

The negative Pell equation and applications

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Around Frobenius distributions and related topics III

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History of Pell's equation

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$$x^2 - dy^2 = 1 \text{ to be solved in } x, y \in \mathbb{Z}$$

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Lagrange was the first to give an algorithm with proof of correctness.

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Question: as we vary d , how often is the negative Pell equation soluble?

Solubility over the rationals

Define \mathcal{D} to be the set of squarefree integers having as odd prime divisors only primes $p \equiv 1 \pmod{4}$ and define \mathcal{D}^- to be the set of squarefree integers for which negative Pell is soluble.

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Refined question: what is the density of \mathcal{D}^- inside \mathcal{D} ?

Conjectures on the negative Pell equation

Nagell (1930s) conjectured that

$$\lim_{X \rightarrow \infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}}$$

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Stevenhagen (1995) conjectured that

$$\lim_{X \rightarrow \infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}} = 1 - \alpha,$$

where

$$\alpha = \prod_{j=1}^{\infty} (1 + 2^{-j})^{-1} \approx 0.41942.$$

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Fouvry and Klüners (2010) proved that

$$\frac{5\alpha}{4} \leq \liminf_{X \rightarrow \infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}} \leq \limsup_{X \rightarrow \infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}} \leq \frac{2}{3}.$$

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CKMP (2019) improved the lower bound to

$$\alpha \cdot \sum_{n=0}^{\infty} 2^{-n(n+3)/2} \approx \alpha \cdot 1.28325.$$

Stevenhagen's conjecture is true

Theorem 1 (K., Pagano (2021))

We have

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Corollary 2

We have

$$\#\{d \leq X : d \in \mathcal{D}^-\} \sim C \cdot (1 - \alpha) \cdot \frac{X}{\sqrt{\log X}}.$$

A criterion for solubility

Recall that the narrow class group $\text{Cl}^+(K)$ is defined as the quotient of the ideal group I_K by the principal ideals P_K^+ admitting a totally positive generator, while the class group is the quotient by the principal ideals P_K .

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$$\begin{aligned}x^2 - dy^2 = -1 \text{ is soluble} &\iff \text{fundamental unit } \epsilon \text{ has negative norm} \\ &\iff (\sqrt{d}) \text{ is trivial in } \text{Cl}^+(\mathbb{Q}(\sqrt{d})).\end{aligned}$$

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There is a fundamental exact sequence

$$1 \rightarrow \frac{P_K}{P_K^+} \rightarrow \text{Cl}^+(K) \rightarrow \text{Cl}(K) \rightarrow 1$$

with $\# \frac{P_K}{P_K^+} \in \{1, 2\}$ and $\frac{P_K}{P_K^+}$ generated by (\sqrt{d}) .

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Goal: study joint distribution of $(\text{Cl}^+(K)[2^\infty], \text{Cl}(K)[2^\infty])$.

Genus theory

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and $\text{Cl}^+(K)[2]$ is generated by the ramified prime ideals of \mathcal{O}_K .

If p divides the discriminant of $\mathbb{Q}(\sqrt{d})$, then p ramifies, so

$$\begin{array}{ccc} \mathbb{Q}(\sqrt{d}) & \mathfrak{p} & \mathfrak{p}^2 = (p). \\ | & | & \\ \mathbb{Q} & p & \end{array}$$

There is precisely one relation between the ramified primes.

Let p be an odd prime. The group $\text{Cl}(K)[p^\infty]$ is believed to behave as a random finite, abelian p -group.

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More formally, Cohen and Lenstra conjectured that

$$\lim_{X \rightarrow \infty} \frac{\#\{K \text{ im. quadr.} : |D_K| < X \text{ and } \text{Cl}(K)[p^\infty] \cong A\}}{\#\{K \text{ im. quadr.} : |D_K| < X\}} = \frac{\prod_{i=1}^{\infty} \left(1 - \frac{1}{p^i}\right)}{\#\text{Aut}(A)}$$

for every finite, abelian p -group A .

Gerth's adaptation

Gerth adapted the conjecture of Cohen and Lenstra to $p = 2$

$$\lim_{X \rightarrow \infty} \frac{\#\{K \text{ im. quadr.} : |D_K| < X, 2\text{Cl}(K)[2^\infty] \cong A\}}{\#\{K \text{ im. quadr.} : |D_K| < X\}} = \frac{\prod_{i=1}^{\infty} (1 - \frac{1}{2^i})}{\#\text{Aut}(A)}$$

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Two major difficulties: \mathcal{D} has density 0 in the squarefree integers, and \mathcal{D} naturally ends up in the error term in Smith's proof!

