Control Systems 1

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1 Definitions

closed loop vs open loop: In an open loop system there are no interconnections within the system
Feedback: Two or more dynamical systems are connected and influence each other and their dynamics are strongly coupled.
Feedforward: Corrective Action before the disturbance has influenced the system.
Positive vs Negative Feedback: Negative feedback decreases the effects of a disturbance. Positive is the opposite.
Control: Compare desired state with current state, calculate and execute corrective action
SISO: Single Input/Single Output Systems
Static vs Dynamic: System is static if for all $t \in T$, $y(t)$ is a function of $u(t)$, Systems represented by ODEs are always dynamic!
Causal vs Non-Causal: System is causal if for any $t \in T$ the output only depends on the values of the Input on $(-\infty, t\,]$ i.e. future inputs are not possible! Almost all real world systems are causal.
Time variance: time invariant system map input and output signals, that are are the same at any point in time. $y(t_1) = y(t_2)$
Linearity: A system is linear if for all input signals u_a, u_b and scalars $\alpha, \beta \in \mathbb{R}$

$$\Sigma(\alpha u_a + \beta u_b) = \alpha(\Sigma u_a) + \beta(\Sigma u_b) = \alpha y_a + \beta y_b$$

differential integrators are **always** linear

2 System Modeling

2.1 Important Equations

$$f[Hz] = \frac{\omega[rad/s]}{2\pi}$$

Systems modeled using ODEs describing physical laws.
Balences

$$\text{LMB: } \Sigma F = \dot{P} = ma = m\ddot{x}$$

$$\text{AMB: } \Sigma M_B = \dot{H}_B \quad \text{with} \quad H_B = I_B \ddot{\varphi}$$

conservation Laws:

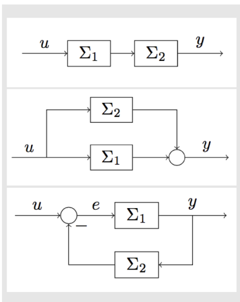
$$\frac{d}{dt} = \Sigma\text{inflows} - \Sigma\text{outflows}$$

2.2 LTI State Space Model

Oder of the governing ODE defines the dimension of $\vec{x}(t)$

$$\dot{x}(t) = Ax(t) + bu(t), \qquad A \in \mathbb{R}^{x \times x}, b \in \mathbb{R}^{n \times 1}$$
$$y(t) = cx(t) + du(t) \qquad c \in \mathbb{R}^{n \times 1}, d \in \mathbb{R}$$

2.3 Block Diagrams



$$w = \Sigma_1 \cdot u, \quad y = \Sigma_2 \cdot w$$
$$y = \Sigma_2 \cdot \Sigma_1 \cdot u$$
$$y = w_1 + w_2$$
$$= (\Sigma_1 + \Sigma_2) \cdot u$$
$$\Sigma_{tot} = \frac{\Sigma_1}{1 + \Sigma_1 \Sigma_2}$$

If sign in diagram switched also change sign in equation! Positive feedback leeds to - in the denominator and vice versa

2.4 Linearize Non-Linear System

1. Find equilibrium points: System described by an ODE $\dot{x}(t) = f(x(t), u(t))$ has an equilibrium point (x_e, u_e) if $f(x_e, u_e) = 0$. Equilibrium Point is always a couple and there is an infinite number of equilibrium points. Normally look for $u = 0$

2. Linearize system around this point:

$$A = \begin{bmatrix} \frac{\partial f_{0,1}}{\partial x_1} & \frac{\partial f_{0,1}}{\partial x_2} & \cdots & \frac{\partial f_{0,1}}{\partial x_n} \\ \frac{\partial f_{0,2}}{\partial x_1} & \frac{\partial f_{0,2}}{\partial x_2} & \cdots & \frac{\partial f_{0,2}}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{0,n}}{\partial x_1} & \frac{\partial f_{0,n}}{\partial x_2} & \cdots & \frac{\partial f_{0,n}}{\partial x_n} \end{bmatrix} \qquad b = \begin{bmatrix} \frac{\partial f_{0,1}}{\partial u} \\ \vdots \\ \frac{\partial f_{0,n}}{\partial u} \end{bmatrix}$$

$$c = \begin{bmatrix} \frac{\partial g_0}{\partial x_1} & \frac{\partial g_0}{\partial x_2} & \cdots & \frac{\partial g_0}{\partial x_n} \end{bmatrix} \qquad d = \begin{bmatrix} \frac{\partial g_0}{\partial u} \end{bmatrix}$$

$$\begin{matrix} \dot{x} = f_0(x, u) \\ y = g_0(x, u) \end{matrix} \approx \begin{matrix} \dot{x} = Ax + bu \\ y = cx + du \end{matrix}$$

3 Analysis

3.1 Solving LTI

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

$$y(t) = \underbrace{Ce^{At}x_0}_{\text{natural response}} + \underbrace{C\int_0^t e^{A(t-\tau)} Bu(\tau)d\tau}_{\text{system dynamics}} + \underbrace{Du(t)}_{\text{feedthrough}}$$

system dynamics + feed through lead also called forced response, natural response also called characteristic response.
Matrix exponential diagonalizable:
$e^A = e^{TDT^{-1}} = T \cdot diag(e^{\lambda_1}, \dots, e^{\lambda_n}$
Matrix exponential diagonal:

$$e^{At} = \begin{bmatrix} exp(\lambda_1 t) & 0 \\ 0 & exp(\lambda_2 t) \end{bmatrix}$$

Matrix exponential Jordan Matrix:

$$e^A t = exp\left(\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} t\right) = \left(\begin{bmatrix} 1 & t & \frac{1}{2!}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} e^{\lambda t}\right)$$

3.2 Modal form

Dependence of response of a system on Eigenvalues of A. Form obtained by applying diagonalizing matrix A.

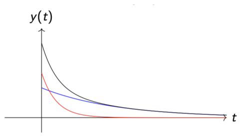
$$\begin{cases} \dot{\tilde{x}} = (T^{-1}AT)\tilde{x} + (T^{-1}B)u = \tilde{A}\tilde{x} + \tilde{B}u \\ y = (CT)\tilde{x} + Du = \tilde{C}\tilde{x} + \tilde{D}u \end{cases}$$

Entries of Matrix $\tilde{A}$ are the Eigenvalues λ_n of A

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} \tilde{x}_i(0) v_i$$

3.3 General response dependant on Eigenvalues

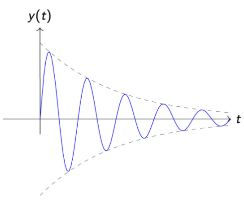
3.3.1 Distinct real Eigenvalues (diagonal, real Matrix)



Response is given by linear combination of exponential of the form $exp(\lambda_i t)$

3.3.2 Complex Conjugate Eigenvalues (Diagonal, complex Matrix)

$$y(t) = C \cdot exp\left(\begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix}\right) x_0$$
$$= e^{\sigma t}[\alpha_1 \sin(\omega t) + \alpha_2 \cos(\omega t)]$$
$$= \alpha e^{\sigma t} \sin(\omega t + \phi)$$



Homogeneous response is given by a sinusoid with frequency ω and in-/decreasing amplitude of $exp(\sigma t)$

3.3.3 Repeated real Eigenvalues (real, Jordan Matrix)

$$y(t) = C\, exp(At)x_0$$
$$= c_1 exp(\lambda_1)x_{0,1} + c_1 t\, exp(\lambda t)x_{0,2} + c_2 t^2 exp(\lambda t)x$$

Often repeated Eigenvalues occur with $\lambda = 0$. In this case the response is a polynominal. The homogeneous response is a linear combination of an exponential $exp(\lambda t)$ and term of the form $t^p exp(\lambda t)$

3.4 Stability

3.4.1 Lyapunov stability

System is Lyapunov stably if for any bounded initial condition and zero input the state remains bounded.
A system is Lyapunov stable if $Re(\lambda_i) \leq 0$ for all i. Also there are no repeated Eigenvalues with 0 real part.

$$\|x_0\| < \epsilon, \text{and} u = 0 \Rightarrow \|x(t)\| < \delta \text{ for all } t \geq 0$$

3.4.2 Asymptotically stable

A system is asymptotically stable if, for any bounded initial condition, and zero input the state converges to zero.
A system is asymptotically stable if $Re(\lambda_i) > 0$ for all i.

$$\|x_0\| < \epsilon, \text{and} u = 0 \Rightarrow \lim_{t \to +\infty} \|x(t)\| = 0$$

3.4.3 BIBO stability

Bounded Input, Bounded Output, for minimal LTI systems asymptotically stable = BIBO stable

$$\|u(t)\| < \epsilon \forall t \geq 0, \text{and } x_0 = 0 \Rightarrow \|y(t)\| < \delta \forall t \geq 0$$

For a non-completely controllable and observable system BIBO stability doesn't imply asymptotic stability.

