D-MAVT
D-MATL

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Analysis III
Exam Solutions

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Exercise	1	2	3	4	5	Total
Value	8	8	11	14	8	49

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1. Periodicity (*8 Points*)

Determine which of the following functions is periodic and which is not. For the periodic ones, determine their fundamental period[1]. For the non-periodic ones, explain/prove why they are not periodic.
[*Hint:* *Recall that every periodic, continuous function is bounded, and that every periodic, differentiable functions has periodic derivative.*]

a) (*2 Points*) $|\sin(x+2)|$

Solution:

It is periodic of fundamental period π.
Explanation: $\sin(x)$ is periodic of period 2π, therefore also $|\sin(x)|$ is. However the fundamental period of $|\sin(x)|$ is π, in fact $|\sin(x+\pi)| = |-\sin(x)| = |\sin(x)|$, and no smaller number of π is a period. Finally, we use the general fact that if a function $f(x)$ is periodic, then also the translated function $f(x+\alpha)$ (for any $\alpha \in \mathbb{R}$) is periodic with the same fundamental period. Therefore $|\sin(x+2)|$ is periodic of fundamental period π.

b) (*2 Points*) $4\cosh(x)$

Solution:

It is not periodic.
Explanation: The function is continuous and not bounded, in fact:

$$\lim_{|x|\to+\infty} \cosh(x) = +\infty\,,$$

therefore it is not periodic.

c) (*2 Points*) $\cos(x^3)$

Solution:

It is not periodic.
Explanation: The function is differentiable and its derivative is $-3x^2\sin(x^3)$. If $\cos(x^3)$ was periodic, then also its derivative would be. However, the derivative is not periodic because it is continuous and not bounded (for example, along the sequence $x_n = \sqrt[3]{2n\pi + \pi/2}$ it assumes increasingly bigger values with limit $+\infty$).

[1] A periodic function of period $P > 0$ is a function f such that $f(x+P) = f(x)$ for all $x \in \mathbb{R}$. The *fundamental period* of a periodic function is the smallest period P.

d) (*2 Points*) $\cos(15x) + 3\sin(6x)$

Solution:

It is periodic of fundamental period $2\pi/3$.
Explanation: If $f(x)$ is periodic of period P_1 and $g(x)$ is periodic of period P_2, then their sum $f(x)+g(x)$ is periodic of period the least common multiple

$$P = \mathrm{LCM}(P_1, P_2)$$

of the two periods[2]. In this case $\cos(15x)$ is periodic of fundamental period $2\pi/15$ while $3\sin(6x)$ is periodic of fundamental period $\pi/3$, therefore their sum is periodic of period

$$P = \mathrm{LCM}\left(\frac{2\pi}{15}, \frac{\pi}{3}\right) = \mathrm{LCM}\,(2,5) \cdot \frac{\pi}{15} = \frac{10}{15}\pi = \frac{2}{3}\pi\,.$$

It is easy to see that no smaller number is a period.

2. Laplace Transform (*8 Points*)

Find the solution $y : [0, +\infty) \to \mathbb{R}$ of the following integral equation with initial condition:

$$\begin{cases} \int_0^t y'(\tau)(t^2 - 2t\tau + \tau^2)\, d\tau = t^3 \\ y(0) = 1 \end{cases}$$

using the Laplace transform.
[*Hint:* *Recognize* $t^2 - 2t\tau + \tau^2 = g(t-\tau)$ *for some function* g*, and the integral as the convolution of* y' *and* $g \ldots$]

Solution:

We recognize that $t^2 - 2t\tau + \tau^2 = (t-\tau)^2$, so that the left hand-side of the integral equation is just the convolution of $y'(t)$ and t^2. With this is mind, we apply the Laplace transform to the equation:

$$\begin{aligned} & \mathcal{L}\left(y' * t^2\right) = \frac{3!}{s^4} \\ \Leftrightarrow\quad & \mathcal{L}(y') \cdot \mathcal{L}(t^2) = \frac{6}{s^4} \\ \Leftrightarrow\quad & (sY - y(0)) \cdot \frac{2}{s^3} = \frac{6}{s^4} \\ \Leftrightarrow\quad & sY = y(0) + \frac{3}{s} \\ \overset{(y(0)=1)}{\Leftrightarrow}\quad & Y = \frac{1}{s} + \frac{3}{s^2} \\ \Leftrightarrow\quad & y(t) = 1 + 3t\,. \end{aligned}$$

[2]By the *least common muliple* of two real numbers we mean the smallest number P such that there are positive *integer* numbers k_1, k_2 such that $P = k_1 P_1 = k_2 P_2$. In the case that there is no such number, we define it to be $+\infty$ and the consequence is that the function is not periodic.

