{0<h\leq g_2, \\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g_2,n+1}(h)}} (-1)^{h-1} F^E_{h,\mathbf{a}} \cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})}.$$

So it remains to check the total contribution of those T_2 to $\mathcal{C}_1$ when we vary $(\mathbf{g}, \mathbf{a}, \mathbf{B})$ and s, t is

$$\sum_{\substack{g_1+g_2=g \\ n_1+n_2=n, n_1>0}} \binom{n}{n_1} (-1)^{g_1} F^{\mathbb{P}^2/E}_{g_1,n_1+1} \sum_{\substack{0<h\leq g_2, \\ (\mathbf{g},\mathbf{a},\mathbf{B})\in\mathcal{P}_{g_2,n_2+1}(h)}} (-1)^{h-1} F^E_{h,\mathbf{a}} \cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})}$$

which is obvious. □

lem-tt **Lemma 5.5.**

$$\mathcal{C}_2 = \mathcal{C}_2'$$

Proof. Given a term

$$\frac{1}{18}(-1)^h F^E_{h-1,\mathbf{a}-\vec{e}_i-\vec{e}_j} \cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g},\mathbf{a},\mathbf{B})}$$

in $\mathcal{C}_2$, it can be rewritten as

$$\frac{|\operatorname{Aut}(\mathbf{g}',\mathbf{a}',\mathbf{B}')|}{|\operatorname{Aut}(\mathbf{g},\mathbf{a},\mathbf{B})|}\frac{1}{2}(-1)^{h'-1} F^E_{h',\mathbf{a}'} \cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g}',\mathbf{a}',\mathbf{B}')}$$

where $h' = h-1$, $\mathbf{g}' = \mathbf{g}$, $\mathbf{a}' = \mathbf{a} - \vec{e}_i - \vec{e}_j$ and $\mathbf{B}'$ is a partition of the set $\{1, 2, \cdots, n+2\}$ which can be determined from $\mathbf{B}$ by adding $n+1$ to the set B_i and adding $n+2$ to the set B_j. Obviously, $(\mathbf{g}', \mathbf{a}', \mathbf{B}') \in P_{g-1,n+2}(h')$. So

$$\frac{1}{2}(-1)^{h'-1} F^E_{h',\mathbf{a}'} \cdot \mathrm{Cont}^{\mathbb{P}^2/E}_{(\mathbf{g}',\mathbf{a}',\mathbf{B}')}$$

becomes one summand in $\mathcal{C}'_2$. Now Lemma 5.5 follows from the fact that for a fixed $(\mathbf{g}', \mathbf{a}', \mathbf{B}') \in P_{g-1,n+2}(h')$, there are exactly $\frac{|\operatorname{Aut}(\mathbf{g},\mathbf{a},\mathbf{B})|}{|\operatorname{Aut}(\mathbf{g}',\mathbf{a}',\mathbf{B}')|}$ choices of $(\mathbf{g}, \mathbf{a}, \mathbf{B}), i, j$ which gives $(\mathbf{g}', \mathbf{a}', \mathbf{B}')$ via the above procedure. Actually, we see that $(\mathbf{g}, \mathbf{a}, \mathbf{B})$ can be determined from $(\mathbf{g}', \mathbf{a}', \mathbf{B}')$. The only flexibility comes from the choices of i and j.

□

Lemma 5.6.

$$\mathcal{C}_3 = \mathcal{C}'_3$$

Proof. The proof is similar to the proof of Lemma 5.5. We omit the details here. □

5.4. ***S*-degree bound on $F_g^{\mathbb{P}^2/E}$.** We prove the S-degree bound of Theorem 1.3 (equivalently the B-degree bound of Theorem 1.6), that is, for every $g \geq 2$,

$$\deg_S F_g^{\mathbb{P}^2/E} \leq 2g-3\,. \tag{80}$$ eq:S_bound

As we only have

$$\deg_S F_g^{K_{\mathbb{P}^2}} \leq 3g-3$$

in general, the bound (80) is not an obvious consequence of Theorem 1.2 and requires a non-trivial cancellation of higher degree terms. For example, we have observed such cancellation in the genus 2 computation of Section 4.7 (vanishing of the terms in S^3/X and S^2). Rather than trying to prove directly this cancellation in general, we show that (80) follows from the holomorphic anomaly equation (22).

