Counting nilpotent extensions

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Arithmetic Statistics

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Malle's conjecture

Conjecture (Malle's conjecture)

Let G be a finite, non-trivial group. Then there exist numbers $a(G) \in \mathbb{Q}_{>0}$, $b(G) \in \mathbb{Z}_{\geq 0}$ and $c(G) > 0$ such that

$$\#\{K/\mathbb{Q} : D_K \leq X, \mathrm{Gal}(K/\mathbb{Q}) \cong G\} \sim c(G) X^{a(G)} (\log X)^{b(G)}.$$

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Sometimes $c(G)$ is an Euler product. This is expected to be true for S_n (Malle–Bhargava principle).

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$$\sum_{ab^2 \le X} 1 = \sum_{b \le \sqrt{X}} \sum_{a \le X/b^2} 1 = \sum_{b \le \sqrt{X}} \left(\frac{X}{b^2} + O(1) \right)$$
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Both observations fail now.

Ramification theory

Let $K/\mathbb{Q}$ be a Galois extension and suppose that p does not divide $[K:\mathbb{Q}]$. Then

$$v_p(D_K) = [K:\mathbb{Q}] \cdot \left(1 - \frac{1}{|\mathcal{I}_p|}\right),$$

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Moral: inertia subgroups tend to "typically" be as small as possible when counting by discriminant.

An example

Example (Non-Galois quartic D_4)

If $L/\mathbb{Q}$ is quartic D_4 with quadratic subfield K, then for all $p \neq 2$

$$v_p(D_L) = \begin{cases} 3 & \text{if } p \text{ is totally ramified} \\ 2 & \text{if } p \text{ is in all other cases} \\ 1 & \text{if } p \text{ is unramified in } K/\mathbb{Q} \text{ but ramifies in the biquadratic} \\ 0 & \text{if } p \text{ is unramified.} \end{cases}$$

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Observations:

- main contribution comes from quadratic fields K with $D_K < \log\log\log\log X$;
- a positive proportion of the quartic D_4-extensions have a given quadratic field K as their subfield.

Difficulties with discriminant counting

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Note that $D_{2^n} = \mathbb{Z}/2^n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. *The elements of minimal order are* $(k, 1)$ *(reflections) and* $(2^{n-1}k, 0)$ *(rotations with order dividing* 2*).*

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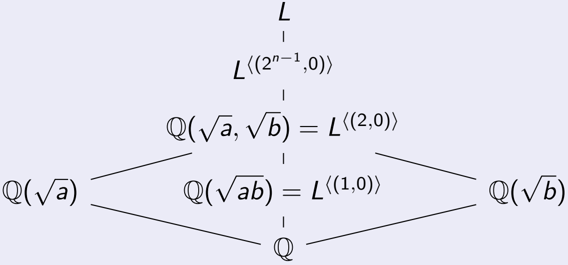

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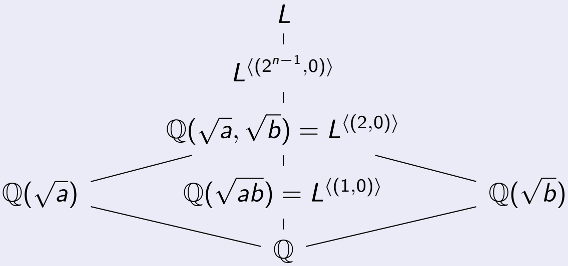

Positive proportion of extensions have $L^{\langle(2^{n-1},0)\rangle}/\mathbb{Q}(\sqrt{ab})$ unramified. So at least as hard as getting distribution of $\mathrm{Cl}(\mathbb{Q}(\sqrt{d}))[2^\infty]$.

Fair counting functions

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Altug–Shankar–Varma–Wilson (2017): Malle's conjecture for D_4 by Artin conductor.

Main result

A group G is called *nilpotent* if it is a direct product of p-groups.

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Theorem (K.–Pagano)

Assume GRH. Let G be a nilpotent group with $\#G$ odd. Then

$$\liminf_{X \to \infty} \frac{\# \left\{ K/\mathbb{Q} : \prod_{p : I_p \neq \{\mathrm{id}\}} p \leq X, \mathrm{Gal}(K/\mathbb{Q}) \cong G \right\}}{c'(G) X (\log X)^{b'(G)}} \geq 1,$$

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Surprisingly, the corresponding asymptotic

$$\lim_{X \to \infty} \frac{\#\left\{K/\mathbb{Q} : \prod_{p : I_p \neq \{\mathrm{id}\}} p \leq X, \mathrm{Gal}(K/\mathbb{Q}) \cong G\right\}}{c'(G) X (\log X)^{b'(G)}} = 1$$

is not true in general. Counterexamples exist for nilpotency class 2.

