Problems and suggested solution

Laplace Transforms: ($F = \mathcal{L}(f)$) (u = Heaviside function, δ = Dirac's delta function)

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geq 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)g(t-a)$	$\mathcal{L}(g)e^{-as}$
12)	$\delta(t-a)$	e^{-as}

Fourier transforms:

	$f(x)$	$\widehat{f}(\omega)$
1)	e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{\frac{-\omega^2}{4a}}$

	$f(x)$	$\widehat{f}(\omega)$
2)	$\begin{cases} e^{-ax}, & x \geq 0, \\ 0, & x < 0. \end{cases}$	$\frac{1}{\sqrt{2\pi}(a+i\omega)}$

	$f(x)$	$\widehat{f}(\omega)$
3)	$\begin{cases} 1, & \lvert x\rvert < 1, \\ 0, & \lvert x\rvert > 1. \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(\omega)}{\omega}$

Indefinite Integrals: $\left(n \in \mathbb{Z}_{\geq 1}\right)$

1) $$\int x\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{\cos\left(\frac{n\pi}{L}x\right) + \left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2} \quad (+\text{constant})$$

2) $$\int x^2\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{\left(\left(\frac{n\pi}{L}\right)^2 x^2 - 2\right)\sin\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right)x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3} \quad (+\text{constant})$$

3) $$\int x\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{\sin\left(\frac{n\pi}{L}x\right) - \left(\frac{n\pi}{L}\right)x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2} \quad (+\text{constant})$$

4) $$\int x^2\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{\left(2 - \left(\frac{n\pi}{L}\right)^2 x^2\right)\cos\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3} \quad (+\text{constant})$$

5) $$\int \frac{1}{1+x^2}\,dx = \arctan(x) \quad (+\text{constant})$$

You can use these formulas without justification.

Question 1

1.MC1 [3 Points] Let f be a solution of the following ordinary differential equation (ODE),

$$\begin{cases} f''(t) + \omega^2 f(t) = 0, & t > 0 \\ f(0) = 1, \quad f'(0) = 2\omega, \end{cases}$$

where $\omega > 0$ is a positive constant. Find the Laplace transform $\mathcal{L}(f) = F$ of the function f.

(A) $F(s) = \frac{s}{s^2+\omega^2} + \frac{2\omega}{s^2+\omega^2}$.

(B) $F(s) = \frac{1}{s^2+\omega^2} + \frac{2s\omega}{s^2+\omega^2}$.

(C) $F(s) = \frac{2\omega}{s+\omega^2}$.

(D) $F(s) = \frac{2\omega}{s^2+\omega}$.

Solution:

The solution is (A) $F(s) = \frac{s}{s^2+\omega^2} + \frac{2\omega}{s^2+\omega^2}$.

We apply the Laplace transform to the ODE in the initial value problem. We denote by $F = \mathcal{L}(f)$ the Laplace transform of the function f, and we denote the variable in the new domain by s as usual (so $F = F(s)$).

The first term to transform is the second derivative f'', for which we use the formula:

$$\mathcal{L}(f'') = s^2 F - sf(0) - f'(0) = s^2 F - s - 2\omega\,.$$

Then we have $\mathcal{L}(\omega^2 f) = \omega^2 F$ (by linearity). In conclusion the ODE becomes the following algebraic equation:

$$s^2 F - s - 2\omega + \omega^2 F = 0 \qquad \Longrightarrow \qquad F = \frac{s}{s^2+\omega^2} + \frac{2\omega}{s^2+\omega^2}.$$

1.MC2 [3 Points] Find the inverse Laplace transform of the following function

$$F(s) = \frac{s+2}{s^2 - 10s + 25}.$$

(A) $f(t) = e^{-5t}(1+7t)$.

(B) $f(t) = e^{5t}(1+7t)$.

(C) $f(t) = e^{5t}(1-7t)$.

(D) $f(t) = e^{-5t}(1-7t)$.