3.5 Controllability

Mathmatically the controlability Matrix R need to have full rank.
$R = \begin{bmatrix} b & A \cdot b & A^2 \cdot b & \dots & A^{n-1} \cdot b \end{bmatrix}$
LTI of form $\dot{x} = Ax + Bu$ is controllable if for any given initial state $x(0) = x_c$ $\exists u$ so that $x(t) = 0$ for a finite time t
If $det(R) \neq 0 \Rightarrow rank(R) = dim(R) = n \Leftrightarrow$ R has full rank

3.6 Observability

Mathematically matrix O need full rank.
$O^T = \begin{bmatrix} c & c \cdot A & c \cdot A^2 & \dots & c \cdot A^{n-1} \end{bmatrix}$
LTI of form $\dot{x} = Ax + Bu$, $y = Cx + Du$ is observable if any given initial condition $x(0) = x_o$ can be reconstructed based on knowledge of input and output signal only over a finite time $[0, t]$ -> $det(O) \neq 0$

3.7 Intuition based on modal form

A system in diagonal from is controllable if $\tilde{b}_i \neq 0, i = 1, \dots, n$
A system in diagonal form is observable if $\tilde{c}_i = 0, i = 1, \dots, n$
A system is stabilizeable if all unstable modes are controllable
A system is detectable if all unstable modes are observable

4 Transfer functions

complex exponentials: All inputs are linear combinations of complex exponentials.

$$u(t) = \sum_i U_i e^{s_i t} \Rightarrow y(t) = \sum_i G(s) U_i e^{s_i t} \quad s \in \mathbb{C}$$

General Solution generic complex exponential as input e^{st}:

$$y(t) = Ce^{At}\underbrace{\left[x(0) - (sI - A)^{-1}B\right]}_{1} + \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{2} e^{st}$$

1 = Transient response $\rightarrow$ 0 if as. stable, 2 = steady-state response y_{ss} $t \rightarrow \infty$
Transfer function can be derived from steady state response:

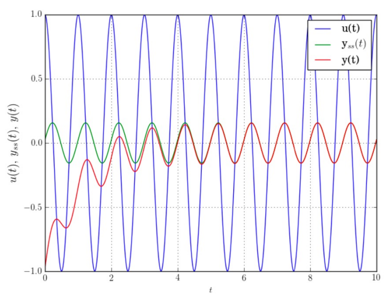
$$y_{ss} = \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{G(s)} e^{st} \qquad y_{ss} = G(s)e^{st}$$

$G(s)$ describes how a system transforms an input into an output at steady state. $G(s) \in \mathbb{C}$

Written in Polar form: $G(s) = Me^{\phi j}$, where M is magnitude and ϕ is the phase $\angle G(s)$

$$z = a + jb = |z|e^{j\angle z}, \ |z| = \sqrt{a^2 + b^2}\ \angle z = \arctan(\frac{b}{a})$$

The behaviour of a system can be completely characterised by its steady state responds to sinusoidal inputs.
Input Output in Frequency Domain:



u and y_{ss} have same frequency but different amplitude and phase.

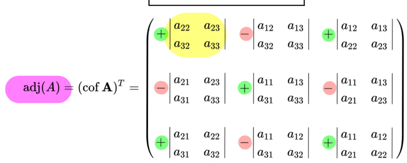
4.0.1 Inverse of a Matrix

2 x 2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3 x 3:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix}$$

General Formula: $A^{-1} = \frac{adj(A)}{det(A)}$



Unterdeterminanten in between lines

4.1 Transfer function via Laplace Transformation

$$X(s) = \int_0^{\infty} u(t) \cdot e^{-st} dt$$

$x(t)$	$X(s)$
$\delta(t)$	1
$h(t)$	$\frac{1}{s}$
$h(t) \cdot t^n \cdot e^{\alpha \cdot t}$	$\frac{n!}{(s-\alpha)^{n+1}}$
$h(t) \cdot \sin(\omega \cdot t)$	$\frac{\omega}{s^2+\omega^2}$
$h(t) \cdot \cos(\omega \cdot t)$	$\frac{s}{s^2+\omega^2}$
$h(t) \cdot \sinh(\omega \cdot t)$	$\frac{\omega}{s^2-\omega^2}$
$h(t) \cdot \cosh(\omega \cdot t)$	$\frac{s}{s^2-\omega^2}$

4.2 From Transfer function to State-space

$$G(s) = \frac{b_{n-1}sn - 1 + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} + d$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & \ddots & & 1 \\ -a_0 & -a_1 & \ldots & & & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \ldots & b_{n-1} \end{bmatrix}, \quad D = [d];$$

only possible if there are no common poles/zeros

4.3 Different ways to write transfer functions

4.3.1 strictly proper rational function

$$G(s) = \frac{b_{n-1}sn - 1 + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} + d$$

For causal, finite-dimensional SISO LTI systems. Possibly a direct feed through term.

4.3.2 Partial fraction expansion

$$G(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \ldots + \frac{r_n}{s - p_n} + r_0$$

Useful to compute transient responses and asses how much different modes contribute to response. p_i are the poles same as Eigenvalues of A. r_i are the residues.

calculate Residues:

for non-repeated pole p_i: $r_i = \lim_{s \to p_i} (s - p_i) G(s)$

for repeated pole of order m:

$$r_i = \frac{1}{(m-1)!} \lim_{s \to p_i} \frac{d^{m-1}}{ds^{m-1}} ((s - p_i)^m G(s)$$

4.3.3 Root-Locus Form

$$G(s) = \frac{k_{rl}}{s^q} \frac{(s - z_1)(s - z_2) \ldots (s - z_m)}{(s - p_1)(s - p_2) \ldots (s - p_{n-q})}$$

z_i are the zeros of $G(s)$

4.3.4 Bode Form

$$\frac{k_{bode}}{s^q} \frac{\left(\frac{s}{-z_1} + 1\right) + \left(\frac{s}{-z_2} + 1\right) \ldots \left(\frac{s}{-z_m} + 1\right)}{\left(\frac{s}{-p_1} + 1\right) + \left(\frac{s}{-p_2} + 1\right) \ldots \left(\frac{s}{-p_{n-q}} + 1\right)}$$

4.4 Steady-state response to a unit step

ONLY FOR STABLE SYSTEMS

Bode Gain can be understood as the steady-state response to a unit step.

Can be obtained using the final value theorem

$\lim_{t \to \infty} x(t) = \lim_{s \to 0^+} s \cdot X(s)$ $\quad \lim_{s \to 0} G(s) = y_{ss}(t)$

4.5 Impulse Response

$D = 0, x(0) = 0$ and $u(t) = \delta(t)$ $y_{imp} = Ce^{At}B$ This is the same as the response to an initial condition $x(0) = B$

4.5.1 First Order system

$A = a,\ B = b,\ C = c,\ D = 0$ The Transfer function is $G(s) = \frac{r}{s-a}$ with $r = bc$ response to a unit impulse has the form $y(t) = re^{at}$

4.5.2 Higher Order systems

$$y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \ldots r_n e^{p_n t}$$

4.6 Pole-Zero cancellation

Pole Matches a zero exactly, example:

$$G(s) = \frac{s+1}{(s+1)(s+1+j)(s+1-j)}$$

Therefore the pole and zero cancel out. Therefore :

$$G(s) = \frac{0}{s+1} + \frac{1/2j}{s+1+j} + \frac{-1/2j}{s+1-j}$$

Since $r_i = 0$ input can't impact output -> controlability issue, pole cant be reconstructed from output -> observability issue. Fine as long as p_i is stable still should be avoided.