3. Dirichlet problem (*11 Points*)

a) (*7 Points*) Consider the Dirichlet problem for the wave equation on the interval $[0, L]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}\,, & x \in [0, L], t > 0 \\ u(0, t) = u(L, t) = 0\,, & t > 0 \end{cases} \tag{1}$$

Use the method of sepation of variables, showing all the steps, until you find the general solution in Fourier series:

$$u(x, t) = \sum_{n=1}^{+\infty} \left(B_n \cos\left(\frac{cn\pi}{L} t\right) + B_n^* \sin\left(\frac{cn\pi}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

Solution:

We find solutions with separated variables $u(x, t) = F(x)G(t)$, for which the wave equation becomes:

$$\underbrace{F\ddot{G}}_{=u_{tt}} = c^2 \underbrace{F''G}_{=u_{xx}} \quad \Leftrightarrow \quad \begin{cases} F'' = kF\,, \\ \ddot{G} = kc^2 G \end{cases}$$

for some constant $k \in \mathbb{R}$. The Dirichlet conditions $u(0, t) = u(L, t) = 0$ force $F(0) = F(L) = 0$ in order to have possibly nontrivial solutions (otherwise $G(t) \equiv 0$ and also $u \equiv 0$). Therefore we have the initial value problem

$$\begin{cases} F'' = kF\,, \\ F(0) = F(L) = 0\,, \end{cases}$$

with different solutions depending on the sign of the constant k. For $k = 0$ the equation $F'' = 0$ has general solution $F(x) = ax + b$ which satisfies the boundary condition only if $F \equiv 0$, so there is no nontrivial solution. For $k > 0$ the equation $F'' = kF$ has general solution $F(x) = ae^{\sqrt{k}x} + be^{-\sqrt{k}x}$. $F(0) = 0$ implies $a = -b$, and now $F(L) = 0$ forces $a = 0$ because $e^{\sqrt{k}L} - e^{-\sqrt{k}L} = (e^{2\sqrt{k}L} - 1)e^{-\sqrt{k}L}$ is never zero, so again there is no nontrivial solution. Finally for $k < 0$ the equation has general solution $F(x) = a\cos(\sqrt{-k}x) + b\sin(\sqrt{-k}x)$. $F(0) = 0$ if and only if $a = 0$, so $F(x) = b\sin(\sqrt{-k}x)$ and now $F(L) = 0$ forces, in order to have nontrivial solution, $\sqrt{-k} = n\pi/L$ for some $n \geqslant 1$ integer number. For each of this values of $k = -n^2\pi^2/L^2$, we call the corresponding solution by $F_n(x)$.

For these values of k we solve the equation $\ddot{G} = kc^2 G$ and we obtain (we call the solution G_n):

$$G_n(t) = \alpha_n \cos\left(\frac{cn\pi}{L} t\right) + \beta_n \sin\left(\frac{cn\pi}{L} t\right).$$

To summarise, renaming the constants, for any integer $n \geqslant 1$ we have a solution

$$u_n(x, t) = F_n(x)G_n(t) = \left(B_n \cos\left(\frac{cn\pi}{L} t\right) + B_n^* \sin\left(\frac{cn\pi}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right),$$

and by the superposition principle, general solution:

$$u(x, t) = \sum_{n=1}^{+\infty} u_n(x, t)\,.$$

b) (*4 Points*) Find the coefficients B_n, B_n^* for the problem (1) with the following initial conditions:

$$\begin{cases} u(x,0) = 0, & x \in [0,L] \\ u_t(x,0) = g(x), & x \in [0,L] \end{cases}$$

where $g(x) = \sum_{n=1}^{100} \frac{n^2}{1+n^2} \sin\left(\frac{n\pi}{L}x\right)$.

Solution:

$u(x,0) = \sum_{n=1}^{+\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = 0$ if and only if $B_n \equiv 0$. On the other hand we have to compare:

$$\sum_{n=1}^{100} \frac{n^2}{1+n^2} \sin\left(\frac{n\pi}{L}x\right) = g(x) = u_t(x,0) = \sum_{n=1}^{+\infty} B_n^* \frac{cn\pi}{L} \sin\left(\frac{n\pi}{L}x\right).$$

By uniqueness of the Fourier series we only need to compare the coefficients, and we obtain:

$$B_n^* = \begin{cases} \frac{L}{c\pi}\frac{n}{1+n^2}, & n = 1,2,\dots,100 \\ 0. & n > 100 \end{cases}$$

For completeness we plug in these values and write down the solution:

$$u(x,t) = \frac{L}{c\pi} \sum_{n=1}^{100} \frac{n}{1+n^2} \sin\left(\frac{cn\pi}{L}t\right) \sin\left(\frac{n\pi}{L}x\right).$$

4. Wave Equation (*14 Points*)

Consider the following 1-dimensional wave equation on the interval $[0,L]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in [0,L],\ t \geqslant 0 \\ u(0,t) = u(L,t) = 0, & t \geqslant 0 \\ u(x,0) = -x^2 + Lx, & x \in [0,L] \\ u_t(x,0) = 0, & x \in [0,L] \end{cases}$$

a) (*6 Points*) Find the solution in Fourier series. You can use the formula from the lecture notes.