Solution:

The solution is (B) $f(t) = e^{5t}(1+7t)$.

We have

$$F(s) = \frac{s+2}{s^2-10s+25} = \frac{s+2}{(s-5)^2} = \frac{(s-5)+7}{(s-5)^2} = \frac{1}{s-5} + \frac{7}{(s-5)^2}.$$

Therefore,

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s-5} + \frac{7}{(s-5)^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-5}\right) + \mathcal{L}^{-1}\left(\frac{7}{(s-5)^2}\right) = e^{5t} + e^{5t}7t = e^{5t}(1+7t).$$

1.MC3 [3 Points] Let f be a continuous function such that $\lim_{x\to\infty} f(x) = 0$. Solve the following differential equation using the Fourier transform

$$f(x) + f'(x) + 4f''(x) = \sqrt{2\pi}e^{-\pi x^2}.$$

(A) $f(x) = \int_{-\infty}^{\infty} \frac{1}{1+i\omega-4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{-i\omega x}\, d\omega.$

(B) $f(x) = \int_{-\infty}^{\infty} \frac{1}{1+i\omega+4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega.$

(C) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+i\omega-4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega.$

(D) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+i\omega+4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{-i\omega x}\, d\omega.$

Solution:

The solution is (C) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+i\omega-4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega.$

We take the Fourier transform on both sides of the equation. (We use the table above and the property of the Fourier transform for a derivative.)

$$\widehat{f}(\omega) + i\omega\widehat{f}(\omega) - 4\omega^2\widehat{f}(\omega) = \sqrt{2\pi}\frac{1}{\sqrt{2\pi}}e^{-\frac{\omega^2}{4\pi}}.$$

Then we solve this algebraic equation

$$\widehat{f}(\omega) = \frac{1}{1+i\omega-4\omega^2}e^{-\frac{\omega^2}{4\pi}}.$$

Finally, we take the inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+i\omega-4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega.$$

1.MC4 [3 Points] The complex Fourier series of the function $\cosh(ax)$ on the interval $[-\pi,\pi)$ is given by

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n a\sinh(a\pi)}{\pi(n^2+a^2)} e^{inx}.$$

Find the value of the numerical series

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2}$$

(A) $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)}$.

(B) $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)} - \frac{1}{2a^2}$.

(C) $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{2\pi}{a\sinh(a\pi)} - \frac{2}{a^2}$.

(D) $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{2\pi}{a\sinh(a\pi)}$.

Solution:

The solution is (B) $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)} - \frac{1}{2a^2}$.

The complex Fourier series will be equal to $\cosh(ax)$ for each $x \in [-\pi,\pi]$, or equivalently:

$$\frac{\pi\cosh(ax)}{a\sinh(a\pi)} = \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n^2+a^2} e^{inx}, \qquad \forall x \in [-\pi,\pi]. \tag{1}$$

In particular for $x=0$ we obtain something very similar to what we need. Observe that the right hand side becomes

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n^2+a^2} \underbrace{e^{in0}}_{=1} = \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \underbrace{\frac{1}{a^2}}_{\substack{n=0\\ \text{term}}} + 2\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} =: \frac{1}{a^2} + 2S,$$

where S is the sum we wanted to compute. Therefore equation (**??**) in $x=0$ becomes:

$$\frac{\pi}{a\sinh(a\pi)} = \frac{1}{a^2} + 2S,$$

from which

$$S = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)} - \frac{1}{2a^2}.$$

1.MC5 [3 Points] Determine if the following function is even, odd, or neither and if it is periodic or not. If the function is periodic, determine the fundamental period.

$$15\cos(3x) + 3\cos(4x).$$

(A) The function is even and periodic of fundamental period 2π.

(B) The function is odd and periodic of fundamental period 2π.

(C) The function is even and periodic of fundamental period π.

(D) The function is odd and periodic of fundamental period π.

Solution:

(A) The function is even and periodic of fundamental period 2π.