4.7 Step Response

Assume $D = 0,\ x(0) = 0,\ u(t) = 1 = e^{0t}, fort \geq 0$ and A is invertible:

$$y_{step}(t) = \int_0^t Ce^{A(t-\tau)} Bd\tau = -CA^{-1}B$$

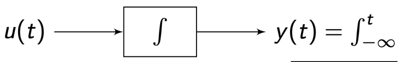
first order scalar A $\quad y_{step}(t) = s_{ss}(t)(1 - e^{at})$

step response is the steady state response minus a scaled impulse response.

4.8 Recipe

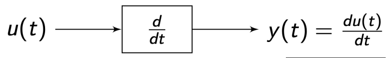
In general compute partial fraction expansion of $Y(s) = G(s)U(s)$ with $U(s) = 1/s$ -> Laplace of unit step

4.9 Integrator



if $u(t) = e^{st}$ then $y(t) = \frac{1}{s}e^{st} \rightarrow$ $G(s) = \frac{1}{s}$

4.9.1 Differentiator



if $u(t) = e^{st}$ then $y(t) = se^{st} \rightarrow$ $G(s) = s$

Note that this is a non causal operator due to definition of numerical differentiation formula.

4.10 Non-minimum-phase zeros

Non-minimum-phase zeros lead to negative derivative action. Meaning the output will tend to move in the wrong direction initially -> opposite of an anticipatory effect.

4.11 Open/Closed-Loop transfer function

Open-Loop in Root Locus Form:

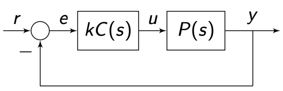
$$kL(s) = k\frac{N(s)}{D(s)} = k\frac{(s - z_1)(s - z_2) \ldots (s - z_m)}{(s - p_1)(s - p_2) \ldots (s - p_n)}$$

Closed-Loop transfer function

$$T(s) = \frac{kL(s)}{1 + kL(s)} = \frac{kN(s)}{D(s) + kN(s)}$$

Therefore the closed-loop poles are: $D(s) + kN(s) = 0$

4.12 Standard Feedback Configuaration



(Open-) Loop gain: $kL(s) = kP(s)C(s)$

Poles and zeros of $L(s)$ include all poles and zeros of plants transfer function $P(s)$ and all poles of dynamic compensator transfer-function $C(s)$

Complementary Sensitivity: (cl) transfer function r to y

$$T(s) = \frac{kL(s)}{1 + kL(s)}$$

Sensitivity: (cl) transfer function from r to e

$$S(s) = \frac{1}{1 + L(s)} = \frac{1}{1 + P(s)C(s)}$$

4.13 Dynamic Compensation

Consider a control system, that itself is a dynamic system with transfer function $C(s)$ -> dynamic compensator.

- PI: $C(s) = k_p + k_I\frac{1}{s} = k_p\frac{s + k_i/k_p}{s}$
- PD: $C(s) = k_p + k_D s$
- PID: $C(s) = k_p + k_I\frac{1}{s} + k_D s$
- Lead: $\frac{s - z}{s - p}$ with $z < p$
- Lag: $\frac{s - z}{s - p}$ with $z < p$

FOR INTERNAL STABILITY ALL TRANSFER FUNCTIONS BETWEEN TWO SIGNALS MUST BE STABLE!!!

5 Root Locus

Root Locus Rules:

- closed-loop poles are symmetric wrt real axis.
- number of closed-loop poles is equal to number of open-loop poles.
- closed-loop poles approach open-loop poles as $k \to 0$
- for $k \to \infty$ closed-loop poles approach open-loop zeros, if degree of $N(s) = D(s)$ else "excess" closed-loop poles "go to infinity". Flow from poles to zeros if more zeros then poles goes to inf. If more zeros then poles line comes from inf to zero

Rewrite closed-loop characteristic equation: $\frac{N(s)}{D(s)} = -\frac{1}{k}$

5.1 Angle Rule

Take argument on both sides:

$$\angle(s - z_1) + \angle(s - z_2) + \ldots + \angle(s - z_m) - \angle(s - p_1) - \angle(s - p_2) - \ldots - \angle(s - p_n) = \begin{cases} 180^\circ(\pm q 360^\circ \text{ if } k > 0 \\ 0^\circ(\pm q 360^\circ) \text{ if } k < 0 \end{cases}$$

5.2 Magnitude Rule

Take Magnitude on both sides:

$$\frac{|s - z_1| \cdot |s - z_2| \cdot \ldots \cdot |s - z_m|}{|s - p_1| \cdot |s - p_2| \cdot \ldots \cdot |s - p_m|} = \frac{1}{|k|}$$

5.3 On Real Axis

- $\angle(s - z) = 0$ if $z \in \mathbb{R}$ and $s > z$
- $\angle(s - z) = 180^\circ$ if $z \in \mathbb{R}$ and $s < z$
- $\angle(s - z) + \angle(s - z^*) = 0$ complex conjugate $z,\ z^*$
- same holds for p_i
- all points on real axis are on root locus
- all points left of odd numbered poles/zeros are positive k root locus and other way around
- two branches coming together on real axis creates breakaway or break-in points

5.4 Asymptotes

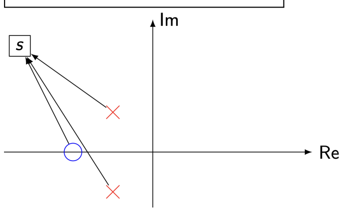
If $k \to \infty$ and more open-loop poles then zeros $\to$ identify what direction goes towards infinity. $n =$ poles, $m =$ zeros

$$k > 0,\ \angle s = \frac{180^\circ \pm q \cdot 360^\circ}{n - m}$$

$$k < 0,\ \angle s = \frac{\pm q \cdot 360^\circ}{n - m}$$

Asymptotes meet in "center of Mass":

$$s_{com} = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n - m}$$



6 Time Domain Specifications

6.1 Steady-state Error to unit step

$r(t) = 1 = e^{0t}$ for $t \geq 0$ and $L(0) = P(0)C(0)$

Ss-Error: $e_{ss} = \lim_{t \to +\infty} e(t) = S(0)e^{0t} = \frac{1}{1 + L(0)}$

6.1.1 Effect of Integrators

if $L(s)$ contains an integrator (pole at 0) and closed-loop system is stable $e_{ss} = 0$

e_{ss}	$q = 0$	$q = 1$	$q = 2$
Type 0	$\frac{1}{1 + K_{Bode}}$	∞	∞
Type 1	0	$\frac{1}{K_{Bode}}$	∞
Type 2	0	0	$\frac{1}{K_{Bode}}$

$q = \{0, 1, 2, 3, \ldots\}$ unit ramps of order q $Type$ 0 etc corresponds to number of integrators

Larger bode gain means smaller e_{ss}, for $e_{ss} = 0$ at least $q + 1$ integrators on path from error to reference input

6.2 Time domain step response of 1st odrder system:

$$y(t) = 1 - e^{-t/\tau}, \quad t \geq 0$$

time constant $\tau = -1/p$ of real pole p time needed to reach 63% of wanted value.

settling time: T_d time needed to reach steady-state within error margin d. $T_d = \tau \log(100/d)$

6.3 Step response of a stable second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

$$\Leftrightarrow \begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} x + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u \\ y &= x \end{cases}$$

nomenclature:

$p = \sigma + j\omega$ $\quad \zeta = \frac{|\sigma|}{\omega_n} = \sin(|\phi|)$ $\quad \phi = \arctan \frac{\sigma}{\omega}$

Time to peak: $T_p = \pi/\omega$

Peak overshoot: $M_p = e^{\frac{\sigma\pi}{\omega}}$ or $\zeta^2 = \frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}$