We prove (80) by induction on g. Using (60), we rewrite the holomorphic anomaly equation (22) as

$$\frac{X}{3}\frac{\partial}{\partial S}F_g^{\mathbb{P}^2/E} = \frac{1}{4}\sum_{\substack{g_1+g_2=g,\\ g_i>0 \text{ for } i=1,2}} \left(q\frac{dF_{g_1}^{\mathbb{P}^2/E}}{dq}\right)\cdot\left(q\frac{dF_{g_2}^{\mathbb{P}^2/E}}{dq}\right). \tag{81}$$ eq:HAE_rw

By induction, we have for $g_j \geq 2$

$$\deg_S F_{g_j}^{\mathbb{P}^2/E} \leq 2g_j - 3\,,$$

and so using (62),

$$\deg_S \left(q \frac{dF_{g_j}^{\mathbb{P}^2/E}}{dq} \right) \leq 2g_j - 2\,.$$

By (68), this bounds also holds for $g_j = 1$. Therefore, the S-degree of the right-hand side of (81) is $\leq (2g_1 - 2) + (2g_2 - 2) = 2g - 4$ and so the S-degree of $F_g^{\mathbb{P}^2/E}$ is $\leq 2g - 3$.

sec:appx

Appendix A. Product formulas for the relative theory

In this section, we mainly want to prove a product formula relating the relative theory of possibly disconnected domain and the one with connected domain (see Lemma A.4). It implies Lemma 2.11 and so completes the last step in the localization calculation of Section 2.4. Before proving Lemma A.4, we need a product formula relating the rubber theory of possibly disconnected domain and the one with connected domain.

Let the rubber target be a chain of $\mathbb{P}_D(L \oplus \mathcal{O})$. Let Γ be a possibly disconnected rubber graph. The rubber theory is relative to the two ends of the rubber target. One of the ends of the rubber target has normal bundle $L^\vee$, and we denote target psi-class associated to this end by Ψ_∞ (see also [GV05, Section 2.5]). The target psi-class associated to the other end will be denoted by Ψ_0.

For a vertex $v \in V(\Gamma)$, recall that γ_v is the graph consisting of a single vertex v plus all the decorations on v. When the curve class of a vertex v is pushed forward to 0 on D and has only two relative markings without absolute markings, we call v an *unstable vertex*. Let $V^{\mathrm{s}}(\Gamma)$ be the set of stable vertices and $V^{\mathrm{us}}(\Gamma)$ the set of unstable vertices. In the following theorem, we need to consider the stabilization map from $\overline{\mathcal{M}}_\Gamma^{\bullet\sim}(D)$ to the moduli of stable maps of D. Note that stabilization does not make sense on components corresponding to unstable vertices. Let $m(v)$ be the number of half-edges on the vertex $v \in V(\Gamma)$. Our convention for the stabilization map

$$\tau : \overline{\mathcal{M}}_\Gamma^{\bullet\sim}(D) \to \prod_{v \in V^{\mathrm{s}}(\Gamma)} \overline{\mathcal{M}}_{g(v),m(v)}(D, b(v)) \times D^{|V^{\mathrm{us}}(\Gamma)|}$$

is that we firstly stabilize stable components and send unstable components to the corresponding points in D. If v is an unstable vertex, the two relative markings have the same multiplicity and we denote by d_v.

thm_appd **Theorem A.1.** *We have the following identity.*

$$\tau_*\Big(\frac{1}{t-\Psi_\infty}\cap[\overline{\mathcal{M}}^{\bullet\sim}_{\Gamma}(D)]^{\mathrm{vir}}\Big)$$
$$=\frac{1}{\prod_{v\in V^{\mathrm{us}}(\Gamma)}d_v}\tau'_*\left(\prod_{v\in V^{\mathrm{s}}(\Gamma)}p_v^*\left(\frac{1}{t-\Psi_\infty}\right)\cap\big[\prod_{v\in V^{\mathrm{s}}(\Gamma)}\overline{\mathcal{M}}^{\sim}_{\gamma_v}(D)\times D^{|V^{\mathrm{us}}(\Gamma)|}\big]^{\mathrm{vir}}\right),$$

where each Ψ_∞ on the left-hand side and right-hand side are the target psi-classes on their corresponding moduli, p_v is the projection of the product of rubber moduli to $\overline{\mathcal{M}}^{\sim}_{\gamma_v}(D)$, and τ,τ' are the corresponding stabilization maps to $\prod_{v\in V^{\mathrm{s}}(\Gamma)}\overline{\mathcal{M}}_{g(v),m(v)}(D,b(v))\times D^{|V^{\mathrm{us}}(\Gamma)|}$.

