TROPICAL GEOMETRY, LOG GEOMETRY AND APPLICATIONS TO GROMOV-WITTEN THEORY

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Warning: nothing in Lectures I, II and III is original. I do not include references in order to not break the flow of the exposition. References to be added later.

1. Lecture I

1.1. **General introduction.** One of the few systematic tools we have in Gromov-Witten theory is the degeneration formula. Given a degeneration of a smooth projective variety X into two smooth projective varieties X_1 and X_2 glued along a common smooth divisor D, the degeneration formula expresses the Gromov-Witten theory of X in terms of the Gromov-Witten theory of X_1 relative to D and of the Gromov-Witten theory of X_2 relative to D.

In particular, one needs to make sense of the Gromov-Witten theory of a smooth projective variety Y relative to a smooth divisor D. Working in some relative setting, the obvious difficulty is the compactness of the moduli spaces of stable maps: stable maps intersecting D properly do not form proper moduli spaces in general, as limiting stable maps can have components falling into D. Naive approaches, such that taking a closure in the space of all stable maps, do not extend at the virtual level. The traditional way to solve this difficulty is to consider stable maps with values in expanded degenerations of Y along D. In other words, the moduli spaces are compactified by allowing stable maps to a modified target space.

Another approach to this compactness issue is to use log geometry. Rather than to modify the target space, the idea is to compactify moduli spaces by stable maps enhanced at some log level: both the source curves and the map will come with some additional information. For Gromov-Witten theory relative to a smooth divisor, expanded degenerations and log approaches are essentially equivalent: they define the same numerical invariants, and simply repackage the same information in different ways, either by remembering how components fall into the bubbles of expanded degenerations, or by remembering some log structure and some non-trivial log map for components falling into the divisor.

The main advantage of the log approach is that it generalizes naturally to more complicated divisors. It gives a way to define Gromov-Witten theory for a smooth projective variety Y relative to a reduced normal crossing divisor D, or for special fibers of normal crossings degenerations. In this more general setting, the analogue of the degeneration formula takes a more complicated form.

The goal of the first two lectures is to explain the "decomposition formula", decomposing the degenerated contributions in a way indexed by some tropical curves. This is roughly the first half towards the general degeneration formula. For some concrete applications, this first half is already enough, as we will see in the last two lectures.

Some comments on log geometry. Log geometry is some useful language to talk in a systematic way about things like normal crossing compactifications and degenerations in algebraic geometry. This framework is not a priori motivated by Gromov-Witten theory. It just happens that it applies nicely to Gromov-Witten theory, essentially because only nodal curves are involved.

An essential point to get some intuitive understanding of log geometry is the realization that log geometry has a combinatorial shadow given by tropical geometry. They are many possible points of view on tropical geometry which should be covered in some elementary introduction to this topic. I am not going to do that. I will simply present tropical objects as coming from log geometry and encoding the basic combinatorics of normal crossing degenerations. The advantage of tropical geometry over log geometry is that it is possible to draw pictures. The main message of this series of lectures is that it is possible to do a lot (but not everything!) using tropical pictures.

1.2. **Toric geometry.** Toric geometry is for log geometry as linear algebra is for differential geometry: a more elementary topic which will be used as local model.

Given $T = (\mathbb{C}^*)^n$ some n-dimensional complex torus, we can construct to lattices:

- The lattice $M = \mathrm{Hom}(T, \mathbb{C}^*) \simeq \mathbb{Z}^n$ of characters of T: $(z_1, \dots, z_n) \mapsto z_1^{b_1} \dots z_n^{b_n}$.
- The lattice $N = \mathrm{Hom}(\mathbb{C}^*, T) \simeq \mathbb{Z}^n$ of cocharacters, or one-paramter subgroups of T, $z \mapsto (z^{a_1}, \dots, z^{a_n})$.

Characters are simply Laurent monomials, i.e. T-invariant functions on T. We have $T = \mathrm{Spec}\ \mathbb{C}[M]$, where $\mathbb{C}[M]$ is the group algebra of M, i.e. simply the algebra of Laurent polynomials in $z_1, \dots, z_n$. Think about cocharacters are ways to go to "infinity" in T.

Composing characters with cocharacters gives a natural duality between M and N: we have canonically $M = \mathrm{Hom}(N, \mathbb{Z})$ and $N = \mathrm{Hom}(M, \mathbb{Z})$.