Rise Time: $T_{100\%} = \frac{\pi/2 - \phi}{\omega} \approx \frac{\pi}{2\omega_n}$

setting time T_s: $\sigma = \frac{-\ln p}{T_s}$ where p= tolerance

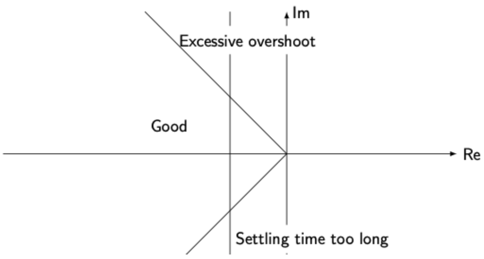
6.3.1 under-damped second-order system $\zeta < 1$

$$y(t) = 1 - \frac{1}{\cos\phi} e^{\sigma t} \cos(\omega t + \phi),\ t \geq 0$$

6.4 Dominant Pole Approximation

Approximate higher order systems with 1st or 2nd order to apply specifications to. Normally dominant poles have least negative real part. Exception if this pole has small residue. Set $s = 0$ for non dominant poles then evaluate for new gain.
$G_{approx} = G(0)$ -> factor out the pole and kürzen with the gain

6.4.1 Time-domain specifications and Root Locus



6.5 Noise and Disturbance Rejection

$T(s) + S(s) = 1$ noise normally high frequency, disturbances low frequency. $|S(s)|$ must be small at low ω, $|T(s)|$ small at high ω. Therefore $|L(s)|$ needs to be large at low ω, small at high ω

7 PID control

Proportional Control: Control input tries to move system in opposite direction to the error, is proportional to error magnitude. Higher proportional gain means: closed loop remains stable, e_{ss} decreases faster, response becomes faster, sensitivity to noise increases. Can change phase margin

$$T(s) = \frac{L(s)}{1 + kL(s)} = \frac{1}{s + 1 + k} \qquad e_{ss} = \frac{1}{1+k}$$

PI control: Integrating error allows detecting biases. integral control reduces biases (see 4.13 for formula). With increase in gain comes: $e_{ss} = 0 \iff k_I \neq 0$, more oscillatory responses, same sensitivity to noise, closed loop poles go from slow, overdamped to fast with low damping
PID control: Differentiating error allows to predict what error will do. Avoids overshooting (formula in 4.13). Is non causal transfer function. With higher derivative gain comes: e_{ss} is not affected, response is less oscillatory but maybe slower, higher sensitivity to noise. Closed-loop poles are pulled far into left plane.

7.1 Design Recipe

1. Assume proportional control P
2. Draw Root Locus
3. If RL doesn't go through good region need D term, back to 2
4. Choose gain so dominant poles are in good region
5. If e_{ss} to large add I back to 2

7.2 PID tuning

$$C(s) = k_{RL}\frac{(s - z_1)(s - z_2)}{s} \cdot \frac{1}{s - p_{fast}}$$

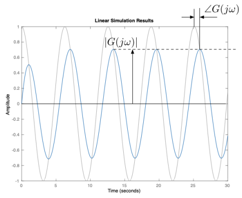
Two zeros one pole at origin with one fast stable pole for proper transfer function. Decide on where want poles, zeros and on root locus gain -> calculate k_P, k_I, k_D

8 Frequency Domain Specifications

8.1 Frequency Response

Steady-state response to a sinusoidal input of frequency ω is a sinusoidal output of the same frequency.

1. Amplitude of Output is $|G(s)| = |G(\omega j)|$ times amplitude of input.
2. Phase of output lages the phase of input by $\angle G(j\omega)$.



8.2 Bode Plot

Bode Plots are composed of two plots: magnitude and phase. On **horizontal axis** of both plots is frequency ω on $\log_{10}$ scale. On vertical axis
1. $\log_{10} G(\omega j)$ or equiv. in dB with convention $|G(j\omega)|[dB] = 20\log_{10}|G(j\omega)|$ therefore one "decade" $= 20dB$ 2. Phase $\angle G(j\omega)$ usualy in degrees, radians also ok. Possible to simply add bode plots of transfer functions. Inverted Transfer function is equal to bode plot reflection about horizontal axis in both plots.

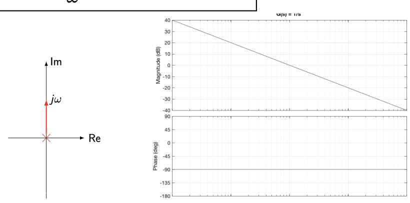
pole/zero type	change in magnitude	phase shift φ
BIBO stable pole	-20dB/dec	$-90°$
BIBO unstable pole	-20dB/dec	$+90°$
minimum phase zero	$+20$dB/dec	$+90°$
non-minimum phase	$+20$dB/dec	$-90°$
time delay	0dB/dec	$-\omega \cdot T$
integrator $\frac{1}{s}$	-20dB/dec	$-90°$
differentiator s	$+20$dB/dec	$+90°$

Dec	dB	dB	Dec
∞	∞	∞	∞
1000	60	1000	$1 \cdot 10^{50}$
100	40	100	$100000 = 10^5$
50	33.98	80	$10000 = 10^4$
20	26.02	60	$1000 = 10^3$
10	20	40	100
9	19.08	30	31.62
8	18.06	20	10
7	16.90	15	5.62
6	15.56	10	$3.16 = \sqrt{10}$
5	13.98	9	2.82
4	12.04	8	2.51
3	9.54	7	2.24
2	6.02	6	≈ 2
1	0	5	$1.78 = \sqrt[4]{10}$
$\frac{1}{2} = 0.5$	-6.02	4	1.58
$\frac{1}{3} \approx 0.33$	-9.54	3	$1.41 \approx \sqrt{2}$
$\frac{1}{4} = 0.25$	-12.04	2	$1.26 = \sqrt[10]{10}$
$\frac{1}{5} = 0.2$	-13.98	1	$1.12 = \sqrt[20]{10}$
$\frac{1}{6} \approx 0.17$	-15.56	0.1	≈ 1.01
$\frac{1}{7} \approx 0.14$	-16.90	0.01	≈ 1.001
0.1	-20	0	1
0.01	-40	$x_{\text{dB}} < 0$	$-\frac{1}{x_{\text{dec}}}$
$\frac{1}{\sqrt{2}}$	-3.0103	-3	$\approx \frac{1}{\sqrt{2}}$
0	$-\infty$	$-\infty$	0

8.2.1 Integrator in Bode Plots

$G(s) = G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega}$

$$|G(j\omega)| = \frac{1}{\omega},\ \angle G(j\omega) = -90°$$



8.2.2 Asymptotic Bode plots - single real, stable pole

Consider $G(s) = \frac{1}{\tau s + 1}$, with $\tau = -1/p > 0$. Construct approximation of Bode plots for $\omega \to 0^+$ and $\omega \to +\infty$.
1. For $\omega \to 0^+$, $G(j\omega) \approx 1$

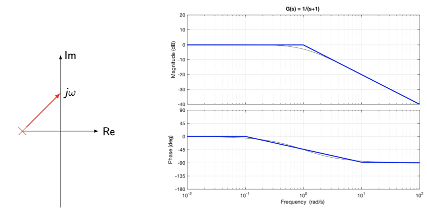
$|G(j\omega)| \approx 1 = 0dB$ $\quad \angle G(j\omega) \approx 0$

2. For $\omega \to +\infty$, $G(j\omega) \approx \frac{1}{j\tau\omega}$

$|G(j\omega)| \approx \frac{1}{\tau\omega}$ $\quad \angle G(j\omega) \approx -90°$

3. For $\omega = 1/\tau$, $G(j\omega) = \frac{1}{j+1}$

$|G(j\omega)| \approx \frac{1}{\sqrt{2}} = -3dB$ $\quad \angle G(j\omega) \approx -45°$



8.2.3 Asymptotic Bode plots - Complex-conjugate, stable poles

$G(s) = \frac{1}{s^2/\omega_n^2 + 2\zeta s/\omega_n + 1}$ $p = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
1. For $\omega \to 0^+$, $G(j\omega) \approx 1$

