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Analysis III – ITET

HS 08

Prüfungsmusterlösung

1(a). *Sei $k \in \mathbb{R}$ eine Konstante. Klassifiziere den Differentialoperator*

$$\frac{k^2}{2}\left(\partial_{xx} + \partial_{yy}\right) + \left(2 - k^2\right)\partial_{xy} + (1+k)\left(\partial_x + 2\,\partial_y\right) + k\,.$$

Using the method (and the notation) encountered[1] in class for second-order operators in two variables, we see that

$$a = \frac{k^2}{2} = c \quad , \quad b = 2 - k^2 \quad , \quad d = 1 + k \quad , \quad e = 2\,(1+k)\,.$$

Accordingly, the discriminant is

$$\Delta = b^2 - 4\,a\,c = \left(2 - k^2\right)^2 - 4\left(\frac{k^2}{2}\right)^2 = 4\left(1 - k^2\right).$$

Therefore:

- if $|k| > 1$, the discriminant is negative, and thus the operator is elliptic.
- if $|k| < 1$, the discriminant is positive, and thus the operator is hyperbolic.
- if $|k| = 1$, the discriminant is zero. To decide whether the operator is parabolic or degenerate, we need to test further. Using the method explained in the Musterlösung 1, we find

$$2\,c\,d - b\,e = k^2\,(1+k) - 2\left(2 - k^2\right)(1+k) = \left(3\,k^2 - 4\right)(1+k)\,.$$

  - if $k = (-1)$, this quantity is zero, and thus the operator is degenerate.
  - if $k = 1$, this quantity is nonzero (i.e. (-2)), and thus the operator is parabolic.

[1] cf. Problems 4 and 5 in Serie 1.

1(b). *Sei H die Heaviside Distribution und $v(x,t,y)$ die Lösung von*

$$\begin{cases} v_t - v_{xx} = 0 & , \quad (x,t,y) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \\ v(x,0,y) = \delta(x-y) & , \quad (x,y) \in \mathbb{R}^2 \,. \end{cases}$$

Zeige, dass

$$G(x,t,y,s) := v(x,t-s,y)\, H(t-s)$$

die Green'sche Funktion vom Operator $\partial_t - \partial_{xx}$ ist. D.h.

$$G_t - G_{xx} = \delta(x-y, t-s)\,, \qquad (x,t,y,s) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \,.$$

We begin the problem by (distributionally) computing $G_t - G_{xx}$. Namely

$$\begin{aligned}
\big(\partial_t - \partial_{xx}\big)\, G(x,t,y,s) &\equiv \big(\partial_t - \partial_{xx}\big)\, v(x,t-s,y)\, H(t-s) \\
&= \big(\partial_{(t-s)} - \partial_{xx}\big)\, v(x,t-s,y)\, H(t-s) && \text{from the chain-rule} \\
&= H(t-s)\, \big(v_{t-s}(x,t-s,y) - v_{xx}(x,t-s,y)\big) \\
&\qquad\qquad + v(x,t-s,y)\, \partial_{t-s} H(t-s) \\
&= v(x,t-s,y)\, \partial_{t-s} H(t-s) && \text{from the PDE of } v \\
&= v(x,t-s,y)\, \delta(t-s) && \text{as } H' = \delta \text{ from class.}
\end{aligned}$$

Let now $\phi(x,t)$ be an arbitrary test-function. Using the above, we find that

$$\begin{aligned}
\int\!\!\int \phi(x,t)\, \big(\partial_t - \partial_{xx}\big)\, G(x,t,y,s)\, dt\, dx &= \int\!\!\int \phi(x,t)\, v(x,t-s,y)\, \delta(t-s)\, dt\, dx \\
&= \int \phi(x,s)\, v(x,0,y)\, dx && \text{by definition of } \delta \\
&= \int \phi(x,s)\, \delta(x-y)\, dx && \text{as given by the problem} \\
&= \phi(y,s) && \text{by definition of } \delta \\
&= \int\!\!\int \phi(x,t)\, \delta(x-y, t-s)\, dx\, dt && \text{by definition of } \delta
\end{aligned}$$