By definition, a toric variety of dimension n is a T-equivariant normal partial compactification of T. Given such toric variety X, we define $\Sigma(X) \subset N_{\mathbb{R}}$ as being the $\mathbb{R}_{\geqslant 0}$-span of the set of cocharacters $\mathbb{C}^* \to T$ having a limit in X as $z \to 0$. In other words, the set of integral points $\Sigma(X) \cap N$ is the set of ways to go to infinity in T which have a limit in the partial compactification X. One can show that $X - T$ is a union of divisors. Intersections of these divisors define a natural stratification of X. Positions in this stratification of the limiting points of cocharacters naturally define a decomposition of $\Sigma(X)$ into striclty convex cones. This cone decomposition of $\Sigma(X)$ is naturally the dual intersection complex of the stratification of X. (Pictures!)

One can show that X is affine if and only if $\Sigma(X)$ contains a unique cone of maximal dimension. If σ is this cone, then $X = \mathrm{Spec}\ \mathbb{C}[P]$, with $P = \sigma_{\mathbb{R}}^{\vee} \cap M$, where $\sigma_{\mathbb{R}}^{\vee}$ is the dual cone of $\sigma_{\mathbb{R}} = \sigma \otimes \mathbb{R}$. Remark that $P = \sigma_{\mathbb{R}}^{\vee} \cap M = \mathrm{Hom}(\sigma_{\mathbb{R}}^{\vee} \cap M, \mathbb{R}_{\geqslant 0})$ is a monoid and that $\mathbb{C}[P]$ is the corresponding monoid algebra. So, in this case, X can be reconstructed from $\sigma_{\mathbb{R}}$. There is a one to one correspondence between affine toric varieties of dimension n and rational convex polyhedral cones in $\mathbb{R}^n$ (up to action of $GL(n, \mathbb{Z})$). If σ is of dimension n, then P has no non-zero invertible elements, and X has a unique T-fixed point. To understand log geometry, it will be essential to have in mind a good picture of affine toric varieties.

More generally, a toric variety X can be reconstructed from its fan, i.e. $\Sigma(X)$ with its cone decomposition, by gluing together affine toric varieties defined by the cones of maximal dimension. Morphisms between toric varieties: torus equivariants morphisms inducing group morphisms between the corresponding tori. Fan morphims: the image of every cone is contained in a cone.

Examples: $(\mathbb{C}^*)^n$, $\mathbb{C}^n$, $\mathbb{P}^n, \dots$ (Pictures!)

A toric variety X is proper if and only if $\Sigma(X) = N_{\mathbb{R}}$.

A toric variety X is smooth if and only if all the cones of $\Sigma(X)$ of maximal dimension are isomorphic to a standard positive quadrant.

Some not completely trivial application of toric geometry: systematic construction of toric degenerations. Given a toric variety X, one can construct toric degeneration of X by drawing rational polyhedral decompositions of $N_{\mathbb{R}}$ of asymptotic fan coinciding with the fan of X. Indeed, given such decomposition $\mathcal{P}$, the closure of the cone over $\mathcal{P} \times \{1\}$ in $N \times \mathbb{Z}$ defines a fan whose corresponding toric variety Y is the total space of a degeneration $\pi: Y \to \mathbb{C}$, whose generic fiber is X and whose special fiber $\pi^{-1}(0)$ has irreducible toric components X_v indexed by vertices of $\mathcal{P}$, of fan the local picture of $\mathcal{P}$ at v. More precisely, if m_v is the smallest positive integer such that $m_v v \in N = \mathbb{Z}^n$, then m_v is the multiplicity of X_v in $\pi^{-1}(0)$, i.e. as divisor class in Y, we have

$$\pi^{-1}(0) = \sum_v m_v X_v \,.$$

In particular, $\pi^{-1}(0)$ is reduced if and only if all $v \in N = \mathbb{Z}^n$.

Remarks:

- It is easy to draw a polyhedral decomposition leading to some very complicated special fiber. Corresponding equations are in general quite complicated.
- The "decomposition formula" will in some sense reduce to

$$\pi^{-1}(0) = \sum_v m_v X_v \,.$$

Exercise: $X = \mathbb{C}^*$, $\mathcal{P}$ a segment of integral length k, get the family $xy = t^k$, local smoothing of a nodal curve, with an A_{k-1}-singularity in the total space.

1.3. **Log geometry.** Monoid: commutative with unit. Invertible elements, associated group. Warning: additive versus multiplicative notations.