$|G(j\omega)| \approx 1 = 0dB$ $\quad \angle G(j\omega) \approx 0°$

2. For $\omega \to +\infty$, $G(j\omega) \approx \frac{\omega_n^2}{-\omega^2}$

$|G(j\omega)| \approx \frac{\omega_n^2}{\omega^2}$ $\quad \angle G(j\omega) \approx -180°$

3. For $\omega = \omega_n$, $G(j\omega) = \frac{1}{2\zeta j}$

$|G(j\omega)| \approx \frac{1}{2\zeta}$ $\quad \angle G(j\omega) \approx -90°$

8.2.4 single, real minimum-phase zero

$G(s) = \tau s + 1$, with $\tau = -1/z > 0$
1. For $\omega \to 0^+$, $G(j\omega) \approx 1$

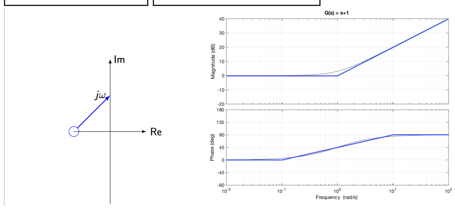
$|G(j\omega)| \approx 1 = 0dB$ $\quad \angle G(j\omega) \approx 0°$

2. For $\omega \to +\infty$, $G(j\omega) \approx -j\tau\omega$

$|G(j\omega)| \approx \tau\omega$ $\quad \angle G(j\omega) \approx +90°$

3. For $\omega = 1/\tau$, $G(j\omega) = -j + 1$

$|G(j\omega)| = \sqrt{2}$ $\quad \angle G(j\omega) = +45°$



8.2.5 single real, non minimum-phase zero

$G(s) = -\tau s + 1$, with $\tau = 1/z > 0$
1. For $\omega \to 0^+$, $G(j\omega) \approx 1$

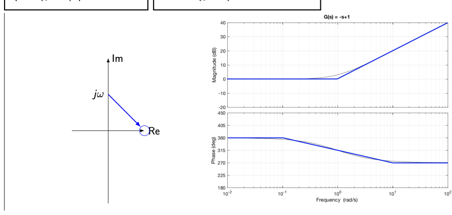
$|G(j\omega)| \approx 1 = 0dB$ $\quad \angle G(j\omega) \approx 0°$

2. For $\omega \to +\infty$, $G(j\omega) \approx -j\tau\omega$

$|G(j\omega)| \approx \tau\omega$ $\quad \angle G(j\omega) \approx -90°$

3. For $\omega = 1/\tau$, $G(j\omega) = -j + 1$

$|G(j\omega)| = \sqrt{2}$ $\quad \angle G(j\omega) = -45°$



8.3 Bode's Law

In Bode plot magnitude slope and phase are not independent. If slope of magnitude plot is $\kappa \cdot 20db/decade$ over a range of more then ≈ 1 decade, phase in that range will be $\kappa \cdot 90°$

8.4 Polar Plot

Frequency response plotted on complex plane as a parametric function of ω. Convenient to sketch Bode Plot first.
Magnitude: Distance form origin, Phase: Angle from real axis

8.5 Frequency Domain Specifications on Bode Plot

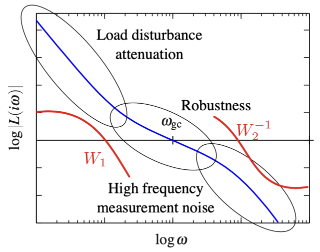
Usually expressed in terms of closed-loop frequency response. For good disturbance rejection $|S(j\omega)| = |1 + L(j\omega)|^{-1}$ small at low frequencies.
Rewritten as $|S(j\omega)| \cdot |W_1(j\omega)| < 1$ for some function $|W_1(j\omega)|$ large at low frequency ($< 10Hz$). Approximated as: $|S(j\omega)| > |W_1(j\omega)|$ Can be observed as low frequency obstacle on magnitude Bode plot.

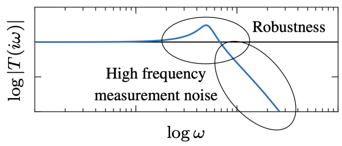
$|T(j\omega)| = \frac{|L(j\omega)|}{|1+L(j\omega)|}$ small at high frequency ($> 100Hz$). Therefor $|T(j\omega)| \approx |L(j\omega)|$ at high frequencies. Typically written as: $|T(j\omega)| \cdot |W_2(j\omega)| < 1$ for some function $|W_2(j\omega)|$ large at high frequencies. Therefore: $|L(j\omega)| < |W_2(j\omega)|^{-1}$

8.5.1 Closed-loop Bandwidth and (open-loop) crossover

Bandwidth of closed-loop system defined as maximum ω for which $|T(j\omega)| > 1/\sqrt{2}$ -> tracks within factor of ≈ 0.7. Open-loop crossover frequency is approximately equal to closed-loop bandwidth.
$|L(j\omega_c)| = 0db = 1$, and $\angle L(j\omega_{pc}) = -180°$

Bode obstacle course:





9 Time Delays

Evaluation of sensory information for deciding course of action requires a finite computation time.

9.1 Transfer function of time delay

$t \to u(t)$ transformed into delayed output $y(t) = u(t - T)$. Delayed version of linear combination of signal is equal to linear combination of delayed signals.
Consider $u(t) = e^{st}$

$$y(t) = e^{s(t-T)} = e^{st}e^{-sT} = e^{-sT}u(t)$$

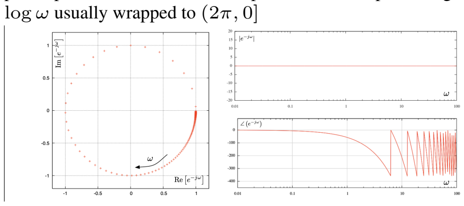
Therefore transfer function of delay T is e^{-sT}
NOT A RATIONAL FUNCTION CAN'T APPLY ROOT LOCUS

9.2 Frequency response of time delay

In terms of frequency response: $|e^{j\omega T}| = 1, \angle(e^{-j\omega T}) = -\omega T$

9.2.1 Polar and Bode plots of time delay

Polar Plot of $e^{-j\omega T}$ corresponds to circle of unit radius. Bode phase plot, linear in ω is an exponential when plotted against $\log\omega$ usually wrapped to $(2\pi, 0]$

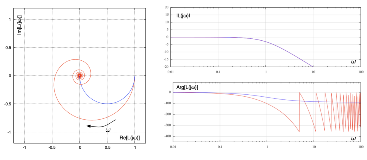


9.3 Effects of time delays on loop transfer function

Effect of time delays on closed-loop stability. $L(s) = C(s)P(s)$ include time delay T seconds. New transfer function $L'(s) = e^{-sT}L(s)$.
Frequency response of system with time delay obtained from ideal frequency response, by shifting the phase back by ωT.
Example: Simple plant $P(s) = \frac{1}{s+1}$ with proportional control $C(s) = k \Rightarrow L(s) = \frac{k}{s+1}$ stable $\forall k > -1$.

With time delay T $L'(s) = e^{-sT}\dfrac{k}{s+1}$

Nyquist and Bode take the forms:



Bode Plot of $L'(s)$ = Bode plot of $L(s)$ + Plot of the time delay.
$\phi_{m,T} = \phi_0 - \omega_c T$ with $\phi_{m,T}$ and ϕ_0 as phase margins, with and without time delays. ω_c crossover frequency. Main effect of time delays is reduction of phase margin. This decreases as crossover frequency increases.

9.4 Design Procedure feedback control in presence of time delay

1. Design feedback control ignoring time delay
2. Check effective phase margin too small or negative phase margin implicate closed-loop instability -> redesign controller by either increasing phase at crossover -> lead controller or decrease crossover frequency -> reduce gain or possibly phase lag controller for command following performance.
3. Iterate until satisfactory

9.5 Time delays and root locus method

For root locus loop transfer function must be rational not the case due to e^{-sT}. To still use root locus must approximate time delay with rational transfer function.