Proof. The theorem is an application of [FWY19, Theorem 4.1]. Note that [FWY19, Theorem 4.1] is stated for connected domains. But our situation requires disconnected domain and it is straightforward to check that the same proof works for disconnected domains. We omit the details here.

Let us recall the content of the theorem. In [FWY19, Theorem 4.1], we consider $P_{D_0,r}$, which is the rth root stack of $P:=\mathbb{P}_D(L\oplus\mathcal{O})$. After the root stack construction, there are two invariant substacks $\mathcal{D}_0,\mathcal{D}_\infty$ under the fiberwise $\mathbb{C}^*$ action. $\mathcal{D}_0$ is the one isomorphic to the root gerbe $\sqrt[r]{D/L}$ and $\mathcal{D}_\infty$ is the one isomorphic to D.

The result [FWY19, Theorem 4.1] compares rubber theory with the orbifold Gromov–Witten theory of the gerbe $\mathcal{D}_0$. The topological data Γ imposes contact orders on the two ends of the rubber target. In [FWY19], the end with normal bundle L is called the 0-side, and the other end with normal bundle $L^\vee$ is called the ∞-side. For the gerbe theory over $\mathcal{D}_0$, we still use Γ to represent the topological data where the weights of roots (contact order conditions) are replaced by suitable ages. A relative marking on the 0-side of order μ corresponds to an orbifold marking of age μ/r, whereas a relative marking on the ∞-side of order μ corresponds to an orbifold marking of age $(r-\mu)/r$. Under this convention, denote by $\overline{\mathcal{M}}^{\bullet}_{\Gamma}(\mathcal{D}_0)$ the corresponding moduli of twisted stable maps to the gerbe $\mathcal{D}_0$.

For simplicity, we assume all vertices on Γ are stable. Denote the forgetful maps by the following.

$$\tau_1:\overline{\mathcal{M}}^{\bullet}_{\Gamma}(\mathcal{D}_0)\to\prod_{v\in V(\Gamma)}\overline{\mathcal{M}}_{g(v),n(v)}(D,\beta(v)),$$
$$\tau_2:\overline{\mathcal{M}}^{\bullet\sim}_{\Gamma}(D)\to\prod_{v\in V(\Gamma)}\overline{\mathcal{M}}_{g(v),n(v)}(D,\beta(v)),$$

where $g(v),n(v),\beta(v)$ are the genus, number of markings and the curve classes of the vertex v. On the root gerbe $\mathcal{D}_0$, there is a universal line bundle L_r. Let $\pi:\mathcal{C}\to\overline{\mathcal{M}}^{\bullet}_{\Gamma}(\mathcal{D}_0)$ be the universal curve and $f:\mathcal{C}\to\mathcal{D}_0$ the map to the target. Let $\mathcal{L}_r=f^*L_r$ and

$$-R^*\pi_*\mathcal{L}_r:=R^1\pi_*\mathcal{L}_r-R^0\pi_*\mathcal{L}_r\in K^0(\overline{\mathcal{M}}^{\bullet}_{\Gamma}(\mathcal{D}_0)).$$

As mentioned above, [FWY19, Theorem 4.1] can be generalized to moduli with disconnected domain using the same proof. Let g be the arithmetic genus of the disconnected curve corresponding to the graph Γ. The disconnected version of the theorem implies that the following identity

$$(\tau_2)_*\left(\frac{1}{t-\Psi_\infty}\cap[\overline{\mathcal{M}}^{\bullet\sim}_{\Gamma}(D)]^{\mathrm{vir}}\right) = \frac{\left[(\tau_1)_*\left(\sum_{i=0}^{\infty}\left(\frac{t}{r}\right)^{g-i-1} c_i(-R^*\pi_*\mathcal{L}_r)\cap[\overline{\mathcal{M}}^{\bullet}_{\Gamma}(\mathcal{D}_0)]^{\mathrm{vir}}\right)\right]_{r^0}}{\prod_{i=1}^{\rho_\infty}\left(1+\frac{\mathrm{ev}_i^*\, c_1(L)-\nu_i\psi_i}{t}\right)}$$

