Equidistribution of Frobenius in nilpotent extensions

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Malle's conjecture

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Let G be a finite, non-trivial group. Then there exist numbers $a(G) \in \mathbb{Q}_{>0}$, $b(G) \in \mathbb{Z}_{\geq 0}$ and $c(G) > 0$ such that

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Sometimes $c(G)$ is an Euler product. This is expected to be true for S_n (Malle–Bhargava principle).

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- nonic Heisenberg extensions by Fouvry–K.;
- direct products $S_n \times A$ for $n \in \{3, 4, 5\}$ and A abelian by Wang (with $\#A$ coprime to some values) and later by Masri–Thorne–Tsai–Wang.

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Main result

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Assume GRH. Let G be a nilpotent group with $\#G$ odd. Then

$$\liminf_{X \to \infty} \frac{\#\left\{K/\mathbb{Q} : \prod_{p : I_p \neq \{\mathrm{id}\}} p \leq X, \mathrm{Gal}(K/\mathbb{Q}) \cong G\right\}}{c'(G) X (\log X)^{b'(G)}} \geq 1,$$

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Surprisingly, the corresponding asymptotic

$$\lim_{X\to\infty} \frac{\#\left\{K/\mathbb{Q} : \prod_{p: I_p \neq \{\mathrm{id}\}} p \leq X, \mathrm{Gal}(K/\mathbb{Q}) \cong G\right\}}{c'(G)X(\log X)^{b'(G)}} = 1$$

is not true in general. Counterexamples exist for nilpotency class 2.

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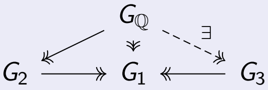

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(ii) *For every place v and for every diagram*

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For every $g_1 \in G_1 - \{\mathrm{id}\}$, every $h_1 \in G_1$ and every α coprime to p satisfying $h_1 g_1 h_1^{-1} = g_1^{\alpha}$, we have

$$(\exists \overline{g_2}, \overline{h_2} \in G_2 : \overline{h_2}\,\overline{g_2}\,\overline{h_2}^{-1} = \overline{g_2}^{\alpha}) \Rightarrow (\exists \overline{g_3}, \overline{h_3} \in G_3 : \overline{h_3}\,\overline{g_3}\,\overline{h_3}^{-1} = \overline{g_3}^{\alpha}),$$

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A non-trivial example is $(G_1, G_2, G_3) = (\mathbb{F}_p^n, U(n+1,p)/Z, U(n+1,p))$.

This is known as the Massey vanishing conjecture (recently proven by Harpaz–Wittenberg for all p and all number fields).

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We have a central exact sequence

$$0 \to \mathbb{F}_2 \to D_4 \xrightarrow{q} \mathbb{F}_2^2 \to 0$$

and a bijection

$$\text{Epi}(G_{\mathbb{Q}}, \mathbb{F}_2^2) \leftrightarrow \{(a, b) \in (\mathbb{Q}^*/\mathbb{Q}^{*2})^2 : a, b \text{ lin. ind.}\}.$$

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It is well-known that a $\mathbb{F}_2^2$-extension $\mathbb{Q}(\sqrt{a}, \sqrt{b})$ of $\mathbb{Q}$ is contained in a D_4-extension if and only if $x^2 = ay^2 + bz^2$ has a non-trivial point.

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If $\rho \in \mathrm{Epi}(G_{\mathbb{Q}}, D_4)$ is a lift of $\pi \in \mathrm{Epi}(G_{\mathbb{Q}}, \mathbb{F}_2)$ and $q : D_4 \twoheadrightarrow \mathbb{F}_2^2$, then

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The variables α_S are squarefree and pairwise coprime, and we have $\text{rad}(|abc|) = \prod_{\emptyset \subset S \subseteq \{a,b,c\}} |\alpha_S|$.

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Now rewrite the above sum as a sum over Legendre symbols involving the variables α_S.

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- Proof can most likely be made unconditional with a suitably strong large sieve for nilpotent extensions.

Questions?

Thank you for your attention!