Because ϕ was arbitrary, the above reveals that (in the sense of distributions of course) there holds

$$\big(\partial_t - \partial_{xx}\big)\, G(x,t,y,s) = \delta(x-y, t-s)\,.$$

2. *Sei $g : \mathbb{R} \to \mathbb{R}$ eine glatte Funktion. Betrachte das Cauchy Problem*

$$\begin{cases} 2\,x\,u_y(x,y) \;-\; u_x(x,y) \;=\; 0 \quad , & (x,y) \in \mathbb{R}^2 \\ u\big(x\,,(p-1)\,x^2\big) \;=\; g\big(x^2\big) \quad , & x \in \mathbb{R}\,, \end{cases}$$

wobei $p \in \mathbb{R}$ eine Konstante ist.

(a) Bestimme die Lösung $u(x,y)$ aus den gegebenen Daten mithilfe der Methode der Charakteristiken.

(b) Zeichne die projezierten Charakteristiken in der (x,y)-Ebene für den Fall $p = 1$.

(c) Erkläre sorgfältig was im Fall $p = 0$ passiert.

(a) We proceed as usual[2] by writing the system of characteristic ODEs along with the parametrized Cauchy datum

$$\begin{cases} \dfrac{d}{dr}\,x(r,s) \;=\; -1 & \dfrac{d}{dr}\,y(r,s) \;=\; 2\,x(r,s) & \dfrac{d}{dr}\,u(r,s) \;=\; 0 \\ x(0,s) \;=\; s & y(0,s) \;=\; (p-1)\,s^2 & u(0,s) \;=\; g\big(s^2\big)\,. \end{cases}$$

We first solve the ODE for $x(r,s)$. This is immediate, namely

$$x(r,s) \;=\; -r \;+\; A(s)\,, \qquad \text{for some function } A(s)\,.$$

Using the data at $r = 0$ shows that $A(s) = s$. Hence

$$x(r,s) \;=\; s \;-\; r\,. \tag{1}$$

Now we substitute this in the ODE for $y(r,s)$ to obtain

$$\frac{d}{dr}\,y(r,s) \;=\; 2\,s \;-\; 2\,r\,,$$

which is easily integrated to yield

$$y(r,s) \;=\; 2\,s\,r \;-\; r^2 \;+\; B(s)\,, \qquad \text{for some function } B(s)\,.$$

Bringing the data at $r = 0$ shows that $B(s) = (p-1)s^2$, so that

$$y(r,s) \;=\; 2\,s\,r \;-\; r^2 \;+\; (p-1)\,s^2\,.$$

Note that

$$y(r,s) \;=\; 2\,s\,r \;-\; r^2 \;-\; s^2 \;+\; p\,s^2 \;=\; p\,s^2 \;-\; (s-r)^2\,.$$

Using (1), we see now that

$$y \;=\; p\,s^2 \;-\; x^2\,. \tag{2}$$

[2]cf. Problems 1 and 2 on Serie 1, and Problems 1, 2, 3, and 4 on Serie 2.

Finally, solving the ODE for $u(r,s)$ gives immediately

$$u(r,s) \;=\; g\big(s^2\big)\,. \tag{3}$$

As seen in (2), there holds

$$s^2 \;=\; \frac{x^2 \,+\, y}{p}\,.$$

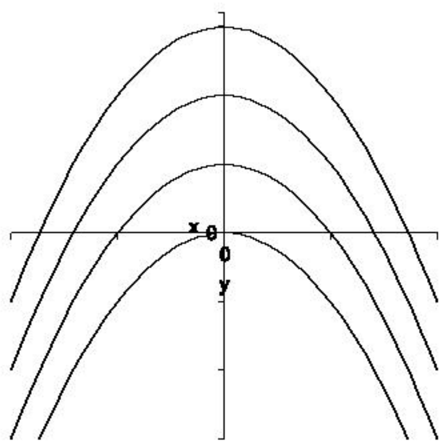
Putting this into (3) gives finally the solution

$$u(x,y) \;=\; g\left(\frac{x^2 \,+\, y}{p}\right)\,. \tag{4}$$

(b) The projected characteristics in the (x,y)-plane are described in (2). With $p = 1$, it is the family of parabolas

$$y \;=\; C \,-\, x^2\,, \qquad \text{where} \qquad C \,\geq\, 0\,.$$

A few of them are shown on the following graph.