Let X be a scheme. Then $\mathcal{O}_X$ is a sheaf in (multiplicative) monoids.

Definition 1.1. *A prelog structure on a scheme X is the data of an (étale) sheaf of monoids M_X on X and of a morphism $\alpha_X: M_X \to \mathcal{O}_X$ of sheaves of monoids.*

Remark: A morphism of monoids sends invertible elements to invertible elements.

Definition 1.2. *A prelog structure (M_X, α_X) is a log structure if the induced morphism $\alpha_X^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism. A log scheme is a scheme with a log structure.*

Remark: Every prelog structure has some associated log structure, pushout of the natural diagram $\mathcal{O}_X^* \leftarrow \alpha_X^{-1}(\mathcal{O}_X^*) \to M$.

Definition 1.3. *Let $f: X \to Y$ be a morphism of schemes. Let (M_Y, α_Y) be some log structure on Y. Then the pullback log structure by f is the log structure on X associated to the prelog structure $f^{-1}M_Y \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.*

Definition 1.4. *A log morphism between two log schemes X and Y is a scheme theoretic morphism $f: X \to Y$, equipped with a lift of the natural map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ as a map of sheaves of monoids $f^{-1}M_Y \to M_X$.*

Remark: In general, a log morphism contains more information than the underlying scheme theoretic morphism.

Definition 1.5. *The ghost sheaf of a log scheme X is the sheaf of monoids $\overline{M}_X = M_X/\alpha_X^{-1}(\mathcal{O}_X^*)$.*

Remark: a log morphism $f: X \to Y$ induces a map at the ghost sheaf level

$$f^\flat : f^{-1}\overline{M}_Y \to \overline{M}_X\,.$$

Definition 1.6. *A fs monoid is a monoid P such that:*

- *P^{gp} is finitely generated as abelian group.*
- *The natural map $P \to P^{gp}$ is injective.*
- *If $x \in P^{gp}$ is such that there is a positive integer n such that $nx \in P$, then $x \in P$.*

Up to torsion, a fs monoid is the same thing than a toric monoid. More precisely, toric monoids are exactly fs monoids P such that P^{gp} is free.

If P is a fs monoid, then $X = \operatorname{Spec} \mathbb{C}[P]$, with the log structure associated to the prelog structure $P \to \mathbb{C}[P]$, is a log scheme. We call this log structure the standard log structure on $\operatorname{Spec} \mathbb{C}[P]$.

Definition 1.7. *A fs log scheme is a log scheme X such that, for every point $x \in X$, there exists an (étale) neighborhood U_x, a fs monoid P_x, a scheme theoretic map $f_x: U_x \to$ Spec $\mathbb{C}[P_x]$, such that the log structure on U_x induced by X is the pullback by f_x of the standard log structure on* Spec $\mathbb{C}[P_x]$.

2. Lecture II

Example: If X is a regular scheme and D is a reduced normal crossing divisor, then the sheaf M_X of functions invertible outside D is a fs log structure. Ghost sheaf: $\mathbb{N}^r$ at the intersection of r components of D.

Exercise: The fs log structures on the point Spec $\mathbb{C}$ are in one to one correspondence with the fs monoids P without invertible non-zero elements. The log structure determined by such monoid P is given explicitly by $\mathbb{C}^* \oplus P \to \mathbb{C}$, $(x,p) \mapsto x$ if $p = 0$, $(x,p) \mapsto 0$ if $p \neq 0$. We denote pt_P the corresponding log point. It is the log structure pullback from the standard log structure on Spec $\mathbb{C}[P]$ to the unique T-fixed point. The ghost sheaf is simply P.

Re-do algebraic geometry putting log everywhere: log smooth, log étale, log differential, log stacks, ...

Proposition 2.1. *A log morphism $f: X \to Y$ is log smooth if and only if it is, étale locally on X, pullback of a log morphism of the form* $\operatorname{Spec} \mathbb{C}[P] \to \operatorname{Spec} \mathbb{C}[Q]$, *induced by a morphism of fs monoids $Q \to P$ such that $Q^{gp} \to P^{gp}$ is injective.*

A log morphism $f: X \to Y$ is log étale if and only if it is, étale locally on X, pullback of a log morphism of the form $\operatorname{Spec} \mathbb{C}[P] \to \operatorname{Spec} \mathbb{C}[Q]$, *induced by a morphism of fs monoids $Q \to P$ such that $Q^{gp} \to P^{gp}$ is injective with finite cokernel.*

Remark: Up to torsion, a log smooth morphism is étale locally pullback of a dominant morphism of toric varieties.