First,

$$15\cos(3(-x)) + 3\cos(4(-x)) = 15\cos(-3x) + 3\cos(-4x) = 15\cos(3x) + 3\cos(4x).$$

Therefore, the function is even.

If $f(x)$ is periodic of period P_1 and $g(x)$ is periodic of period P_2, then their sum $f(x) + g(x)$ is periodic of period the least common multiple $P = \mathrm{LCM}(P_1, P_2)$ of the two periods.

In this case, $15\cos(3x)$ is periodic of fundamental period $\frac{2\pi}{3}$ while $3\cos(4x)$ is periodic of fundamental period $\frac{2\pi}{4}$, therefore their sum is periodic of period

$P = \mathrm{LCM}\left(\frac{2\pi}{3}, \frac{\pi}{2}\right) = \mathrm{LCM}\,(4, 3) \cdot \frac{\pi}{6} = \frac{12}{6}\pi = 2\pi$

It is easy to see that no smaller number is a period.

1.MC6 [3 Points] Consider the following PDE (partial differential equation) for the function $u = u(x, y)$:

$$4u_{xx} + xu_x + 6u_{xy} - yu_y = -7u_{yy} + u_x.$$

Is the PDE hyperbolic, parabolic, elliptic or of mixed type ?

(A) hyperbolic.

(B) elliptic.

(C) parabolic.

(D) mixed type.

Solution:

(B) The PDE is elliptic because, $A = 4$, $B = 3$, and $C = 7$ therefore,

$$AC - B^2 = 4 \cdot 7 - 9 = 19 > 0.$$

The sign of the coefficient is positive, therefore the PDE is elliptic.

1.MC7 [3 Points] Wave equation with D'Alembert solution.

Let $u(x,t)$ be the solution of the following problem

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x) = \begin{cases} 2, & |x| \le 14 \\ 0, & |x| > 14 \end{cases} & x \in \mathbb{R}, \\ u_t(x,0) = g(x) = \begin{cases} 1, & 3 \le x \le 4 \\ 0, & x \notin [3,4] \end{cases} & x \in \mathbb{R}. \end{cases}$$

Find the values of u at the point $(x,t) = (10,7)$, i.e. find $u(10,7)$

(A) $u(10,7) = 8$.
(B) $u(10,7) = 10$.
(C) $u(10,7) = \frac{3}{2}$.
(D) $u(10,7) = \frac{5}{2}$.

Solution:

The solution is (C) $u(10,7) = \frac{3}{2}$.

D'Alembert's formula for the solution of the wave equation is:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

With our given initial conditions, $c = 1$, $x = 10$, and $t = 7$, we get

$$\begin{aligned} u(10,7) &= \frac{1}{2}\left(f(17) + f(3)\right) + \frac{1}{2}\int_3^{17} g(s)ds \\ &= \frac{1}{2}(0+2) + \frac{1}{2}\int_3^4 1ds \\ &= 1 + \frac{1}{2} \\ &= \frac{3}{2}. \end{aligned}$$

1.MC8 [3 Points] Let $u = u(x,y)$ be a harmonic function in D_2 the disk of radius 2 centred at 0. The maximum value of u is at $(x,y) = (\sqrt{2}, -\sqrt{2})$, i.e. $\max_{D_2} u(x,y) = u(\sqrt{2}, -\sqrt{2})$.

Which of the following statements is true?

(A) u is not constant in D_2.
(B) u is constant in D_2.
(C) There exists another point (x', y') in D_2 such that $u(x,y) = u(x', y')$.
(D) We cannot conclude that (A), (B) and (C) are true for every u.

Solution:

The solution is (D): We cannot conclude that (A), (B) and (C) are true.

The point $(x,y) = (\sqrt{2}, -\sqrt{2})$ is on the boundary of D_2.

Therefore, the maximum of u is taken on the boundary. Hence, we cannot conclude anything about u.