(c) On the other hand, if $p = 0$, then (2) reveals that there is only one characteristic curve, namely

$$y \;=\; -\,x^2\,.$$

This degenerate case has a simple explanation. Note that when $p = 0$, the Cauchy datum is given along the curve

$$y \;=\; -\,x^2\,.$$

Thus, the Cauchy datum is precisely given along a characteristic curve. Therefore, the problem is *ill-posed*. This phenomenon is also reflected in the solution (4), which no longer makes sense when $p = 0$.

3. *Betrachte das Problem*

$$u_{tt} - \frac{1}{4} u_{xx} = 0 \,, \qquad (t,x) \in \mathbb{R}^2 \,,$$

mit den Bedingungen

$$u(0,x) = 2f(x) \,, \qquad u_t(0,x) = 0 \,, \qquad u_t\left(y \,, \frac{y}{2}\right) = \frac{1}{2} h'(y) \,, \qquad \forall\, x, y \in \mathbb{R} \,.$$

Die Funktionen f und h sind glatt und besitzen die Eigenschaften

$$f(0) = h(0) = h'(0) = 0 \qquad \text{und} \qquad f'(0) = 1 \,.$$

Drücke die Lösung $u(t,x)$ nur in Abhängigkeit von h aus.

This is a one-dimensional wave equation problem. We will thus use the method of d'Alembert[3]. Because here $c = 1/2$, it states that there exist functions ψ and ϕ with

$$u(t,x) = \phi\left(x + \frac{t}{2}\right) + \psi\left(x - \frac{t}{2}\right) . \tag{5}$$

Substituting this form in the first two given conditions yields

$$\phi(x) + \psi(x) = 2\, f(x) \qquad \text{and} \qquad \phi'(x) - \psi'(x) = 0 \,, \qquad \forall\; x \in \mathbb{R} \,.$$

This shows immediately that

$$\phi(x) = f(x) + K \qquad \text{and} \qquad \psi(x) = f(x) - K \,,$$

for some constant K. Hence the solution (5) becomes

$$u(t,x) = f\left(x + \frac{t}{2}\right) + f\left(x - \frac{t}{2}\right) . \tag{6}$$

The remaining condition to match reads

$$\frac{1}{2} h'(y) = u_t\left(y \,, \frac{y}{2}\right) \overset{(6)}{=} \frac{1}{2}\left[f'\left(\frac{y}{2} + \frac{y}{2}\right) - f'\left(\frac{y}{2} - \frac{y}{2}\right)\right] = \frac{1}{2}\Big[f'(y) - 1\Big] \,,$$

where we have used the given fact that $f'(0) = 1$.
Therefore, we see that

$$h'(y) = f'(y) - 1 \,.$$

Integrating this simple equation on the interval $[0, z]$ yields

$$h(z) = f(z) - z \,, \qquad \text{where we have used the given facts } h(0) = 0 = f(0) \,.$$

Substituting this into (6) produces finally the desired answer

$$u(t,x) = h\left(x + \frac{t}{2}\right) + h\left(x - \frac{t}{2}\right) + 2\,x \,.$$

[3] cf. Problem 6 on Serie 2, and Problems 2 and 4 on Serie 3.

4. *Löse mithilfe von Separation der Variablen*

$$\left\{\begin{array}{rclcrcl} \Delta\, u(x,y) & = & u(x,y) & & & & \\ u(x,0) & = & 0 & , & u(x,\pi) & = & 0 \\ u(0,y) & = & \sin y & , & u(\pi,y) & = & 0 \end{array}\right\} \quad \textit{wobei} \quad (x,y) \in [0,\pi]\times[0,\pi]\,.$$

We will proceed as we have often done in the Serien. You are invited to consult the Musterlösungen for further details.