Remark: Up to torsion, a log smooth morphism is étale locally pullback of a birational morphism of toric varieties. In particular, toric blow-ups are log étale (but no flat!).

2.1. Tropicalization.

Definition 2.1. *A cone is a pair $\sigma = (\sigma_{\mathbb{R}}, N)$, with N a finite rank lattice, $N \simeq \mathbb{Z}^n$, and $\sigma_{\mathbb{R}} \subset N_{\mathbb{R}}$ is a top dimensional strictly convex rational polyhedral cone.*

A cone morphism $\varphi: \sigma_1 \to \sigma_2$ is a lattice map $\varphi: N_1 \to N_2$ such that $\varphi(\sigma_{1,\mathbb{R}}) \subset \sigma_{2,\mathbb{R}}$.

Definition 2.2. *A cone complex is a topological space with a presentation as colimit (gluing) of a finite diagram in the category of cones, with all morphisms being face morphisms.*

Definition 2.3. *Let X be a fs log scheme. The tropicalization $\Sigma(X)$ of X is the cone complex obtained by gluing together the cones $((\overline{M}_{X,\eta})^\vee_{\mathbb{R}}, (\overline{M}_{X,\eta})^*)$, for η generic points of the strata on which the ghost sheaf $\overline{M}_X$ is constant.*

Notations: $(\overline{M}_{X,\eta})^\vee = \operatorname{Hom}(\overline{M}_{X,\eta}, \mathbb{N})$, $(\overline{M}_{X,\eta})^* = \operatorname{Hom}(\overline{M}_{X,\eta}, \mathbb{Z})$.

Remark: A log morphism $f: X \to Y$ induces a morphism of cone complexes

$$\Sigma(f): \Sigma(X) \to \Sigma(Y)$$

at the level of tropicalizations.

Examples:

- If X is a toric variety, with its standard log structure, then $\Sigma(X)$ is the fan of X (as abstract cone complex, without embedding in a global $\mathbb{R}^n$).
- Tropicalization of log points: $\Sigma(\mathrm{pt}_P) = P^\vee_{\mathbb{R}}$.
- Tropicalization of a reduced normal crossing divisor: dual intersection complex.

2.2. Stable log maps.

Definition 2.4. *A prestable log curve is a flat proper log smooth morphism between fs log schemes, $\pi: C \to W$, whose fibers are reduced and connected curves.*

Proposition 2.2. *Description of a prestable log curve $\pi: C \to \mathrm{pt}_Q$:*

- *C is nodal.*
- *For general $x \in C$, we have $\overline{M}_{C,x} \simeq Q$.*
- *For finitely many smooth point $x \in C$, we have $\overline{M}_{C,x} = Q \oplus \mathbb{N}$ ("marked points").*
- *If x is a node of C, then there is a non-zero element $\rho_x \in Q$ such that $\overline{M}_{C,x} = Q \oplus_{\mathbb{N}} \mathbb{N}^2$, where $\mathbb{N} \to \mathbb{N}^2$ is the diagonal map, and $\mathbb{N} \to Q$, $1 \mapsto \rho_x$.*

Remark: If C has nodes, then the log structure on the base has to be non-trivial.

Tropical picture: $\Sigma(\pi)\colon \Sigma(C) \to \Sigma(\mathrm{pt}_Q)$, family of tropical curves. Generically, nodes correspond to bounded edges, marked points to unbounded edges, components to vertices.

Geometric/tropical interpretation of ρ_x as the length of the bounded edge dual to the node x, viewed as a function on $\Sigma(\mathrm{pt}_Q)$.

Definition 2.5. *A stable log curve is a prestable log curve whose underlying curve is stable.*

Moduli space of stable log curves? Log space such that to give a stable log curve over a log scheme W is equivalent to give a log morphism from W to this log space.

Proposition 2.3. *The moduli space of stable log curves is $\overline{M}_{g,n}$, viewed as a fs log stack for the natural divisorial log structure given by the boundary divisors.*

Remark: Scheme theoretic points of $\overline{M}_{g,n}$ parametrize stable log curves $\pi\colon C \to W$ with log structure on W pullback by $W \to \overline{M}_{g,n}$ of the log structure on $\overline{M}_{g,n}$. Given a stable curve C, the corresponding stable log curve with basic log structure is $C \to \mathrm{pt}_{Q^{bas}}$, where Q^{bas} has one copy of $\mathbb{N}$ for each node. Equivalently, $\Sigma(\overline{M}_{g,n})$ is a moduli space of tropical curves/metrized graphs. Remark that this says nothing new about $\overline{M}_{g,n}$, it is simply some rewriting of the well-known boundary structure of $\overline{M}_{g,n}$.

