ETH Zürich
Prof. Dr. A. Iozzi

Winter 2022
29 January

Exam Analysis III
D–MAVT, D–MATL

Surname:	
First Name:	
Student Card Nr.:	
Exam Nr.:	

This page contains the generalities of the student: surname, first name, student card (Legi) number and exam number. The exam number is a number that identifies uniquely the student.

By signing this page the student confirms that the above personal data are correct.

Signature

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Prof. Dr. A. Iozzi

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Exam Nr.:	

Please do not fill!

Exercise	Value	Points	Control
1	8		
2	10		
3	10		
4	12		
5	15		
6	15		
Total	70		

Please do not fill!

Completeness	

Please turn!

Before the exam:

- Turn off your mobile phone and place it inside your briefcase/backpack.
- Put your bags on the floor. No bags on the desk!
- Place your Student Card (Legi) on the desk.

During the exam, please:

- Start every exercise on a new piece of paper.
- Put your **exam number** on the top right corner of every page.
- Motivate your answers. Write down calculations and intermediate results.
- Provide at most **one** solution to each exercise.
- **Do not** write with pencils. Please avoid using **red** or **green** ink pens.

After the exam:

- Make sure that every solutions sheet has your exam number on it.
- Place back the exam sheets, together with your solutions, in the envelope.

Allowed aids:

- 20 pages (= 10 sheets) DIN A4 handwritten or typed personal summary.
- An English (or English-German) dictionary.
- **No** further aids are allowed. In particular neither communication devices, nor pocket calculators.

Good Luck!

See the next page!

Laplace Transforms: $\quad (F = \mathcal{L}(f))$

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geq 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)$	$\frac{1}{s}e^{-as}$
12)	$\delta(t-a)$	e^{-as}

(Γ = Gamma function, u = Heaviside function, δ = Delta function)

Indefinite Integrals (you may use): $\quad \left(n \in \mathbb{Z}_{\geq 1}\right)$

1) $$\int x\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{\cos\left(\frac{n\pi}{L}x\right) + \left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2} \quad \text{(+ constant)}$$

2) $$\int x^2\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{\left(\left(\frac{n\pi}{L}\right)^2 x^2 - 2\right)\sin\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right)x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3} \quad \text{(+ constant)}$$

3) $$\int x\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{\sin\left(\frac{n\pi}{L}x\right) - \left(\frac{n\pi}{L}\right)x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2} \quad \text{(+ constant)}$$

4) $$\int x^2\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{\left(2 - \left(\frac{n\pi}{L}\right)^2 x^2\right)\cos\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3} \quad \text{(+ constant)}$$

5) $$\int \frac{1}{1+x^2}\,dx = \arctan(x) \quad \text{(+ constant)}$$

If you need to use these integrals without the $\frac{n\pi}{L}$, just put $\frac{n\pi}{L} = 1$ in the left hand side.

You can use these formulas without justification.

1. Classification of PDEs (*8 Points*)

Definition : The *order* of a PDE is the order of the highest derivative in the PDE.

Consider the following PDEs (in what follows, $u = u(x, y)$ is a function of two variables x and y).

State the order of the PDE and for second order PDE, classify each of them (hyperbolic, parabolic, elliptic, mixed type (you don't need to draw the region in this case)).

Write the answer in the box

a) $u_{xx} + yu_{yy} = \tan(u)$.

b) $u_x + au_y + u^3 = 0$, where $a > 0$ is a positive constant.

c) $y^2 u_{xxx} + (\pi + 2)u_{yy}u_{xx} + u_x = 2u_y + u$.

d) $\pi u_{xx} + 2eu_{xy} + \pi u_{yy} = 0$, where e is the Euler's number ($e \approx 2.718$).

2. Laplace Transform (*10 Points*)

Find the solution $f(t)$ of the following initial value problem:

$$\begin{cases} f''(t) = 3 + u(t-a) - \delta(t-\pi), & t > 0, \\ f(0) = b, \quad f'(0) = c, \end{cases}$$

where $a, b, c > 0$ are positive constants.

Write the final answer in the box.

$f(t) =$

3. Fourier Series (*10 Points*)

Compute the real Fourier series of the function $f(x) = \sin(\frac{5\pi x}{L}) + \cos(\frac{4\pi x}{L}) + |x|$ on the interval $[-L, L]$. Where $|x|$ is the absolute value of x.

$$|x| = \begin{cases} x, & 0 \leq x \leq L, \\ -x, & -L \leq x \leq 0. \end{cases}$$

Write the answer $f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \right)$ **in the box.**

$f(x) =$

4. Wave Equation with D'Alembert solution (*12 Points*)

Let $c > 0$. Consider the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \geq 0, \\ u(x,0) = \frac{1}{c}\left((x^2-2)\sin(x) + 2x\cos(x)\right), & x \in \mathbb{R}, \\ u_t(x,0) = x^2\cos(x), & x \in \mathbb{R}. \end{cases}$$

Find the solution $u(x,t)$. You may use D'Alembert formula.
[Simplify the expression up to the point of solving the integral and rearranging the terms if possible. But you don't need to use trigonometric formula to simplify the cos and sin terms].

5. Heat Equation with inhomogeneous boundary conditions (*15 Points*)

Find the general solution of the Heat equation (**with inhomogeneous boundary conditions**) for the following problem:

$$\begin{cases} u_t = c^2 u_{xx}, & 0 \leq x \leq L, t \geq 0, \\ u(0,t) = 0, & t \geq 0, \\ u(L,t) = L, & t \geq 0, \\ u(x,0) = f(x) + x, & 0 \leq x \leq L, \end{cases} \tag{1}$$

where $L > 0$ is a constant, and $f(x)$ is any (twice differentiable) function such that $f(0) = 0, f(L) = 0$.

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w'' = 0$, $w(0) = 0$, and $w(L) = L$.

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

c) The Fourier series of the $2L$ periodic odd extension of f is given by

$$f(x) := \sum_{n=1}^{+\infty} \frac{(4n+2)}{(n^2 + \pi n - 1)^3} \sin\left(\frac{n\pi}{L} x\right).$$

(i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (1).

6. PDE with Fourier transform (*15 Points*)

Find the solution $u = u(x,t)$ of the following equation using the Fourier transform:

$$\begin{cases} u_x + u_t + u = 0\,, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x)\,, & x \in \mathbb{R}. \end{cases}$$

[*Hint:* *You can proceed as follow:*

a) Take the Fourier transform with respect to the x variable of the PDE and the initial condition and transform them into an ODE.

b) Solve the ODE.

c) Take the inverse Fourier transform of the solution of the ODE to find the solution of the PDE.]