Strategy for Stevenhagen's conjecture

Example 1 (Definition of 2^k -rank)

Take

$$A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} .$$

Then $\text{rk}_2 A = 3$, $\text{rk}_4 A = \text{rk}_8 A = 1$, $\text{rk}_{2^k} A = 0$ for $k > 3$.

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Find for every integer $m \geq 1$, the density of $d \in \mathcal{D}$ for which

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Duality of abelian groups

For a finite abelian group A , define

$$A^\vee := \text{Hom}(A, \mathbb{C}^*).$$

There is a natural pairing

$$\text{Art}_1 : A[2] \times A^\vee[2] \rightarrow \{\pm 1\}, \quad (a, \chi) \mapsto \chi(a).$$

Left kernel of Art_1 is $2A[4]$ and right kernel is $2A^\vee[4]$.

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So 4-rank is determined by the pairing Art_1 . We start by describing $\text{Cl}^{+, \vee}(K)[2]$.

The dual class group

Theorem 4 (Class field theory)

We have an isomorphism

$$\text{Cl}^+(K) \cong \text{Gal}(H^+(K)/K)$$

given by sending a prime ideal \mathfrak{p} to $\text{Art}(\mathfrak{p})$. Furthermore, if K is Galois, this isomorphism respects the natural Galois action of $\text{Gal}(K/\mathbb{Q})$ on both sides.

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From this we get a bijection

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If K is quadratic with odd discriminant, then $\text{Cl}^{+, \vee}(K)[2]$ is generated by the quadratic characters $\chi_{\mathfrak{p}^*}$, where \mathfrak{p}^* satisfies $|\mathfrak{p}^*| = |\mathfrak{p}|$ and $\mathfrak{p}^* \equiv 1 \pmod{4}$.

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If K is quadratic with odd discriminant, then $\mathrm{Cl}^{+, \vee}(K)[2]$ is generated by the quadratic characters χ_{p^*} , where p^* satisfies $|p^*| = |p|$ and $p^* \equiv 1 \pmod{4}$. In particular

$$d \in \mathcal{D} \iff \mathrm{rk}_2 \mathrm{Cl}^+(\mathbb{Q}(\sqrt{d})) = \mathrm{rk}_2 \mathrm{Cl}(\mathbb{Q}(\sqrt{d})) \iff (\sqrt{d}) \in 2\mathrm{Cl}^+(\mathbb{Q}(\sqrt{d}))[4].$$

The Artin pairing

Under the earlier identifications, we have that

$$\text{Art}_1 : \text{Cl}^+(K)[2] \times \text{Cl}^{+,\vee}(K)[2] \rightarrow \{\pm 1\}, \quad (\mathfrak{p}, \chi) \mapsto \chi(\text{Art } \mathfrak{p}).$$

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Let ρ_1, \dots, ρ_t be the prime divisors of d . The Rédei matrix is

$$\begin{array}{ccccc} & \chi_{\rho_1^*} & \chi_{\rho_2^*} & \cdots & \chi_{\rho_t^*} \\ \rho_1 & * & \left(\frac{\rho_2^*}{\rho_1}\right) & \cdots & \left(\frac{\rho_t^*}{\rho_1}\right) \\ \rho_2 & \left(\frac{\rho_1^*}{\rho_2}\right) & * & \cdots & \left(\frac{\rho_t^*}{\rho_2}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_t & \left(\frac{\rho_1^*}{\rho_t}\right) & \left(\frac{\rho_2^*}{\rho_t}\right) & \cdots & * \end{array}.$$

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$$\begin{array}{ccccc} & \chi_{\mathfrak{p}_1^*} & \chi_{\mathfrak{p}_2^*} & \cdots & \chi_{\mathfrak{p}_t^*} \\ \mathfrak{p}_1 & * & \left(\frac{\mathfrak{p}_2^*}{\mathfrak{p}_1}\right) & \cdots & \left(\frac{\mathfrak{p}_t^*}{\mathfrak{p}_1}\right) \\ \mathfrak{p}_2 & \left(\frac{\mathfrak{p}_1^*}{\mathfrak{p}_2}\right) & * & \cdots & \left(\frac{\mathfrak{p}_t^*}{\mathfrak{p}_2}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathfrak{p}_t & \left(\frac{\mathfrak{p}_1^*}{\mathfrak{p}_t}\right) & \left(\frac{\mathfrak{p}_2^*}{\mathfrak{p}_t}\right) & \cdots & * \end{array}.$$

Left kernel subjects on $2\text{Cl}^+(K)[4]$ with 1-dimensional kernel.

