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Analysis III

Exam Solutions

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1. We will use the following properties of the Laplace transform:

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0), \tag{1}$$
$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g), \tag{2}$$
$$\mathcal{L}^{-1}\left(e^{-as}\mathcal{L}(f)\right) = u(t-a)f(t-a). \tag{3}$$

Note that
$$\int_0^t y(\tau)\cos(t-\tau)d\tau = (y * \cos)(t). \tag{4}$$

We transform both sides of the equation

$$\mathcal{L}(y' + y * \cos) = s\mathcal{L}(y) - y(0) + \mathcal{L}(y)\frac{s}{s^2+1} \tag{5}$$
$$= \mathcal{L}(y)\frac{s(s^2+2)}{s^2+1}, \tag{6}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}. \tag{7}$$

This leads to an algebraic equation

$$\mathcal{L}(y) = e^{-as}\frac{s^2+1}{s(s^2+2)}. \tag{8}$$

Using the partial fraction decomposition

$$\frac{s^2+1}{s(s^2+2)} = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+2}\right), \tag{9}$$

and the uniqueness of the Laplace transform, we can invert the transformation and get

$$\mathcal{L}^{-1}\left(\frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+2}\right)\right) = \frac{1}{2}(1+\cos(\sqrt{2}t)), \tag{10}$$

$$y(t) = \mathcal{L}^{-1}\left(e^{-as}\frac{s^2+1}{s(s^2+2)}\right) \tag{11}$$
$$= \frac{1}{2}u(t-a)(1+\cos(\sqrt{2}(t-a))). \tag{12}$$

2. a) Since the function is odd, the Fourier coefficients of the cosine terms in the Fourier expansion of f vanish. The coefficients of the sine terms can be computed as follows. Let $n \geq 1$, then we have with $2L = 2\pi$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx \\ &= \frac{2}{\pi} \left(\int_0^{\frac{\pi}{2}} x \sin(nx)\, dx - \int_{\frac{\pi}{2}}^{\pi} (x - \pi) \sin(nx)\, dx \right). \end{aligned} \tag{13}$$

Integration by parts leads to

$$\begin{aligned} \int_a^b x \sin(nx)\, dx &= -\frac{1}{n} \left(x \cos(nx)|_a^b - \int_a^b \cos(nx)\, dx \right) \\ &= -\frac{1}{n} x \cos(nx)|_a^b + \frac{1}{n^2} \sin(nx)|_a^b. \end{aligned} \tag{14}$$

Consequently, the Fourier coefficients are given by

$$b_n = \frac{4}{n^2 \pi} \sin\left(n \frac{\pi}{2}\right). \tag{15}$$

Finally, the Fourier series reads as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) = \sum_{n=1}^{\infty} \frac{4}{n^2 \pi} \sin\left(n \frac{\pi}{2}\right) \sin(nx). \tag{16}$$

b) Yes, the Fourier series converges pointwise to the function f, since the function f is continuous.

3. The formula for D'Alambert's solution is given by

$$u(x,t) = \frac{1}{2} \left(f(x + ct) + f(x - ct) + \frac{1}{c} \int_{x-ct}^{x+ct} g(y)\, dy \right). \tag{17}$$

a) Setting $(x,t) = \left(0, \frac{\pi}{c}\right)$ gives

$$\begin{aligned} u\left(0, \frac{\pi}{c}\right) &= \frac{1}{2} \left(f(0 + \pi) + f(0 - \pi) + \frac{1}{c} \int_{0-\pi}^{0+\pi} g(y)\, dy \right) \\ &= \frac{1}{2} \left(2(2\pi - \pi) + \frac{1}{c} \int_{-\pi}^{\pi} \cos^2(y)\, dy \right) \\ &= \frac{1}{2} \left(2\pi + \frac{1}{2c} \int_{-\pi}^{\pi} (1 + \cos(2y))\, dy \right) \\ &= \frac{1}{2} \left(2\pi + \frac{1}{2c} (2\pi + 0) \right) \\ &= \pi \left(1 + \frac{1}{2c} \right). \end{aligned} \tag{18}$$