Inverse Galois problem

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For 2-groups, the situation is much more involved. The only known proof is a famous result of Shafarevich (inverse Galois for solvable groups).

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Suppose that we have a H-extension $\pi : G_{\mathbb{Q}} \to H$, and consider

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It is well-known that we have a local-to-global for the above diagram, which roughly means that we have to control $\pi(\mathrm{Frob}_v)$ for all v.

Scholz–Reichardt sketch II

Note that H also fits in an exact sequence

$$1 \to \mathbb{F}_p \to H \to H' \to 1.$$

Therefore we may twist our H-extension $\pi : G_{\mathbb{Q}} \to H$ by $\chi : G_{\mathbb{Q}} \to \mathbb{F}_p$ to get $\pi + \chi : G_{\mathbb{Q}} \to H$.

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Here we use that p is odd in an essential way: $\chi_\ell(\mathrm{Frob}_q)$ and $\chi_q(\mathrm{Frob}_\ell)$ are independent.

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and a bijection

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It is well-known that a $\mathbb{F}_2^2$-extension $\mathbb{Q}(\sqrt{a}, \sqrt{b})$ of $\mathbb{Q}$ is contained in a D_4-extension if and only if $x^2 = ay^2 + bz^2$ has a non-trivial point.

Step 1b: parametrization

If $\rho \in \mathrm{Epi}(G_{\mathbb{Q}}, D_4)$ is a lift of $\pi \in \mathrm{Epi}(G_{\mathbb{Q}}, \mathbb{F}_2)$ and $q : D_4 \twoheadrightarrow \mathbb{F}_2^2$, then

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It turns out to be more convenient to work with seven variables α_S for $\emptyset \subset S \subseteq \{a, b, c\}$, where α_S is the product over all primes p dividing the variables in S and not dividing the variables in $\{a, b, c\} - S$.

Step 1b: parametrization

If $\rho \in \text{Epi}(G_{\mathbb{Q}}, D_4)$ is a lift of $\pi \in \text{Epi}(G_{\mathbb{Q}}, \mathbb{F}_2)$ and $q : D_4 \twoheadrightarrow \mathbb{F}_2^2$, then

$$\{f \in \text{Epi}(G_{\mathbb{Q}}, D_4) : f \circ q = \pi\} = \{\rho \cdot \chi : \chi \in \text{Hom}(G_{\mathbb{Q}}, \mathbb{F}_2)\}.$$

Therefore we have a bijection

$$\text{Epi}(G_{\mathbb{Q}}, D_4) \leftrightarrow \{(a, b, c) \in (\mathbb{Q}^*/\mathbb{Q}^{*2})^3 : a, b \text{ ind.}, x^2 = ay^2 + bz^2 \text{ sol.}\}.$$

Under this parametrization, the product of ramified primes maps to $\text{rad}(|abc|)$ (ignoring minor issues with ramification at 2).

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The variables α_S are squarefree and pairwise coprime, and we have $\text{rad}(|abc|) = \prod_{\emptyset \subset S \subseteq \{a,b,c\}} |\alpha_S|$.

Step 2: character sums

Define $T(a)$ to be the subsets of $\{a, b, c\}$ containing a. Then we have

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$$\sum_{\substack{\prod_{\emptyset \subset S \subseteq \{a,b,c\}} |\alpha_S| \leq X \\ a,b \text{ lin. ind.}}} \mu^2\left(\prod_S |\alpha_S|\right) \cdot \mathbf{1}_{x^2 = \alpha_a \alpha_{a,b} \alpha_{a,c} \alpha_{a,b,c} y^2 + \alpha_b \alpha_{a,b} \alpha_{b,c} \alpha_{a,b,c} z^2 \text{ sol.}} \cdot$$

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Now rewrite the above sum as a sum over Legendre symbols involving the variables α_S.

Step 3: equidistribution

Evaluate the resulting character sum using Chebotarev and the large sieve.

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- Proof can most likely be made unconditional with a suitably strong large sieve for nilpotent extensions.

Arithmetic Statistics

I would like to thank the organizers Elisa, Jan, Peter and Samuele for a wonderful conference.

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Happy birthday Peter!