Interlude: Stevenhagen's conjecture

For $d \in \mathcal{D}$, recall that $(\sqrt{d}) \in 2\text{Cl}^+(\mathbb{Q}(\sqrt{d}))[4]$.

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Conjecture 1 (Stevenhagen's conjecture)

We have

$$\lim_{X \rightarrow \infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}} = \sum_{j=0}^{\infty} \frac{\mathbb{P}(4\text{-rank of } d \in \mathcal{D} \text{ equals } j)}{2^{j+1} - 1}.$$

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Furthermore,

$$\mathbb{P}(\text{4-rank of } d \in \mathcal{D} \text{ equals } j) = \lim_{t \rightarrow \infty} \mathbb{P}(t \times t \text{ sym. matrix has ker. of dim. } j).$$

The second Artin pairing

There is a natural pairing

$$\text{Art}_2 : 2A[4] \times 2A^\vee[4] \rightarrow \{\pm 1\}, \quad (a, \chi) \mapsto \psi(a), \quad 2\psi = \chi.$$

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Such extensions are of the shape $\mathbb{Q}(\sqrt{d}, \sqrt{a}, \sqrt{\alpha})$, where

$$x^2 = ay^2 + \frac{d}{a}z^2 \text{ with } x, y, z \in \mathbb{Z} \text{ and } \gcd(x, y, z) = 1, \quad \alpha := x + y\sqrt{a}.$$

Reflection principles

In the literature there are many known results that compare different class groups. For example, we have

$$\mathrm{rk}_3\mathrm{Cl}(\mathbb{Q}(\sqrt{d})) \leq \mathrm{rk}_3\mathrm{Cl}(\mathbb{Q}(\sqrt{-3d})) \leq 1 + \mathrm{rk}_3\mathrm{Cl}(\mathbb{Q}(\sqrt{d})),$$

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How can we find such reflection principles?

Smith's idea is to look for situations where the compositum of various Hilbert class fields is in some sense *small*.

Intersections of Hilbert class fields

Take primes p_1, p_2, q_1, q_2 . Now suppose that we have a degree 4 unramified, abelian extension of $\mathbb{Q}(\sqrt{dp_i q_j})$ each lifting the character χ_a .

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Recall that we then get $\alpha_{i,j} \in \mathbb{Q}(\sqrt{a})$ with

$$\text{Norm}_{\mathbb{Q}(\sqrt{a})/\mathbb{Q}}(\alpha_{i,j}) = \frac{dp_iq_j}{a} z_{i,j}^2.$$

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$$\text{Art}_{2,dp_1q_1}(b, \chi_a) + \text{Art}_{2,dp_1q_2}(b, \chi_a) + \text{Art}_{2,dp_2q_1}(b, \chi_a) + \text{Art}_{2,dp_2q_2}(b, \chi_a) = 0$$

for $b \in 2\text{Cl}(\mathbb{Q}(\sqrt{dp_iq_j}))[4]$ a fixed divisor of d .

Another reflection principle

With similar techniques, Smith proves another reflection principle

$$\begin{aligned} & \text{Art}_{2,dp_1q_1}(b, \chi_{ap_1}) + \text{Art}_{2,dp_1q_2}(b, \chi_{ap_1}) + \\ & \text{Art}_{2,dp_2q_1}(b, \chi_{ap_2}) + \text{Art}_{2,dp_2q_2}(b, \chi_{ap_2}) = \sum_{r|b} \text{Frob}_{K_{p_1p_2, q_1q_2}/\mathbb{Q}}(r). \end{aligned}$$

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We develop two new reflection principles. Unlike Smith's work, they make essential use of Hilbert reciprocity in multiquadratic fields.

Bonus slide: new reflection principles

For the Artin pairing with $dp_i q_j$ we have (following Smith's ideas)

$$\begin{aligned} & \text{Art}_{2, dp_1 q_1}(dp_1 q_1, \chi_{ap_1}) + \text{Art}_{2, dp_1 q_2}(dp_1 q_2, \chi_{ap_1}) + \\ & \text{Art}_{2, dp_2 q_1}(dp_2 q_1, \chi_{ap_2}) + \text{Art}_{2, dp_2 q_2}(dp_2 q_2, \chi_{ap_2}) = \text{Frob}_{K_{p_1 p_2, q_1 q_2} / \mathbb{Q}}(\infty). \end{aligned}$$

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Our reciprocity law shows that

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For the pairing between a and χ_a we also develop a new reflection principle.

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Thank you for your attention!