9.5.1 Padé Approximation (Time delays for Root Locus)

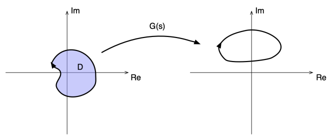
Represent exponential as ratio of two polynomials (only first order needed in this).

$$e^{-sT} \approx k\frac{2/T - s}{2/T + s}$$

Magnitude of frequency response always one. Using Padé, represent time delay on root locus as pole and zero respectively at $\pm 2/T$. Non-minimum-phase zero present! Can't increase gain arbitrarily pole converges to n.m.p. zero for large gains.
Only tool that always provide correct answer in all cases when time delay is present is Nyquist plot

10 Niquist Theorem

10.1 Principle of variation of the Argument



Number of times $G(s)$ encircles the origin/total variation of argument moving along Γ counts number of poles and zeros of $G(s)$ in D
Remember: $G(s) = (s - z)/(s - p)$ then $\angle G(s) = \angle(s - z) - \angle(s - p)$

- no poles/zeros in D net variation across one cycle is 0
- One 0 in D net variation across one cycle is 2π
- One pole in D net variation across one cycle is -2π

Number N of times $G(s)$ encircles origin of complex plane as s moves along boundary Γ of a bounded simply-connected region of the plane satisfies: $N = Z - P$ Where P are poles and Z are zeros.

10.2 Nyquist or D contour

Assess stability of a system using Nyquist:
Construct arbitrarily large but finite D shaped region D on right half plane. s moving along the boundary of D $1 + kL(s)$ encircles origin $N = Z - P$ times. $Z = N + P \overset{!}{=} 0$

- Z number of unstable closed-loop poles (zeros of $1 + kL(s)$ in rhp)
- P = # unstable open-loop poles (poles of $1 + kL(s)$ in rhp)
- N = # encirclements, $CW = +1, CCW = -1$

10.3 Nyquist Plot

Rephrasing Nyquist contour: As s moves along boundary of D $L(s)$ encircles $-1/k$ point $N = Z - P$ times where:

- Z number of unstable closed-loop poles (zeros of $L(s)$ in Nyquist Contour)
- P number of unstable open-loop poles (poles of $L(s)$ in Nyquist Contour)

Symmetry about real axis states:
$\angle L(-j\omega) = -\angle L(j\omega)$ i.e. plot s moving along boundary of NC just polar plot + symmetric plot about real axis.

10.4 Nyquist Condition

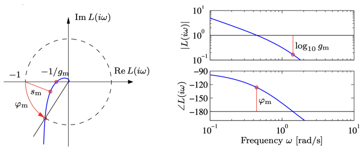
If open-loop system is stable, closed loop system is stable as long as Nyquist doesn't encircle $-1/k$ point.
If open-loop system has P unstable poles, closed loop is stable as long as Nyquist plot of $L(s)$ encircles point $-1/k$ point P times in clockwise direction.

10.4.1 open-loop poles on imaginary axis

Make indentations in Contour -> leaving imaginary poles on left.

10.5 Nyquist condition and robustness margins

Gain margin and phase margin measure how close the system is to closed-loop instability. φ_m from -180, g_m from Graph



10.6 Nyquist condition on Bode plots

If open-loop is stable Nyquist should not encircle -1 point for closed-loop to be stable.

$|L(j\omega)| < 1$ whenever $\angle L(j\omega) = 180°$

On Bode this means magnitude should be below $0dB$ when phase plot crosses $-180°$ line. Only valid if open-loop is stable.
Distance form Nyquist plot to -1 Point measure of robustness. Easily measured on Bode, distance of point from 0 for magnitude and from $-180°$ for phase.

11 Control Synthesis

11.1 Loop Shaping

If we have frequency domain specs like W_1, W_2^{-1} or ω_{gc} steer open-loop frequency response like Bode obstacle course -> modify $C(s)$. Mostly done using following elements:

11.1.1 Integrators

Add as many as needed to track n^{th}-Order ramp with $e_{ss} = 0$. Increases magnitude at low freq. and decreases at high freq. Decreases phase by $\#integrators \cdot 90°$ everywhere -> phase margin.

11.1.2 Gain k, proportional static compensation.

Choose gain so that low freq. asymptote clears command-tracking/disturbance spec. -> increase/decrease magnitude everywhere. $C(s) = k$ Small enough k yields stable closed loop.

11.1.3 Lead Compensator

Approximates PD control as $b \to +\infty$

$$\frac{(s/a) + 1}{s/b + 1} = \frac{b}{a}\frac{s + a}{s + b}, \ (0 < a < b)$$

Increase phase around $\sqrt{ab}$ -> midpoint between a and b on bode plot by a maximum of $90°$

max phase increase: $\phi_{max} = 2 \cdot \arctan\left(\sqrt{\frac{b}{a}}\right) - 90°$

Magnitude and phase at lower freq not affected. Increase slope of magnitude at freq between a and b by $20db/dec$ increase magnitude at high freq by b/a -> noise sensitivity.

Use case: Increase phase margin:
Pick $\sqrt{ab}$ at desired crossover freq. Pick b/a depending on desired phase increase. Adjust k put crossover at desired frequency. Side effect noise sensitivity

11.1.4 Lag Compensator

Approximates PI control as $b \to 0$

$$\frac{(s/a) + 1}{s/b + 1} = \frac{b}{a}\frac{s + a}{s + b}, \ (0 < b < a)$$

Decrease slope of magnitude at freq between a and b by $-20db/dec$ -> Decrease magnitude at high freq by b/a. Mag at low freq not affected.
Decrease phase around $\sqrt{ab}$ -> midpoint between a and b on bode plot up to $90°$

max phase decrease: $\phi_{max} = 2 \cdot \arctan\left(\sqrt{\frac{b}{a}}\right) - 90°$

Use case: Improve command tracking/disturbance rejection
Pick a/b as desired increase in mag at low freq. Pick a so smaller than crossover freq. Multiply gain k by a/b

11.1.5 General procedure for open-loop stable

Proceed from left to right.

1. Figure out how many integrators needed in $C(s)$ -> dependant on order of ramp signal
2. Fix gain so low freq. asymptote clears bode obstacle
3. Add terms of form $(\tau s + 1)$ at numerator or denominator Bode magnitude plot intersects $0dB$ with $\approx 20dB/s$ -> $90°$ phase margin. poles steer down and zeros up. Normalizing zero order term to 1 makes it so it doesn't affect Bode plot on left of pole/zero.

11.2 PID as Lead/Lag

Implementable PID as proper transfer-function with p as fast pole $p \gg 1$

$$PID(s) = k\frac{(s/z_1 + 1)(s/z_2 + 1)}{s(s/p + 1)}$$

Can be interpreted as:

$$PID = k \cdot \underbrace{\frac{s/z_1 + 1}{s + 0}}_{Lag} \cdot \underbrace{\frac{s/z_2 + 1}{s/p + 1}}_{Lead}$$

PID corresponds to extreme lead-lag-compensator one pole at $s = 0$ and one pole at $-p = p \gg 1$

11.3 Loop shaping for non-minimum-phase/unstable systems

ALWAYS CHECK WITH RL OR NY
$P(s) = P_{mp}(s) \cdot D(s)$ where $p_{mp}(s)$ obtained by replacing all poles/zeros of $P(s)$ in right half plane with their mirror image wrt imaginary axis. $D(s)$ contains all poles and zeros of $P(s)$ in right half plane times inverse all mirror images introduced. $|D(j\omega)| = 1 \forall\omega$ $D(s)$ is an all-pass filter. Choose sign $D(s)$ so phase is negative, doesn't affect magnitude.
Example:

$$P(s) = \frac{s - z}{s - p} \text{ with } z, p > 0$$

$$P_{mp}(s) = \frac{s + z}{s + p} \text{ and } D(s) = \frac{z - s}{s + z} \cdot \frac{s + p}{s - p}$$

11.3.1 Loop shaping non-minimum-phase system

$P(s)$ open-loop stable, has non-minimum-phase system $s - z$ with $z > 0$ in $D = -\frac{s-z}{s+z}$

Results in phase lag $\angle D(j\omega) = -2\arctan\frac{\omega}{z}$

This forces closed-loop system to be slow, slow n.m.p zero is worse than fast. Also limits gain as large k will lead to unstable system.