holds in $A_*(\prod_{v\in V(\Gamma)}\overline{\mathcal{M}}_{g(v),n(v)}(D,\beta(v)))[t,t^{-1}]$ as Laurent polynomials in the formal variable t. The right-hand side decomposes into a product of such expressions according to the connected component of domain curves, because the disconnected moduli in orbifold theory is a product of connected moduli, and $-R^*\pi_*\mathcal{L}_r$ decomposes into a sum accordingly. Applying the original connected version of [FWY19, Theorem 4.1] to each factor, we conclude that the right-hand side is nothing but the pushforward of a product of $(\frac{1}{t-\Psi_\infty})\cap[\overline{\mathcal{M}}^{\sim}_{\gamma_v}(D)]^{\mathrm{vir}}$, where Ψ_∞ should be treated as the target psi-class of $\overline{\mathcal{M}}^{\sim}_{\gamma_v}(D)$.

If there are unstable vertices in Γ, the unstable vertices in Γ correspond to unstable vertices in the localization of the root stack $P_{D_0,r}$. It is straightforward to add in factors of D in the statement of the thereom and match with the statement that we want to prove.

□

Theorem A.1 has interesting corollaries. If we take the $1/t$ coefficient of the theorem, we find the following corollary.

Corollary A.2. *If Γ has more than two stable vertices, we have that*

$$\tau_*([\overline{\mathcal{M}}^{\bullet\sim}_{\Gamma}(D)]^{\mathrm{vir}})=0\,.$$

If we take the coefficient of $1/t^{|V^s(\Gamma)|}$, we obtain the following corollary.

Corollary A.3.

$$\tau_*(\Psi_\infty^{(|V^{\mathrm{s}}(\Gamma)|-1)}\cap[\overline{\mathcal{M}}^{\bullet\sim}_{\Gamma}(D)]^{\mathrm{vir}}) = \frac{1}{\prod_{v\in V^{\mathrm{us}}(\Gamma)} d_v}\tau'_*\left([\prod_{v\in V^{\mathrm{s}}(\Gamma)}\overline{\mathcal{M}}^{\sim}_{\gamma_v}(D)\times D^{|V^{\mathrm{us}}(\Gamma)|}]^{\mathrm{vir}}\right).$$

More importantly, we can use Theorem A.1 to deduce Lemma 2.11. Let us repeat the statement here:

lem_split1 **Lemma A.4.** *We have the following identity:*

$$\begin{aligned}&\tau_*\left(\frac{t}{t+\Psi}\cap[\overline{\mathcal{M}}^{\bullet}_{\Gamma_2'}(X,D)]^{\mathrm{vir}}\right)\\ =&\tau'_*\left(\prod_{v\in V(\Gamma_2')}p_v^*\left(\frac{t}{t+\Psi}\right)\cap\Big[\prod_{v\in V(\Gamma_2')}\overline{\mathcal{M}}_{\gamma_v}(X,D)\Big]^{\mathrm{vir}}\right),\end{aligned}$$

where p_v is the projection to the factor corresponding to v, and τ,τ' are the corresponding stabilization maps to

$$\prod_{v\in V(\Gamma_2')}\overline{\mathcal{M}}_{g(v),m(v)}(X,b(v))\times_{X^{r(v)}}D^{r(v)}$$

with $m(v)$ the number of half-edges on v and $r(v)$ the number of roots on v.

Lemma 2.11 is a special case of Lemma A.4 with $m(v)=r(v)=1$.