We begin by searching for a solution in the separated form

$$u(x,y) \;=\; X(x)\,Y(y)\,,$$

for two nontrivial functions $X(x)$ and $Y(y)$ to be determined. Substituting this Ansatz in the first set of boundary conditions reveals that

$$Y(0) \;=\; 0 \;=\; Y(\pi)\,. \tag{7}$$

Substitution in the PDE gives

$$X''(x)Y(y) \,+\, X(x)Y''(y) \;=\; X(x)Y(y)\,.$$

Equivalently,

$$\frac{Y''(y)}{Y(y)} \;=\; 1 \,-\, \frac{X''(x)}{X(x)}\,.$$

The left-hand side depends only on y, whereas the right-hand side depends only on x. Consequently, there exists some constant λ satisfying

$$\frac{Y''_\lambda}{Y_\lambda} \;=\; -\,\lambda \qquad \text{and} \qquad 1 \,-\, \frac{X''_\lambda}{X_\lambda} \;=\; -\,\lambda\,.$$

We have appended a subscript to indicate the dependence on the parameter λ.
Hence, we will have to solve

$$Y''_\lambda \;=\; -\,\lambda\, Y_\lambda \qquad \text{and} \qquad X''_\lambda \;=\; (1+\lambda)\, X_\lambda\,. \tag{8}$$

If $\lambda = 0$, we find the solution of the first ODE to be

$$Y_0(y) \;=\; A_0\, y \,+\, B_0\,.$$

The constants A_0 and B_0 are easily found to be zero from the condition (7). Thus, when $\lambda = 0$, we have only produced the uninteresting trivial solution.

If $\lambda \neq 0$, the solution of the first ODE in (8) is

$$Y_\lambda(y) \;=\; A_\lambda \cos\left(y\sqrt{\lambda}\right) \,+\, B_\lambda \sin\left(y\sqrt{\lambda}\right)$$

for some constants A_λ and B_λ to be determined. Using the boundary condition (7), we find that

$$A_\lambda \;=\; 0 \qquad \text{and thus} \qquad \sin\left(\pi\sqrt{\lambda}\right) \;=\; 0\,.$$

Accordingly, we see that $\sqrt{\lambda}$ must be an integer. Hence

$$\lambda \;=\; n^2 \qquad \text{for } \; n \in \mathbb{N}^*\,. \tag{9}$$

Finally, the solution may be written

$$Y_n(y) \;=\; B_n\,\sin(n\,y)\,. \tag{10}$$

We have appended the index n to indicate the dependence on this parameter.

We now return to the second ODE in (8). Knowing now (9), we see that

$$X_n'' \;=\; \left(1+n^2\right) X_n\,.$$

Its general solution is

$$X_n(x) \;=\; C_n \cosh\left(x\sqrt{1+n^2}\right) \;+\; D_n \sinh\left(x\sqrt{1+n^2}\right)\,, \tag{11}$$

for some constants C_n and D_n .

Combining now (10) and (11) into the superposition principle gives the solution

$$\begin{aligned} u(x,y) \;&=\; \sum_{n\geq 1} X_n(x)\,Y_n(y) \\ &=\; \sum_{n\geq 1}\left[P_n \cosh\left(x\sqrt{1+n^2}\right) \;+\; Q_n \sinh\left(x\sqrt{1+n^2}\right)\right]\sin(n\,y)\,, \end{aligned} \tag{12}$$

where we have set $P_n = B_n C_n$ and $Q_n = B_n D_n$ for notational convenience.

We have yet to match the second set of boundary data given by the problem, namely

$$u(0,y) \;=\; \sin y \qquad \text{and} \qquad u(\pi,y) \;=\; 0\,.$$

Substituting the solution (12) yields first

$$\sum_{n\geq 1} P_n\,\sin(n\,y) \;=\; \sin y\,,$$

so that

$$P_1 \;=\; 1\,, \qquad \text{and} \qquad P_n \;=\; 0 \qquad \forall\; n \;>\; 1\,. \tag{13}$$