b)

$$
\begin{aligned}
\lim_{a\to\infty} u\left(a, \frac{a}{c}\right) &= \lim_{a\to\infty, 2|a|>2\pi} \frac{1}{2}\left(f(2a) + f(0) + \frac{1}{c}\int_0^{2a} g(y)\,dy\right) \\
&= \lim_{a\to\infty, 2|a|>2\pi} \frac{1}{2}\left(0 + 2\pi + \frac{1}{c}\int_0^{2\pi} \cos^2(y)\,dy + \frac{1}{c}\int_{2\pi}^{2a} \frac{1}{y^2}\,dy\right) \\
&= \frac{1}{2}\left(2\pi + \frac{\pi}{c} + \frac{1}{c}\int_{2\pi}^{\infty} \frac{1}{y^2}\,dy\right) \\
&= \frac{1}{2}\left(2\pi + \frac{\pi}{c} + \frac{1}{2\pi c}\right) \\
&= \pi\left(1 + \frac{1}{2c}\right) + \frac{1}{4\pi c}.
\end{aligned} \tag{19}
$$

Since $a \to \infty$, we assumed without loss of generality that $|2a| > 2\pi$.

4. a) First we determine the stationary solution v, which fulfils the following boundary value problem:

$$
\left\{ \begin{array}{rcl|l} a^2 v_{xx}(x) + b &=& 0 & x \in \mathbb{R} \\ v(0) = v(L) &=& 0 & \end{array} \right. \tag{20}
$$

The unique solution of this problem is a polynom of second order with zeros 0 und L, i.e.

$$
v(x) = -\frac{b}{2a^2}(x - L)x. \tag{21}
$$

b) Set $w(x,t) = u(x,t) - v(x)$. w is the solution of the following homogeneous PDE with boundaries:

$$
\left\{ \begin{array}{rcl|l} w_t &=& u_t = a^2(w_{xx} + v_{xx}) + b = a^2 w_{xx} & x \in \mathbb{R},\ t > 0 \\ w(0,t) = w(L,t) &=& 0 & t \geq 0 \\ w(x,0) &=& \sin\left(\frac{\pi x}{L}\right) - v(x) & x \in \mathbb{R} \end{array} \right. \tag{22}
$$

In order to solve this homogeneous problem we use separation of variables. Inserting the Ansatz $w(x,t) = X(x)T(t)$ in the PDE for w leads to

$$
T'(t)X(x) = a^2 T(t) X''(x) \tag{23}
$$

with the homogeneous boundary conditions

$$
\begin{aligned}
w(0,t) &= T(t)X(0) = 0 \\
w(L,t) &= T(t)X(L) = 0
\end{aligned} \tag{24}
$$

Since we are interested in non trivial solutions we get two differential equations including the homogeneous boundary conditions

$$
T'(t) = a^2 C T(t) \tag{25}
$$

and

$$\left\{ \begin{array}{rl|l} X''(x) &= CX(x) & x \in (0,1) \\ X(0) = X(L) &= 0 & \end{array} \right. \tag{26}$$

for some constant $C \in \mathbb{R}$. First we solve the differential equation for X with homogeneous boundary conditions and distinguish the three cases for C:

- $C > 0$: In this case the general solution for X is given by

$$X(x) = A_1 \sinh(\sqrt{C}x) + A_2 \cosh(\sqrt{C}x). \tag{27}$$

  The boundary condition $X(0) = 0$ demands

$$0 = X(0) = A_1 \sinh(\sqrt{C}0) + A_2 \cosh(\sqrt{C}0) = A_2, \tag{28}$$

  i.e. $A_2 = 0$. The other boundary condition $X(L) = 0$ leads to

$$0 = X(L) = A_1 \sinh(\sqrt{C}L), \tag{29}$$

  which is only possible for $A_1 = 0$. Therefore we get only the trivial solution $X(x) = 0$.
- In the case $C = 0$, we have

$$X''(x) = 0, \tag{30}$$

  which leads to linear solutions

$$X(x) = A_1 x + A_2. \tag{31}$$

  The first boundary condition $X(0) = 0$ demands

$$0 = X(0) = A_2. \tag{32}$$

  Therefore we have $X(x) = A_1 x$. The second boundary condition $X(L) = 0$, i.e.

