D–MAVT, D–MATL

Mock Exam Analysis III

401-0363-10L

Last Name **XX** *First Name* **XX** *Legi-Nr.* **XX-000-000** *Exam-No.* ***000***

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Please take note of the information on the answer-booklet.

Laplace Transforms: $\left(F = \mathcal{L}(f) \right)$

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geq 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)g(t-a)$	$\mathcal{L}(g)e^{-as}$
12)	$\delta(t-a)$	e^{-as}

(Γ = Gamma function, u = Heaviside function, δ = Dirac's delta function)

Fourier transforms:

	$f(x)$	$\widehat{f}(\omega)$
1)	e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{\frac{-\omega^2}{4a}}$

	$f(x)$	$\widehat{f}(\omega)$
2)	$\begin{cases} e^{-ax}, & x \geq 0, \\ 0, & x < 0. \end{cases}$	$\frac{1}{\sqrt{2\pi}(a+i\omega)}$

	$f(x)$	$\widehat{f}(\omega)$
3)	$\begin{cases} 1, & \lvert x \rvert < 1, \\ 0, & \lvert x \rvert > 1. \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(\omega)}{\omega}$

Indefinite Integrals: $\left(n \in \mathbb{Z}_{\geq 1} \right)$

1)	$\int x \cos\left(\frac{n\pi}{L} x\right) \, dx = \frac{\cos\left(\frac{n\pi}{L} x\right) + \left(\frac{n\pi}{L}\right) x \sin\left(\frac{n\pi}{L} x\right)}{\left(\frac{n\pi}{L}\right)^2} \quad (+\text{constant})$
2)	$\int x^2 \cos\left(\frac{n\pi}{L} x\right) \, dx = \frac{\left(\left(\frac{n\pi}{L}\right)^2 x^2 - 2\right) \sin\left(\frac{n\pi}{L} x\right) + 2\left(\frac{n\pi}{L}\right) x \cos\left(\frac{n\pi}{L} x\right)}{\left(\frac{n\pi}{L}\right)^3} \quad (+\text{constant})$
3)	$\int x \sin\left(\frac{n\pi}{L} x\right) \, dx = \frac{\sin\left(\frac{n\pi}{L} x\right) - \left(\frac{n\pi}{L}\right) x \cos\left(\frac{n\pi}{L} x\right)}{\left(\frac{n\pi}{L}\right)^2} \quad (+\text{constant})$
4)	$\int x^2 \sin\left(\frac{n\pi}{L} x\right) \, dx = \frac{\left(2 - \left(\frac{n\pi}{L}\right)^2 x^2\right) \cos\left(\frac{n\pi}{L} x\right) + 2\left(\frac{n\pi}{L}\right) x \sin\left(\frac{n\pi}{L} x\right)}{\left(\frac{n\pi}{L}\right)^3} \quad (+\text{constant})$
5)	$\int \frac{1}{1+x^2} \, dx = \arctan(x) \quad (+\text{constant})$

Question 1

1.MC1 [3 Points] Let f be a solution of the following ordinary differential equation (ODE),

$$\begin{cases} f''(t) + \omega^2 f(t) = 0, & t > 0 \\ f(0) = 1, \quad f'(0) = 2\omega, \end{cases}$$

where $\omega > 0$ is a positive constant. Find the Laplace transform $\mathcal{L}(f) = F$ of the function f.

(A) $F(s) = \frac{s}{s^2+\omega^2} + \frac{2\omega}{s^2+\omega^2}$.

(B) $F(s) = \frac{1}{s^2+\omega^2} + \frac{2s\omega}{s^2+\omega^2}$.

(C) $F(s) = \frac{2\omega}{s+\omega^2}$.

(D) $F(s) = \frac{2\omega}{s^2+\omega}$.

1.MC2 [3 Points] Find the inverse Laplace transform of the following function

$$F(s) = \frac{s+2}{s^2 - 10s + 25}.$$

(A) $f(t) = e^{-5t}(1 + 7t)$.

(B) $f(t) = e^{5t}(1 + 7t)$.

(C) $f(t) = e^{5t}(1 - 7t)$.

(D) $f(t) = e^{-5t}(1 - 7t)$.

1.MC3 [3 Points] Let f be a continuous function such that $\lim_{x\to\infty} f(x) = 0$. Solve the following differential equation using the Fourier transform

$$f(x) + f'(x) + 4f''(x) = \sqrt{2\pi}e^{-\pi x^2}.$$

(A) $f(x) = \int_{-\infty}^{\infty} \frac{1}{1+i\omega-4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{-i\omega x}\, d\omega$.

(B) $f(x) = \int_{-\infty}^{\infty} \frac{1}{1+i\omega+4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega$.

(C) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+i\omega-4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega$.

(D) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+i\omega+4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{-i\omega x}\, d\omega$.