The second condition gives

$$\sum_{n\geq 1}\left[P_n \cosh\left(\pi\sqrt{1+n^2}\right) \;+\; Q_n \sinh\left(\pi\sqrt{1+n^2}\right)\right]\sin(n\,y) \;=\; 0\,,$$

so that

$$P_n \cosh\left(\pi\sqrt{1+n^2}\right) \;+\; Q_n \sinh\left(\pi\sqrt{1+n^2}\right) \;=\; 0\,.$$

Bringing in (13) now shows that

$$Q_1 \;=\; -\,\frac{\cosh\left(\pi\,\sqrt{1+1^2}\right)}{\sinh\left(\pi\,\sqrt{1+1^2}\right)} \;=\; -\coth\big(\pi\sqrt{2}\big)$$

and

$$Q_n \;=\; 0 \qquad\qquad \forall\; n \;>\; 1\,.$$

Putting altogether the above findings gives finally the solution to the problem:

$$u(x,y) \;=\; \Big[\cosh\big(x\sqrt{2}\big) \;-\; \coth\big(\pi\sqrt{2}\big)\;\sinh\big(x\sqrt{2}\big)\Big]\;\sin y\,. \tag{14}$$

Those of you who prefer to express "cosh and sinh" solutions in terms of "e^+ and e^-" may not like the way (14) looks. But it can easily be converted using the (defining) identities

$$\cosh s \;=\; \frac{\mathrm{e}^s + \mathrm{e}^{-s}}{2}\,, \qquad \sinh s \;=\; \frac{\mathrm{e}^s - \mathrm{e}^{-s}}{2}\,, \qquad \text{and} \qquad \coth s \;=\; \frac{\mathrm{e}^{2s}+1}{\mathrm{e}^{2s}-1}\,.$$

These identity yield

$$\begin{aligned}
2\,\cosh z \;-\; 2\coth s\,\sinh z \;&=\; \mathrm{e}^z + \mathrm{e}^{-z} \;-\; \frac{\mathrm{e}^{2s}+1}{\mathrm{e}^{2s}-1}\,\big(\mathrm{e}^z - \mathrm{e}^{-z}\big)\\
&=\; \frac{1}{\mathrm{e}^{2s}-1}\,\Big[\big(\mathrm{e}^z + \mathrm{e}^{-z}\big)\,\big(\mathrm{e}^{2s}-1\big) \;-\; \big(\mathrm{e}^{2s}+1\big)\,\big(\mathrm{e}^z - \mathrm{e}^{-z}\big)\Big]\\
&=\; \frac{2}{\mathrm{e}^{2s}-1}\,\big(\mathrm{e}^{2s-z} \;-\; \mathrm{e}^z\big)\,.
\end{aligned}$$

With $z = x\sqrt{2}$ and $s = \pi\sqrt{2}$, the solution (14) can be equivalently expressed in the form

$$u(x,y) \;=\; \frac{\sin y}{\mathrm{e}^{2\pi\sqrt{2}}-1}\,\left(\mathrm{e}^{(2\pi-x)\sqrt{2}} \;-\; \mathrm{e}^{x\sqrt{2}}\right).$$

5. *Betrachte einen Stab der in $0 \leq x \leq 1$ liegt. Der Stab besteht aus radioaktivem Material, welches die Konstante Wärme 1 (in geeigneten Einheiten) abgibt. Zusätzlich ist der Stab an beiden Enden isoliert. Am Anfang ist die Temperaturverteilung $f(x)$. Mit $u(x,t)$ bezeichnen wir die Temperatur an der Stelle $x \in [0,1]$ zum Zeitpunkt $t \geq 0$. Nimm an, dass u die Wärmeleitungsgleichung mit spezifischer Wärme 1 erfüllt.*

(a) Bestimme das Problem (Gleichung, Rand-/Anfangsbedingung) welches $u(x,t)$ löst.

(b) Löse das Problem mit den gegebenen Daten.

(c) Was passiert mir der Lösung für grosse Zeiten? Rechtfertige dies physikalisch.

This problem is a clone of Problem 5 from Serie 4.