$$0 = X(L) = A_1 L, \tag{33}$$

  sets $A_1 = 0$ and hence allows again only the trivial solution $X(x) = 0$.
- Therefore we are left with the case $C := -\lambda^2 < 0$ and $X(x)$ has the form

$$X(x) = A_1 \sin(\lambda x) + A_2 \cos(\lambda x). \tag{34}$$

  From the first boundary condition we deduce $0 = X(0) = A_2$ und from the second $X(L) = 0$ follows

$$\lambda = \lambda_n = \frac{n\pi}{L}, \tag{35}$$

  for $n \in \mathbb{N}$. The possible nontrivial solutions for X are therefore given by

$$X_n(x) = \alpha_n \sin\left(\frac{n\pi}{L}x\right). \tag{36}$$

Inserting λ_n in the differential equation for $T(t)$

$$T'(t) = -\left(\frac{n\pi a}{L}\right)^2 T(t) \tag{37}$$

leads to the possible solutions:

$$T_n(t) = e^{-\left(\frac{n\pi a}{L}\right)^2 t}. \tag{38}$$

Using the superposition principle for linear PDEs we deduce that the general solution $w(x,t)$ is of the form

$$w(x,t) = \sum_{n\geq 1} \alpha_n e^{-\left(\frac{n\pi a}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right). \tag{39}$$

The coefficients α_n are determined by the initial condition for w:

$$w(x,0) = \sum_{n\geq 1} \alpha_n \sin\left(\frac{n\pi}{L}x\right) = \sin\left(\frac{\pi x}{L}\right) - v(x). \tag{40}$$

We extend v to an odd $2L$-periodic function $\widetilde{v} : \mathbb{R} \to \mathbb{R}$ and determine its Fourier coefficients:

$$B_n := \frac{1}{L}\int_{-L}^{L} \widetilde{v}(x) \sin\left(\frac{n\pi}{L}x\right) dx. \tag{41}$$

Partial integration (see below) leads to

$$B_n = \frac{4L^2}{(n\pi)^3}\frac{b}{2a^2}(1-(-1)^n). \tag{42}$$

By comparing the coefficients we get

$$\alpha_n = -B_n = \frac{4L^2}{(n\pi)^3}\frac{b}{2a^2}((-1)^n - 1) \tag{43}$$

for $n \neq 1$, and

$$\alpha_1 = 1 - B_1 = 1 - \frac{4L^2}{\pi^3}\frac{b}{a^2}. \tag{44}$$

c) Combining the two steps the solution of the inhomogeneous heat equation is given as

$$\begin{aligned} u(x,t) &= w(x,t) + v(x) \\ &= \left(\left(1 - \frac{4L^2}{(\pi)^3}\frac{b}{a^2}\right) e^{-(\frac{\pi a}{L})^2 t} + \frac{4L^2}{(\pi)^3}\frac{b}{a^2}\right) \sin\left(\frac{\pi}{L}x\right) \\ &\quad + \sum_{n>1} \frac{4L^2}{(n\pi)^3}\frac{b}{2a^2}((-1)^n - 1)\left(e^{-(\frac{n\pi a}{L})^2 t} - 1\right) \sin\left(\frac{n\pi}{L}x\right) \\ &= \left(1 - \frac{4L^2}{(\pi)^3}\frac{b}{a^2}\right) e^{-(\frac{\pi a}{L})^2 t} \sin\left(\frac{\pi}{L}x\right) \\ &\quad + \sum_{n>1} \frac{4L^2}{(n\pi)^3}\frac{b}{2a^2}((-1)^n - 1) e^{-(\frac{n\pi a}{L})^2 t} \sin\left(\frac{n\pi}{L}x\right) + v(x). \end{aligned} \tag{45}$$

Please turn!