1.MC9 [3 Points] Consider the Dirichlet problem for the Laplace equation,

$$R = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 2\},$$

$$\begin{cases} \Delta u = 0, & (x,y) \in R, \\ u(0,y) = u(1,y) = 0, & 0 \le y \le 2, \\ u(x,0) = 0, & 0 \le x \le 1, \\ u(x,2) = f(x) & 0 \le x \le 1, \end{cases}$$

where f is a continuous function.

(A) There are infinitely many solutions.

(B) We cannot say anything.

(C) There is a unique solution.

(D) There is no solution.

Solution:

(C) There is a unique solution.

By the lecture notes we know that there exists a solution. We can show that there is a unique solution in the following way:

Let assume that u_1 and u_2 are two solutions of the Laplace equation. And consider $v = u_1 - u_2$. Then by linearity of the Laplace equation we have,

$$\Delta v = \Delta u_1 - \Delta u_2 = 0.$$

And for the boundary conditions we have

$$\begin{aligned} v(0,y) &= u_1(0,y) - u_2(0,y) = 0, && 0 \le y \le 2, \\ v(1,y) &= u_1(1,y) - u_2(1,y) = 0, && 0 \le y \le 2, \\ v(x,0) &= u_1(x,0) - u_2(x,0) = 0, && 0 \le x \le 1, \\ v(x,2) &= u_1(x,2) - u_2(x,2) = f(x) - f(x) = 0 && 0 \le x \le 1. \end{aligned}$$

Therefore, we get for v,

$$\begin{cases} \Delta v = 0, & (x,y) \in R, \\ v_(0,y) = v(1,y) = 0, & 0 \le y \le 2, \\ v(x,0) = 0, & 0 \le x \le 1, \\ v(x,2) = 0 & 0 \le x \le 1. \end{cases}$$

Hence, $v = 0$ and $u_1 = u_2$.

Question 2

2.Q1 [15 Points] Heat Equation with inhomogeneous boundary conditions

Find the general solution of the Heat equation (**with inhomogeneous boundary conditions**) for the following problem:

$$(1) \qquad \begin{cases} u_t = c^2 u_{xx}, & 0 \le x \le \pi, t \ge 0, \\ u(0,t) = 5, & t \ge 0, \\ u(\pi,t) = 8, & t \ge 0, \\ u(x,0) = f(x) + w(x), & 0 \le x \le \pi, \end{cases}$$

where w is the function that you have to find in point a) and f is given by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \in [0,\pi), \\ 0 & \text{if} \quad x = \pi. \end{cases}$$

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w'' = 0$, $w(0) = 5$, and $w(\pi) = 8$.

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

c) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

d) Write down explicitly the solution $u(x,t)$ of the original problem (1).

Solution:

a) The only functions with second derivative zero are the linear functions

$$w(x) = \alpha x + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

Imposing the boundary conditions we find the right coefficients

$$\begin{cases} 5 = w(0) = \alpha \cdot 0 + \beta \\ 8 = w(\pi) = \alpha \cdot \pi + \beta \end{cases} \quad \Leftrightarrow \quad \begin{cases} \alpha = \frac{8-5}{\pi} = \frac{3}{\pi} \\ \beta = 5 \end{cases} \quad \Leftrightarrow \quad w(x) = \frac{3}{\pi}x + 5.$$

b) The PDE doesn't change because w is independent of time and has second derivative in x zero. The boundary conditions become homogeneous (that's why we chose this w)

$$v(0,t) = u(0,t) - w(0) = 5 - 5 = 0 \quad \text{and} \quad v(\pi,t) = u(\pi,t) - w(\pi) = 8 - 8 = 0.$$