Exercise: Draw $\Sigma(\overline{M}_{0,4})$ and $\Sigma(\overline{M}_{1,1})$.

3. Lecture III

Definition 3.1. *Let $X \to B$ be a log morphism between fs log schemes. A n-marked stable log map is a diagram*

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ W & \longrightarrow & B\,, \end{array}$$

in the category of fs log schemes, with n sections p_i of π, such that

- *$\pi\colon C \to W$ is a prestable log curve,*
- *if U is the open locus of C on which π is smooth, then*

$$\overline{M}_C|_U = \pi^* \overline{M}_W \oplus \bigoplus_{i=1}^n \mathbb{N}_{p_i(W)}\,,$$

- *the underlying scheme theoretic diagram is a stable map.*

Information contained in the map at the ghost sheaf level. Assume $W = \mathrm{pt}_Q$. For every point $x \in C$, we denote $P_x \coloneqq \overline{M}_{X,f(x)}$.

- For $x = \eta$ a generic point of C, we have a morphism of monoid $\varphi_\eta\colon P_\eta \to Q$.
- For $x = p$ a marked point of C, denote $u_p\colon P_p \to Q \oplus \mathbb{N} \to \mathbb{N}$.

- For x a node of C, intersection of two components with generic points η_1 and η_2, denoting $\chi_i\colon P_x \to P_{\eta_i}$ the specialization morphisms, there exists $u_x\colon P_x \to \mathbb{Z}$ (defined up to sign) such that
$$\varphi_{\eta_2}(\chi_2(m)) - \varphi_{\eta_1}(\chi_1(m)) = u_x(m)\rho_x\,.$$

Tropical interpretation, dualize, family of tropical curves parametrized by $\Sigma(\mathrm{pt}_Q)$ mapping to $\Sigma(X)$.

- Image of the vertex dual to η.
- $u_p \in (\Sigma(X))(\mathbb{Z})$, asymptotic direction of the image of the unbounded edge, contact order.
- u_x: direction and multiplication factor of the length of the edge dual to the node x

Type $\beta = (g, n, A, u_{p_1}, \dots, u_{p_n})$, $A \in H_2(X)$, $u_{p_i} \in (\Sigma(X))(\mathbb{Z})$ contact orders.

Moduli space of stable log maps of type β? Log space such that to give a stable log map to $X \to B$ over a log scheme W is equivalent to give a log morphism from W to this log space. Notation: $\mathcal{M}(X/B, \beta)$.

Theorem 3.1. *If $X \to B$ is proper, then $\mathcal{M}(X/B,\beta)$ is a proper Deligne-Mumford fs log stack of finite type.*

Remark: Scheme theoretic points of $\mathcal{M}(X/B,\beta)$ parametrize stable log maps
$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"] & X \arrow[d] \\ W \arrow[r] & B\,, \end{tikzcd}$$
with log structure on W pullback by $W \to$ of the log structure on $\mathcal{M}(X/B,\beta)$.

Given a dual graph, contact orders u_p for marked points and contact data u_x for nodes, we can consider the monoid Q^{bas} defined as the dual of the monoid underlying a moduli space of tropical maps:
$$(Q^{bas})^\vee = \{((v_\eta)_\eta), (\rho_x)_x) \in \bigoplus_\eta P_\eta^\vee \oplus \bigoplus_{x\,,\mathrm{node}} \mathbb{N} | v_{\eta_2} - v_{\eta_1} = \rho_x u_x\}\,.$$

A stable log map over pt_Q defines a family of tropical maps, and so we have a natural morphism $Q \to Q^{bas}$. We say that the stable log map is basic if this morphism is an isomorphism.

Proposition 3.1. *Scheme theoretic points of $\mathcal{M}(X/B,\beta)$ parametrize basic stable log maps.*

Theorem 3.2. *If $X \to B$ is proper and log smooth, then there is a natural perfect obstruction theory on $\mathcal{M}(X/B,\beta)$, defining a virtual fundamental class $[\mathcal{M}(X/B,\beta)]^{\mathrm{virt}}$.*

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