(a) The heat equation (specific heat 1) with the constant generation of heat rate 1 gives the PDE

$$u_t - u_{xx} = 1 .$$

The condition that the rod be insulated at its ends tells us that no heat can either enter or leave the rod (no heat flux). Hence

$$u_x(0,t) = 0 = u_x(1,t) .$$

Finally, we have the initial data

$$u(x,0) = f(x) .$$

Accordingly, we shall solve the problem

$$\begin{cases} u_t - u_{xx} = 1 , & (x,t) \in [0,1] \times [0,\infty) \\ u(x,0) = f(x) , & x \in [0,1] \\ u_x(0,t) = 0 , & t \geq 0 \\ u_x(1,t) = 0 , & t \geq 0 . \end{cases} \tag{15}$$

(b) Because this is an inhomogeneous problem, we start by solving its homogeneous counterpart

$$\begin{cases} v_t - v_{xx} = 0 , & (x,t) \in [0,1] \times [0,\infty) \\ v_x(0,t) = 0 , & t \geq 0 \\ v_x(1,t) = 0 , & t \geq 0 . \end{cases} \tag{16}$$

This is handled with the method of separation of variables, as usual. We start with the Ansatz

$$v(x,t) = X(x)\,T(t) , \qquad \text{for some functions } X \text{ and } T \text{ to be determined.}$$

The boundary conditions read

$$X'(0) = 0 = X'(1) . \tag{17}$$

The PDE gives

$$\frac{T'(t)}{T(t)} \;=\; \frac{X''(x)}{X(x)}\,.$$

As usual, the left-hand side depends only on t, while the right-hand side depends only on x. Hence there exists some constant λ (to be determined) such that

$$T'_\lambda(t) \;=\; -\,\lambda\,T_\lambda(t) \qquad \text{and} \qquad X''_\lambda(x) \;=\; -\,\lambda\,X_\lambda(x)\,. \tag{18}$$

We have appended the subscript λ to indicate the dependence on λ.

The first ODE is immediately solved, namely

$$T_\lambda(t) \;=\; C_\lambda\,\mathrm{e}^{-\lambda\,t}\,,$$

for some constant C_λ to be determined later.

We consider next the second ODE in (18) with its boundary conditions (17), namely

$$X''_\lambda(x) \;=\; -\,\lambda\,X_\lambda(x) \qquad \text{with} \qquad X'_\lambda(0) \;=\; 0 \;=\; X'_\lambda(1)\,.$$

If $\lambda = 0$, we see that $X_0(x) = A_0$, is the constant solution.

The case $\lambda \neq 0$ gives:

$$X_\lambda(x) \;=\; A_\lambda\cos\left(x\sqrt{\lambda}\right) \;+\; B_\lambda\sin\left(x\sqrt{\lambda}\right)\,, \qquad \text{for some constants } A_\lambda \text{ and } B_\lambda.$$

Matching the boundary conditions (17) requires that

$$B_\lambda \;=\; 0 \qquad \text{and thus} \qquad \sqrt{\lambda}\,A_\lambda\sin\left(\sqrt{\lambda}\right) \;=\; 0\,.$$

To avoid the trivial solution, and since $\lambda \neq 0$, we thus need

$$\sin\left(\sqrt{\lambda}\right) \;=\; 0\,.$$

Hence

$$\sqrt{\lambda} \;=\; \pi\,n \qquad \text{for some } n \in \mathbb{Z}^*\,.$$

This gives the solution

$$X_n(x) \;=\; A_n\cos\left(n\,\pi\,x\right)\,.$$

We can actually restrict our attention to $n \in \mathbb{N}^*$. Indeed, changing the sign of n amounts to changing the sign of A_n. But because the coefficients A_n are arbitrary, we see that $n \geq 1$ suffices to encompass all possibilities. Note that we have again appended a subscript to indicate the dependence on $n \in \mathbb{N}^*$.

Combining the above results, we have found the solutions

$$P_0\,, \qquad \text{and} \qquad P_n\,\mathrm{e}^{-n^2\pi^2 t}\cos\left(n\,\pi\,x\right) \qquad \forall\, n \;\geq\; 1\,.$$

where we have set $P_n = C_n A_n$ for convenience.