The initial condition becomes

$$v(x,0) = u(x,0) - w(x) = f(x) + w(x) - w(x) = f(x).$$

Finally, the boundary value problem for v with homogeneous boundary conditions reads as

$$\begin{cases} v_t = c^2 v_{xx}, & 0 \le x \le \pi,\ t \ge 0, \\ v(0,t) = 0, & t \ge 0, \\ v(\pi,t) = 0, & t \ge 0, \\ v(x,0) = f(x) & 0 \le x \le \pi. \end{cases}$$

c) By the lecture notes the general solution of the heat equation in an interval is given by

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-c^2 n^2 t} \sin(nx).$$

By imposing the initial condition $v(x,0) = f(x)$, we have

$$v(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) = f(x).$$

Hence the A_n are the Fourier coefficients of the extension of f to be an odd periodic function of period 2π, that is

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin(nx)\,dx \\ &= \frac{2}{\pi}\left.\frac{(2-n^2x^2)\cos(nx) + 2nx\sin(nx)}{n^3}\right|_0^{\pi} \\ &= \frac{2}{\pi}\frac{(2-n^2\pi^2)\cos(n\pi)}{n^3} - \frac{2}{\pi}\frac{2}{n^3} \\ &= \frac{2}{\pi}\frac{(2-n^2\pi^2)(-1)^n}{n^3} - \frac{2}{\pi}\frac{2}{n^3} \\ &= \frac{4((-1)^n - 1)}{\pi n^3} - \frac{2\pi(-1)^n}{n}. \end{aligned}$$

Hence,

$$v(x,t) = \sum_{n=1}^{\infty}\left(\frac{4((-1)^n-1)}{\pi n^3} - \frac{2\pi(-1)^n}{n}\right) e^{-c^2n^2t}\sin(nx).$$

d) The solution $u(x,t)$ is given by

$$u(x,t) = v(x,t) + w(x) = \sum_{n=1}^{\infty}\left(\frac{4((-1)^n-1)}{\pi n^3} - \frac{2\pi(-1)^n}{n}\right) e^{-c^2n^2t}\sin(nx) + \frac{3}{\pi}x + 5.$$

Question 3

3.Q1 [10 Points] Dirichlet problem on a region with symmetries

Find the solution $u(r,\theta)$ of the following Dirichlet problem on the disk of radius R in polar coordinates:

$$\begin{cases} \Delta u = 0, & 0 \le r \le R, 0 \le \theta \le 2\pi, \\ u(R,\theta) = \sin^2(\theta) + 8\cos^3(\theta), & 0 \le \theta \le 2\pi. \end{cases}$$

You should give the answer without unsolved integral and you can use the formulas developed in the lecture.

[*Hint:* Don't try to find the solution in the Poisson integral form.]

[*Hint:* Remember the trigonometric formulas

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta) \qquad \text{and} \qquad \cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta).\,]$$

Solution:

The solution will be

$$u(r,\theta) = \sum_{n=0}^{+\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right),$$

with coefficients found imposing

$$u(R,\theta) = \sum_{n=0}^{+\infty} R^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) = \sin^2(\theta) + 8\cos^3(\theta).$$

Using the trigonometric formulas

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta) \qquad \text{and} \qquad 8\cos^3(\theta) = 6\cos(\theta) + 2\cos(3\theta),$$

we have,

$$u(R,\theta) = \sum_{n=0}^{+\infty} R^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) = \frac{1}{2} + 6\cos(\theta) - \frac{1}{2}\cos(2\theta) + 2\cos(3\theta).$$

We obtain coefficients

$$\begin{cases} B_n = 0, & \forall n \geq 0, \\ A_n = 0, & \forall n \geq 0,\ n \neq 0,1,2, \text{ and } 3, \\ A_0 = \frac{1}{2}, & \\ A_1 = \frac{6}{R}, & \\ A_2 = -\frac{1}{2R^2}, & \\ A_3 = \frac{2}{R^3}. & \end{cases}$$

Finally

$$u(r,\theta) = \frac{1}{2} + \frac{6r}{R}\cos(\theta) - \frac{r^2}{2R^2}\cos(2\theta) + \frac{2r^3}{R^3}\cos(3\theta).$$