11.3.2 Loop-shaping unstable open-loop system

$P(s)$ has an unstable open-loop pole, no non-minimum-phase zeros $(s-p)$ with $p > 0$ all-pass filter has form:

$$D = \frac{s+p}{s-p}$$

Results in phase lag $\angle D(j\omega) = -2\arctan\frac{p}{z}$

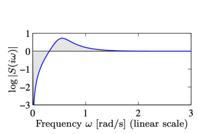
For closed-loop stability magnitude as $\omega \rightarrow 0^+$ must be > 0 ONLY BODE WONT SHOW THIS
Forces gain and crossover frequency to be large. Require fast controllers and powerful actuators. Fast unstable poles are worse.

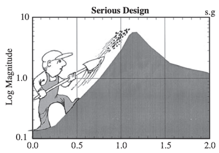
11.4 Bode's integral

Theorem: $S(s)$ is sensitivity function of an internally stable close-loop system with loop transfer function $L(s)$ assume $\lim_{s\rightarrow\infty} sL(s) = 0$. Then:

$$\int_0^{+\infty} \log|S(j\omega)|d\omega = \pi\sum_p p_k$$

Sum is over unstable poles p_k of $L(s)$
Impossible to reject e.g. disturbances at all frequencies. If in some range attenuated, then in some other freq range they must be amplified -> **waterbed effect**. Unstable open-loop poles make amplification higher.





11.5 Choosing sampling time dt

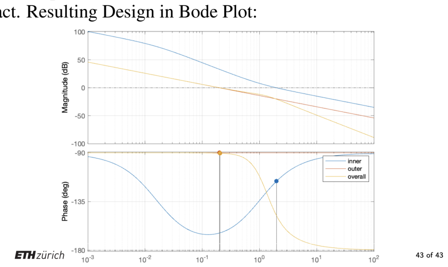
In common implementations commanded value for input maintained over sampling period dt -> Zero-Order, Hold having a similar effect as time delay by $dt/2$. Therefor if $2/dt \gg$ bandwidth expect digital implementation to work well. Slow PC wrt system lead to large decrease in phase margin and possible instability.

11.6 Cascade control

Example Adaptive Cruise Control: Maintain distance and speed -> one input two outputs (speed and distance). 2 loops needed:

- inner loop, control speed using throttle
- outer loop, use speed control position

I.e. output of outer controller is reference speed for inner controller -> Cascade control
Bandwidth of inner loop must be much faster then outer loop. Inner loop is closer to the disturbance therefore better able to react. Resulting Design in Bode Plot:



12 Nonlinear Systems

Most real systems are nonlinear, therefore principle of superposition doesn't hold, since behavior changes depending on initial conditions, amplitude and shape of input.

General model continuous-time nonlinear system:

$$\frac{d}{dt}x(t) = f(t, x(t), u(t))$$
$$y(t) = h(t, x(t), u(t))$$

Time-invariant systems:

$$\frac{d}{dt}x(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

12.1 Analysis of nonlinear systems

12.1.1 Equilibrium points

Nonlinear system $\frac{d}{dt}x = f(x,u),\ y = h(x,u)$

State $\bar{x}$ is an equilibrium point if input $\bar{u}$ exists so $f(\bar{x},\bar{u}) = 0$. Therefore if system is at state $\bar{x}$ at some time $\bar{t}$ and control input $\bar{u}$, then system will remain at $\bar{x}\forall t \geq \bar{t}$. Also $y(\bar{x},\bar{u}) =$ const. $\forall t \geq \bar{t}$

12.1.2 Jacobian Linearization

always possible to change coordinate such that eq point at $(0,0)$:

$\xi = x - \bar{x}$ $\quad \nu = u - \bar{u}$ $\quad \Rightarrow \quad \frac{d}{dt}\xi = \bar{f}(\xi,\nu)$

$y = \bar{h}(\xi,\nu)$ with $\bar{f}(0,0) = 0$ & $\bar{h}(0,0) = h(\bar{x},\bar{u}) = \bar{y}$

If $\bar{f}, \bar{h}$ both continuous and differentiable at eq point:

$$\frac{d}{dt}\xi = \bar{f}(\xi,\nu) \approx \underbrace{\bar{f}(0,0)}_{=0} + \underbrace{\frac{\partial \bar{f}(\xi,\nu}{\partial \xi}\Big|_{(0,0)}}_{A}\xi + \underbrace{\frac{\partial \bar{f}(\xi,\nu}{\partial \nu}\Big|_{(0,0)}}_{B}\nu$$

With similar calculations for $y = \bar{h}(\xi,\nu)$:

$\frac{d}{dt}\xi \approx A\xi + B\nu$, $\quad y - \bar{y} \approx C\xi + D\nu$

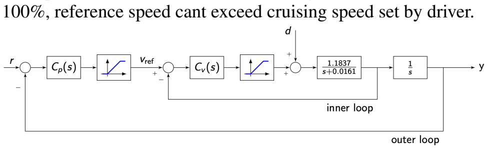
12.1.3 Validity

Approximation only valid for very small ξ, ν. Hartmann-Grossman Theorem states if linearized system is closed-loop BIBO stable so is the non-linear system for very small ν, ξ i.e. in neighborhood of $(0,0)$

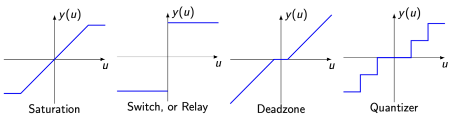
12.2 Nonlinear Elements

12.2.1 Saturation

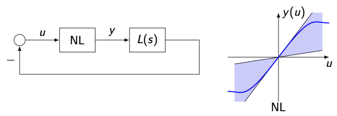
Example of cruise control: throttle cant go below zero or above 100%, reference speed cant exceed cruising speed set by driver.



12.2.2 static non-linearities



12.3 Absolute Stability



Feedback interconnection linear system $L(s)$ with static nonlinear gain element $NL : u \rightarrow y(u)$ such that $y(0) = 0$ and $k_1 \leq y(u)/u \leq k_2 \forall u \neq 0$. I.e. graph of NL must be contained in sector $k_1 \leq y(u)/u \leq k_2$.
System is absolutely stable if for any NL $u = 0$ is a globally asymptotically stable equilibrium point for closed-loop system. i.e. for any initial condition of dynamic system $L(s), u, y \rightarrow 0$ as $t \rightarrow +\infty$

12.3.1 Necessary conditions: Circle Criterion

Functions in NL include all linear gains $k_1 \leq k \leq k_2$. necessary condition for absolute stability of feedback is Nyquist plot encircles segments $[-1/k_1, -1/k_2]$ counterclockwise equal times to number of poles of $L(s)$ with positive real part. -> Nyquist condition with segment instead of point.

12.4 Frequency response of static non-linearity

$NL : u \rightarrow y(u) = u$ with sinusoidal input: $u(t) = a\sin(\omega t)$. Output will be of form $y(t) = f(A\sin(\omega t))$

$$y = \begin{cases} 1 & \text{if } u \geq 1; \\ u & \text{if } -1 < u < 1; \\ -N & \text{if } u \leq -1 \end{cases}$$

12.5 Output harmonics

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}[a_i\cos(i\omega t) + b_i\sin(i\omega t)]$$

$$a_i = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t)\cos(i\omega t)d(\omega t)$$

$$b_i = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t)\sin(i\omega t)d(\omega t)$$

12.6 Describing function

approximate output of non-linearity by first harmonic $y(t) \approx b_1\sin(\omega t)$. Amplitude of the first output harmonic b_1 is a function of the amplitude of input. Ratio b_1/A id describing function.

$$N(A) = \frac{b_1(A)}{A} = \frac{1}{\pi A}\int_{-\pi}^{\pi} y(t)\sin(i\omega t)d(\omega t)$$

With describing function we can approximate non-linearity as an amplitude-dependant gain.
Reasoning: if non-linearity is in feedback loop with physical plant, all higher-order harmonics will be attenuated -> physical systems act as low-pass filters