Proof. First of all, we have a weaker statement of the lemma as follows.

lem_split **Lemma A.5.** *We have*

$$\tau_*[\overline{\mathcal{M}}^{\bullet}_{\Gamma_2'}(X,D)]^{\mathrm{vir}}=\tau'_*\left(\Big[\prod_{v\in V(\Gamma_2')}\overline{\mathcal{M}}_{\gamma_v}(X,D)\Big]^{\mathrm{vir}}\right).$$

In the literature, this product rule is mentioned multiple times (e.g. in [MP06, Section 1.8]) but we are unable to find a complete proof so far. Here we provide another two-sentence proof using the main result of [FWY19]. Note that both $\tau_*[\overline{\mathcal{M}}^{\bullet}_{\Gamma_2'}(X,D)]^{\mathrm{vir}}$ and $\tau'_*\left(\big[\prod_{v\in V(\Gamma_2')}\overline{\mathcal{M}}_{\gamma_v}(X,D)\big]^{\mathrm{vir}}\right)$ are equal to the r^0 parts of the pushforward of their corresponding orbifold virtual cycle of the corresponding root stacks. The lemma holds because the product rule of virtual cycles holds in the orbifold Gromov–Witten theory.

Next, observe that $\dfrac{\Psi}{-t-\Psi}$ is in fact $\dfrac{\delta}{-t-\Psi}$ where δ is the divisor corresponding to the locus where the target degenerates. Expanding δ, we have

$$\begin{aligned}&\left(\frac{\delta}{-t-\Psi}\right)\cap[\overline{\mathcal{M}}^{\bullet}_{\Gamma_2'}(X,D)]^{\mathrm{vir}}\\ =&\sum_{\mathfrak{i}=((\Gamma_2')_1,(\Gamma_2')_2)}\frac{\prod_{i=1}^{m(\mathfrak{i})}d_i}{\mathrm{Aut}(\mathfrak{i})}(\tau_{\mathfrak{i}})_*\left(\left(\frac{1}{-t-\Psi_0}\right)\cap[\overline{\mathcal{M}}^{\bullet\sim}_{(\Gamma_2')_1}(D)\times_{D^{k_{\mathfrak{i}}}}\overline{\mathcal{M}}^{\bullet}_{(\Gamma_2')_2}(X,D)]^{\mathrm{vir}}\right),\end{aligned} \tag{82}$$

eqn:productrule

where it is easy to see that the restriction of Ψ to $\overline{\mathcal{M}}^{\bullet\sim}_{(\Gamma_2')_1}(D)$ becomes Ψ_0. The splitting of Γ_2' into $(\Gamma_2')_1,(\Gamma_2')_2$ are determined by the splitting of each component of Γ_2'. Thus, the next step is to split the virtual classes according to the components of $(\Gamma_2')_1$ and $(\Gamma_2')_2$. First of all, Lemma A.5 tells us that the pushforward of $[\overline{\mathcal{M}}^{\bullet}_{(\Gamma_2')_2}(X,D)]^{\mathrm{vir}}$ splits into a product of cycles according to each component (product rule). As to the rubber moduli, Theorem A.1 implies that $\dfrac{1}{-t-\Psi_0}\cap[\overline{\mathcal{M}}^{\bullet\sim}_{(\Gamma_2')_1}(D)]^{\mathrm{vir}}$ also satisfies the product rule. Note that the statement of the theorem uses Ψ_∞ in order to match with [FWY19, Theorem 4.1]. But switching the rubber target upside down turns Ψ_∞ into Ψ_0 with the rest unchanged. To fit the exact statement of the theorem, we also need to change t into $-t$.

Now we can apply (82) to the left-hand side of Lemma A.4. Note that a vertex in $(\Gamma_2')_1$ can be either stable or unstable. On the right-hand side, we expand the product $\prod_{v\in V(\Gamma_2')} p_v^*\left(1+\dfrac{\Psi}{-t-\Psi}\right)$ and apply (82) again to each $\dfrac{\Psi}{-t-\Psi}\cap[\overline{\mathcal{M}}_{\gamma_v}(X,D)]^{\mathrm{vir}}$. The summand 1 in $p_v^*\left(1+\dfrac{\Psi}{-t-\Psi}\right)$ covers the case when γ_v splits into $((\Gamma_2')_1,(\Gamma_2')_2)$ where all the vertices in $(\Gamma_2')_1$ are unstable. Such cases appear on the left-hand side of Lemma A.4 but are missing when applying (82) to each $\dfrac{\Psi}{-t-\Psi}\cap[\overline{\mathcal{M}}_{\gamma_v}(X,D)]^{\mathrm{vir}}$ on the right-hand side. It is straightforward to check that both sides match. □

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