1.MC4 [3 Points] The complex Fourier series of the function $\cosh(ax)$ on the interval $[-\pi, \pi)$ is given by

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n a \sinh(a\pi)}{\pi(n^2 + a^2)} e^{inx}.$$

Find the value of the numerical series

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2 + a^2}$$

(A) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)}$.

(B) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)} - \frac{1}{2a^2}$.

(C) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{2\pi}{a\sinh(a\pi)} - \frac{2}{a^2}$.

(D) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{2\pi}{a\sinh(a\pi)}$.

1.MC5 [3 Points] Determine if the following function is even, odd, or neither and if it is periodic or not. If the function is periodic, determine the fundamental period.

$$15\cos(3x) + 3\cos(4x).$$

(A) The function is even and periodic of fundamental period 2π.

(B) The function is odd and periodic of fundamental period 2π.

(C) The function is even and periodic of fundamental period π.

(D) The function is odd and periodic of fundamental period π.

1.MC6 [3 Points] Consider the following PDE (partial differential equation) for the function $u = u(x, y)$:

$$4u_{xx} + xu_x + 6u_{xy} - yu_y = -7u_{yy} + u_x.$$

Is the PDE hyperbolic, parabolic, elliptic or of mixed type ?

(A) hyperbolic.

(B) elliptic.

(C) parabolic.

(D) mixed type.

1.MC7 [3 Points] Wave equation with D'Alembert solution.

Let $u(x,t)$ be the solution of the following problem

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x) = \begin{cases} 2, & |x| \le 14 \\ 0, & |x| > 14 \end{cases} & x \in \mathbb{R}, \\ u_t(x,0) = g(x) = \begin{cases} 1, & 3 \le x \le 4 \\ 0, & x \notin [3,4] \end{cases} & x \in \mathbb{R}. \end{cases}$$

Find the values of u at the point $(x,t) = (10,7)$, i.e. find $u(10,7)$

(A) $u(10,7) = 8$.
(B) $u(10,7) = 10$.
(C) $u(10,7) = \frac{3}{2}$.
(D) $u(10,7) = \frac{5}{2}$.

1.MC8 [3 Points] Let $u = u(x,y)$ be a harmonic function in D_2 the disk of radius 2 centred at 0. The maximum value of u is at $(x,y) = (\sqrt{2}, -\sqrt{2})$, i.e. $\max_{D_2} u(x,y) = u(\sqrt{2}, -\sqrt{2})$.

Which of the following statements is true?

(A) u is not constant in D_2.
(B) u is constant in D_2.
(C) There exists another point (x', y') in D_2 such that $u(x,y) = u(x', y')$.
(D) We cannot conclude that (A), (B) and (C) are true for every u.

1.MC9 [3 Points] Consider the Neumann problem for the following PDE,

$$\begin{cases} \nabla^2 u = f, & \text{in } D_2, \\ \frac{\partial u}{\partial n} = g, & \text{on } \partial D_2, \end{cases}$$

with D_2 the disk of radius 2 centred at 0 and f and g are two given functions such that

$$\int_{D_2} f(x)\ dx = 2, \qquad \text{and} \qquad \int_{\partial D_2} g(x)\ dx = 2.$$

Which of the following is true:

(A) There are infinitely many solutions.
(B) There is no solution.
(C) There are two solutions.
(D) We cannot conclude that (A), (B), or (C) are true.

Question 2

2.Q1 [15 Points] Separation of variables for the Heat equation

Consider the following time-dependent version of the Heat equation on the interval $[0,1]$. We also impose boundary conditions and we look for a solution $u = u(x,t)$ such that:

$$\begin{cases} u_t(x,t) = a(t)u_{xx}(x,t), & x \in [0,1], t \in [0,+\infty), \\ u(0,t) = 0, & t \in [0,+\infty), \\ u(1,t) = 0, & t \in [0,+\infty), \\ u(x,0) = 2\sin(3\pi x) + \sin(7\pi x), & x \in [0,1], \end{cases}$$

where $a(t)$ is a given continuous function. Find the solution $u(x,t)$ using separation of variable. Proceed as in the lecture and adapt the steps if necessary.

Question 3

3.Q1 [10 Points] Dirichlet problem on a region with symmetries

Find the solution $u(r,\theta)$ of the following Dirichlet problem on the disk of radius R in polar coordinates:

$$\begin{cases} \nabla^2 u = 0, & 0 \leq r \leq R, 0 \leq \theta \leq 2\pi, \\ u(R,\theta) = \sin^2(\theta) + 8\cos^3(\theta), & 0 \leq \theta \leq 2\pi. \end{cases}$$

You should give the answer without unsolved integral and you can use the formulas developed in the lecture.

[*Hint:* Don't try to find the solution in the Poisson integral form.]

[*Hint:* Remember the trigonometric formulas

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta) \qquad \text{and} \qquad \cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta).\,]$$