12.6.1 General Definition of Describing function

Definition above only holds for odd, static non-linearities. In general way first harmonic can be written as:

$$y(t) \approx c_1(A,\omega)e^{j(\omega t + \phi_1(A,\omega))}$$

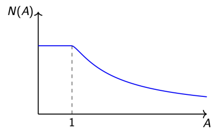
Therefore describing function will be complex number defined as approximate transfer function:

$$N(A,\omega) = \frac{c_1(A,\omega)}{A}e^{\phi_1(A,\omega)}$$

12.6.2 Output Harmonics for saturation non-linearity

for odd non-linearity :

$$N(A) = \begin{cases} \frac{2}{\pi}\left[\arcsin(\frac{1}{A} + \frac{1}{A\sqrt{1-\left(\frac{1}{2}\right)^2}}\right] & \text{, if } A > 1; \\ 1 & \text{, if } A \leq 1 \end{cases}$$

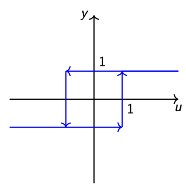


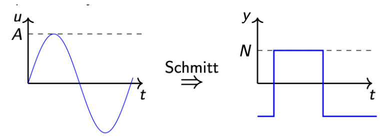
Think of saturation as gain decreasing with increasing amplitude once it exceeds saturation level.

12.7 (Odd) Non-linearities with memory (Schmitt trigger)

Non-linearities with hysteresis often used in applications, not static since output depends on movement direction of system.

12.7.1 Relay with hysteresis - Schmitt trigger





$$N(A) = \frac{4}{\pi A}\left(\sqrt{1 - \left(\frac{1}{A}\right)^2} - j\frac{1}{A}\right)$$

12.8 Stability analysis using describing functions

Approximate new loop transfer function

$$L'(A,s) \approx N(A)L(s)$$

Assume input to non-linearity has complex form $e(t) = Ae^{j\omega t}$. Oscillation is self-sustained if $A = -AN(A)L(j\omega)$, i.e. if $-\frac{1}{N(A)} = G(j\omega)$

12.8.1 Checking limit cycles

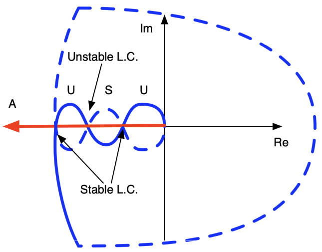
$$L(j\omega) = \frac{1}{j\omega(j\omega+1)} = \frac{-\omega - j}{\omega(\omega^2+1)} = \frac{1}{N(A)}$$

Periodic orbit to which system converges over time.

- sketch polar plot of frequency response $L(j\omega)$
- Sketch polar plot of $-1/N(A)$
- Limit cycles could exist at intersections between these curves
- Intersection gives estimate of freq and amp.

12.8.2 Stability of limit cycles

Use $-1/N(A)$ as the -1 or $-1/K$ point for Nyquist. If this point is in unstable region amplitude of oscillations will increase. If in stable region amplitude will decrease.
If small amplitude decreases make amplitude increase and vice verso limit cycle is stable otherwise unstable.



13 Robustness

13.1 Anti Wind-up

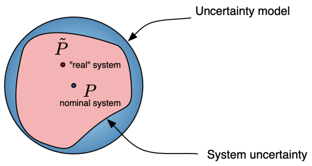
Once input saturates integral error keeps increasing. When error decreases large integral prevents resuming normal operations

13.2 Anti-windup logic

Logic for integral gain: $K_I' \begin{cases} K_I & \text{if input deosn't saturate} \\ 0 & \text{if input saturates} \end{cases}$

Anti wind-up guarantees stability of compensator when feedback loop is opened by saturation. Maintain "small" integral error.

13.3 System uncertainty and uncertainty models

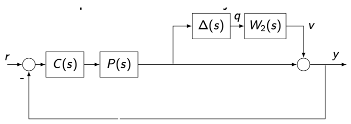


To take uncertainty into account first create uncertainty model. Made of nominal model and set of models guaranteed to contain systems uncertainty.
Design control system that meets stability and performance specs for all possible models in uncertainty model.

13.4 Uncertainty models for SISO linear systems

All methods aim at writing transfer function of real system in terms of transfer function of nominal system and unknown transfer function Δ representing uncertainty as perturbation of nominal system. Perturbation Δ assumed to be stable minimum-phase transfer function so it doesn't cancel e any unstable poles of nominal system.

13.5 SISO with multiplicative uncertainty



True but unknown loop transfer function is:

$$\tilde{L}(s) = (1 + W_2(s)\Delta(s))P(s)C(s)$$

Nominal loop transfer function for $\Delta = 0$

$$L(s) = P(s)C(s)$$

Such that closed-loop system is stable.

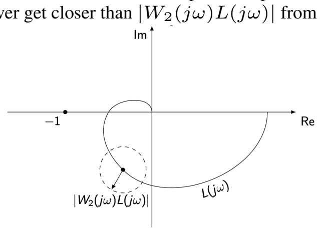
13.6 Nyquist condition for robust stability

$$\tilde{L}(j\omega) = L(j\omega) + W_2(j\omega)\Delta(j\omega)L(j\omega)$$

Although not known what $\Delta(j\omega)$ magnitude must be bounded by 1

$$\tilde{L}(j\omega) - L(j\omega) = |W_2(j\omega)\Delta(j\omega)L(j\omega)| \leq |W_2(j\omega)L(j\omega)|$$

For Nyquist not to encircle -1 point loop transfer function should never get closer than $|W_2(j\omega)L(j\omega)|$ from -1 point.



14 Appendix

14.1 system types

SISO - Single Input - Singe Output	**MIMO** - Multiple Input - Multiple Output
linear - condition: $\Sigma(\alpha u_1 + \beta u_2 = \alpha\Sigma(u_1) + \beta\Sigma(u_2)$ Ex. $y(t) = \frac{d}{dt}u(t)$ $y(t) = \int_0^t u(\tau)d\tau$ $y(t) = a \cdot x(t) + u(t)$	**non linear** - otherwise Ex. $y(t) = \alpha u(t) + \beta$ $y(t) = \sin(t)$ $y(t) = x(t) \cdot u(t)$ $y(t) = x^2(t)$ $y(t) = x(t) \cdot u(t)$ $\frac{d}{dt}x = x \cdot u$
time-invariant same input - same output? (no matter the time) for static: $u_1(t) = u_2(t - T_t) \mapsto y_1(t) = y_1(t - T_t)$ for dynamic: No explicit time dependency Ex. $y(t) = 3u(t) + 5$ $y(t) = \frac{d}{dt}u(t)$ $y(t) = \sin(u(t))$ $y(t) = u(t-3)$	**time varying** if t is part of coefficients of x and u Ex. $y(t) = \sin(t)u(t)$ $y(t) = u(t) + t$
dynamic If output is influenced by past inputs ("memory") - always given if differential equation is used **integrals, derrivatives, delays** Ex. $y(t) = \int_0^t u(\tau)dt$ $y(t) = u(t-2)$	**static** If Output is only dependent on present (current) inputs ("memoryless") Ex $y(t) = 3u(t)$ $y(t) = \sqrt{u(t)}$ $2^{-(t-1)}u(t)$

14.2 Trigonometrische Werte 97

Bogenmass Gradmass	0 $0°$	$\frac{\pi}{6}$ $30°$	$\frac{\pi}{4}$ $45°$	$\frac{\pi}{3}$ $60°$	$\frac{\pi}{2}$ $90°$	π $180°$
$\sin(\alpha)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos(\alpha)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\tan(\alpha)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0

		ROOT LOCUS	BODE	NYQUIST
STABILITY IN FUNCTION OF K	IF THE CLOSE LOOP CAN BE STABLE	✓	ONLY IF L(S) STABLE	✓
	FOR WHICH K IS THE CLOSE LOOP STABLE	✗	✗	✓
	POSITION OF THE CLOSE-LOOP POLES IN FUNCTION OF K	✓	✗	✗
	|G| AND ∠G IN FUNCTION OF ω	✗	✓	✓ BUT IMPLICITELY
	ROBUSTNESS	✗	✓	✓