Using the superposition principle, the solution of the homogeneous problem (16) is

$$v(x,t) \;=\; P_0 \;+\; \sum_{n\geq 1} P_n \, \mathrm{e}^{-n^2\pi^2 t} \cos{(n\,\pi\,x)}\;. \tag{19}$$

To get our hands on the solution of the inhomogeneous problem (15), we vary the constants by making the Ansatz

$$u(x,t) \;=\; P_0(t) \;+\; \sum_{n\geq 1} P_n(t) \, \mathrm{e}^{-n^2\pi^2 t} \cos{(n\,\pi\,x)}\;, \tag{20}$$

where $P_0(t)$ and $P_n(t)$ are now functions to be determined.
Substituting this Ansatz in our PDE (15) yields easily

$$P_0'(t) \;+\; \sum_{n\geq 1} P_n'(t) \, \mathrm{e}^{-n^2\pi^2 t} \cos{(n\,\pi\,x)} \;=\; 1\;.$$

Matching both sides is immediate[4], and we find

$$P_0'(t) \;=\; 1\;, \qquad \text{and} \qquad P_n'(t) \;=\; 0 \qquad \forall\; n \,>\, 1\;.$$

Therefore,

$$P_0(t) \;=\; t \,+\, Q_0\;, \qquad \text{and} \qquad P_n(t) \;=\; Q_n \qquad \forall\; n \,>\, 1\;, \tag{21}$$

where Q_n are constants to be determined.
Substituting (21) into (20) gives now the solution

$$u(x,t) \;=\; t \;+\; Q_0 \;+\; \sum_{n\geq 1} Q_n \, \mathrm{e}^{-n^2\pi^2 t} \cos{(n\,\pi\,x)}\;. \tag{22}$$

There only remains to accommodate the initial data. Namely

$$f(x) \;=\; Q_0 \;+\; \sum_{n\geq 1} Q_n \, \cos{(n\,\pi\,x)}\;.$$

We thus see that Q_0 and Q_n are the coefficients of the 2-periodic Fourier cosine series of the function $f(x)$. Hence

$$Q_0 \;=\; \int_0^1 f(x)\,dx \qquad \text{and} \qquad Q_n \;=\; 2\int_0^1 f(x)\,\cos{(\pi\,n\,x)}\,dx\;. \tag{23}$$

Substituting this into (22) gives the desired complete description of the solution in terms of the data provided by the problem.

[4] note indeed that the constant 1 is already expressed in a Fourier cosine series!

Remark. A "faster" method consists in setting in (15)

$$u(x,t) \;=\; t \,+\, v(x,t)\,. \tag{24}$$

Then $v(x,t)$ is found to satisfy the problem

$$\begin{cases} v_t \,-\, v_{xx} &=\; 0 \qquad , \qquad (x,t) \,\in\, [0,1]\times[0,\infty) \\ v_x(0,t) &=\; 0 \qquad , \qquad t \,\geq\, 0 \\ v_x(1,t) &=\; 0 \qquad , \qquad t \,\geq\, 0 \\ v(x,0) &=\; f(x) \qquad , \qquad x \,\in\, [0,1]\,. \end{cases}$$

This homogeneous problem is then solved by separation of variables as we obtained (19) to produce

$$v(x,t) \;=\; Q_0 \;+\; \sum_{n\geq 1} Q_n\, \mathrm{e}^{-n^2\pi^2 t} \cos\left(n\,\pi\,x\right).$$

The initial data is used to determine Q_0 and Q_n just like in (23).

Therefore, from (24), we recover directly

$$u(x,t) \;=\; t \;+\; Q_0 \;+\; \sum_{n\geq 1} Q_n\, \mathrm{e}^{-n^2\pi^2 t} \cos\left(n\,\pi\,x\right).$$

(c) As seen in (22), when t is very large, the solution $u(x,t)$ behaves like t. Thus,

$$\lim_{t\to\infty} u(x,t) \;=\; \infty \qquad \text{for all values of } x\,.$$

This mathematical consequence is evident physically. Indeed, because the rod is insulated, no heat can escape from it. Yet, the radioactive reaction taking place inside the rod produces heat continuously at the positive rate 1. Accordingly, the heat inside the rod keeps increasing indefinitely.