Partial integration:
Let $f(x)$ denote the odd $2L$ periodic function with $f(x) = x^2$ for $x \in [0, L]$.

$$\begin{aligned}
\int_{-L}^{L} f(x) \sin(n\pi x/L) dx = 2 \int_0^L x^2 \sin(n\pi x/L) dx \\
&= 2x^2 \frac{-L}{n\pi} \cos(n\pi x/L)\Big|_0^L - 2 \int_{0,L} 2x \frac{-L}{n\pi} \cos(n\pi x/L) dx \\
&= 2 \frac{-L^3}{n\pi}(-1)^n + 4 \int_0^L \frac{-L^2}{(n\pi)^2} \sin(\frac{n\pi x}{L}) dx \\
&= 2 \frac{-L^3}{n\pi}(-1)^n + 4 \frac{L^3}{(n\pi)^3} \cos(n\pi x/L)\Big|_0^L \\
&= 2 \frac{-L^3}{n\pi}(-1)^n + 4 \frac{L^3}{(n\pi)^3}((-1)^n - 1).
\end{aligned} \tag{46}$$

Furthermore, we have

$$\begin{aligned}
\int_{-L}^{L} x \sin(n\pi x/L) dx &= x \frac{-L}{n\pi} \cos(n\pi x/L)|_{-L}^{L} - \int_{-L}^{L} \frac{-L}{n\pi} \sin(n\pi x/L) dx \\
&= \frac{-2L^2}{n\pi}(-1)^n
\end{aligned} \tag{47}$$

Therefore the Fourriercoefficients of the odd $2L$-periodic extension of v are given by

$$\begin{aligned}
\int_{-L}^{L} \widetilde{v}(x) \sin(n\pi x) dx &= -\frac{b}{2a^2}(2\frac{-L^3}{n\pi}(-1)^n + 4\frac{L^3}{(n\pi)^3}((-1)^n - 1) + \frac{2L^3}{n\pi}(-1)^n) \\
&= -\frac{b}{2a^2} 4 \frac{L^3}{(n\pi)^3}((-1)^n - 1).
\end{aligned} \tag{48}$$

5. a) The coefficients are given as $a = -1$, $b = 2$, $c = -1$ und $d = 1$, i.e. for v we get

$$v(x, y) = -x + 2xy - y + 1. \tag{49}$$

b) $w = u - v$ fulfils the following boundary value problem with only one inhomogeneous boundary

$$\begin{cases}
\Delta w = 0 & x \in (0,1) \times (0,1) \\
w(x, 0) = 0 & x \in [0, 1] \\
w(0, y) = 0 & y \in [0, 1] \\
w(1, y) = 0 & y \in [0, 1] \\
w(x, 1) = \sin^3(k\pi x) & x \in [0, 1].
\end{cases} \tag{50}$$

In order to determine the solution of this boundary value problem for w we use the separation of variables Ansatz $w(x, y) = X(x)Y(y)$ and insert it in the PDE

$$X''(x)Y(y) + X(x)Y''(y) = 0. \tag{51}$$

Therefore we have

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = C \tag{52}$$

for some constant $C \in \mathbb{R}$ with the homogeneous boundary conditions

$$\begin{aligned} w(x,0) &= X(x)Y(0) = 0 \\ w(0,y) &= X(0)Y(y) = 0 \\ w(1,y) &= X(1)Y(y) = 0. \end{aligned} \tag{53}$$

Since we are interested in nontrivial solutions, this leads to the following differential equations including the homogeneous boundary conditions:

$$\left\{ \begin{array}{rcl|l} X''(x) &=& CX(x) & x \in \ (0,1) \\ X(0) = X(1) &=& 0 & \end{array} \right. \tag{54}$$

and

$$\left\{ \begin{array}{rcl|l} Y''(y) &=& -CY(y) & y \in \ (0,1) \\ Y(0) &=& 0. & \end{array} \right. \tag{55}$$

We determine the possible solutions for X distinguishing three cases for C:

- $C > 0$: In this case we get

$$X(x) = A_1 \sinh(\sqrt{C}x) + A_2 \cosh(\sqrt{C}x). \tag{56}$$

  From the boundary condition $X(0) = 0$ follows

$$0 = X(0) = A_1 \sinh(\sqrt{C}0) + A_2 \cosh(\sqrt{C}0) = A_2, \tag{57}$$

  i.e. $A_2 = 0$. Furthermore, since $X(1) = 0$, also

$$0 = X(1) = A_1 \sinh(\sqrt{C}1). \tag{58}$$

  and hence $A_1 = 0$. Therefore, in this case we get only the trivial solution $X(x) = 0$.
- In the case $C = 0$ the equation

$$X''(x) = 0 \tag{59}$$

  has linear solutions

$$X(x) = A_1 x + A_2. \tag{60}$$

  From the boundary conditions we deduce

$$0 = X(0) = A_2, \tag{61}$$

  and

$$0 = X(1) = A_1, \tag{62}$$

  which leads again to the trivial solution $X(x) = 0$.

- We are left with the case $C = -\lambda^2 < 0$ and the general solution

$$X(x) = A_1 \sin(\lambda x) + A_2 \cos(\lambda x) \tag{63}$$

as well as

$$Y(y) = B_1 \sinh(\lambda y) + B_2 \cosh(\lambda y). \tag{64}$$

Using the homogeneous boundary conditions we deduce that $X(0) = A_2 = 0$ and $Y(0) = B_2 = 0$. Furthermore since $X(1) = A_1 \sin(\lambda x) = 0$ we need $\lambda = \lambda_n = n\pi$, with $n \in \mathbb{N}$, in order to get non trivial solutions.

Therefore possible nontrivial solutions are given by

$$w(x,y)_n = \sinh(n\pi y) \sin(n\pi x) \tag{65}$$

for $n \in \mathbb{N}$. We still have to fulfil the last inhomogeneous boundary condition for w:

$$w(x,1) = \sin^3(k\pi x) \text{ , for } x \in [0,1] \tag{66}$$

Since the PDE is linear the superposition principle allows us to take a series of possible solutions as Ansatz for the inhomogeneous boundary condition:

$$w(x,y) = \sum_{n \geq 1} \alpha_n \sinh(n\pi y) \sin(n\pi x). \tag{67}$$

The coefficients are then determined by the inhomogeneous boundary condition

$$w(x,1) = \sum_{n \geq 1} \alpha_n \sinh(n\pi) \sin(n\pi x) = \sin^3(k\pi x). \tag{68}$$

Since the Ansatz is an expansion in sine terms, we should extend the boundary function as odd function with period 2 and determine its Fourier series. The coefficients of the superposition Ansatz are then obtained by a comparison of coefficients. But notice that the boundary function is already a sine function and we can use the following formula as given in the hint:

$$\sin^3(k\pi x) = \frac{1}{4}(3\sin(k\pi x) - \sin(3k\pi x)). \tag{69}$$

Hence $\alpha_n = 0$, if $n \notin \{k, 3k\}$. Furthermore, $\alpha_k = \frac{3}{4}\frac{1}{\sinh(k\pi)}$ and $\alpha_{3k} = \frac{-1}{4}\frac{1}{\sinh(3k\pi)}$. Finally, the solution for w is given as

$$w(x,y) = \frac{3}{4}\frac{\sinh(k\pi y)}{\sinh(k\pi)} \sin(k\pi x) - \frac{1}{4}\frac{\sinh(3k\pi y)}{\sinh(3k\pi)} \sin(3k\pi x). \tag{70}$$

c) Combining the two steps leads us to the solution u of the boundary value problem:

$$\begin{aligned} u(x,y) &= v(x,y) + w(x,y) \\ &= -x + 2xy - y + 1 + \frac{3}{4}\frac{\sinh(k\pi y)}{\sinh(k\pi)} \sin(k\pi x) - \frac{1}{4}\frac{\sinh(3k\pi y)}{\sinh(3k\pi)} \sin(3k\pi x). \end{aligned} \tag{71}$$