Solution:

The formula for the solution in Fourier series is

$$u(x,t) = \sum_{n=1}^{+\infty} \left(B_n \cos\left(\frac{cn\pi}{L}t\right) + B_n^* \sin\left(\frac{cn\pi}{L}t\right)\right) \sin\left(\frac{n\pi}{L}x\right),$$

However B_n^* are zero because the initial datum $u_t(x,0)$ is zero. The coefficients B_n are given by:

$$
\begin{aligned}
B_n =& \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx = \qquad\qquad\qquad (f(x) = -x^2 + Lx)\\
=& -\frac{2}{L}\int_0^L x^2\sin\left(\frac{n\pi}{L}x\right)\,dx + 2\int_0^L x\sin\left(\frac{n\pi}{L}x\right)\,dx =\\
=& -2\frac{L^2}{\pi^3}\int_0^\pi x^2\sin(nx)\,dx + 2\frac{L^2}{\pi^2}\int_0^\pi x\sin(nx)\,dx\\
=& -2\frac{L^2}{\pi^3}\left(\frac{(2-n^2x^2)\cos(nx)+2nx\sin(nx)}{n^3}\right)\Bigg|_0^\pi + 2\frac{L^2}{\pi^2}\left(\frac{\sin(nx)-nx\cos(nx)}{n^2}\right)\Bigg|_0^\pi\\
=& -2\frac{L^2}{\pi^3}\left(\frac{(2-\pi^2n^2)(-1)^n-2}{n^3}\right) + 2\frac{L^2}{\pi^2}\left(\frac{\pi(-1)^{n+1}}{n}\right) =\\
=& \frac{4L^2}{\pi^3n^3}(1-(-1)^n) =\\
=& \begin{cases} \frac{8L^2}{\pi^3n^3}, & n\,\text{odd}\\ 0, & n\,\text{even}\end{cases}
\end{aligned}
$$

For completeness we plug in the values for B_n in the formula of the solution:

$$
u(x,t) = \frac{8L^2}{\pi^3}\sum_{m=0}^{+\infty}\frac{1}{(2m+1)^3}\cos\left(\frac{c\pi}{L}(2m+1)t\right)\sin\left(\frac{\pi}{L}(2m+1)x\right)
$$

b) (*4 Points*) Remember that the solution can also be written as

$$
u(x,t) = \frac{1}{2}\left(f^*(x-ct)+f^*(x+ct)\right),
$$

where f^* is the odd, 2L-periodic extension of the initial datum $f = u(\cdot,0)$. Use this formula to compute

$$
u\left(\frac{L}{2},\frac{2L}{c}\right) = \,?
$$

Solution:

Let us call $(x^*,t^*) = (L/2, 2L/c)$. We need to evaluate the odd, 2L-periodic extension f^* of the initial datum f in the points $x^* \pm ct^* = L/2 \pm 2L$. Using the 2L-periodicity we have that in both cases:

$$
f^*(x^* \pm ct^*) = f^*(L/2 \pm 2L) = f(L/2)\,.
$$

Now to evaluate the latter we can use formula $f(x) = -x^2 + Lx$ because the point $L/2$ is in $[0,L]$, the original interval of definition of the function. It follows that

$$
u(x^*,t^*) = \frac{1}{2}(f^*(x^*-ct^*)+f^*(x^*+ct^*)) = \frac{1}{2}(f(L/2)+f(L/2)) = f(L/2) = -\frac{L^2}{4}+\frac{L^2}{2} = \frac{L^2}{4}
$$

c) (*4 Points*) Compare the result from **b)** with the formula from **a)** evaluated in the point $(x,t) = (L/2, 2L/c)$ to find the value of the following numerical series:

$$\sum_{m=0}^{+\infty} \frac{(-1)^m}{(2m+1)^3} = \ ?$$

Solution:

For $t = t^*$ the cosine terms in the solution are identically 1:

$$\cos\left(\frac{c\pi}{L}(2m+1)t^*\right) = \cos(2\pi(2m+1)) \equiv 1\,,$$

while the sine terms, for $x = x^*$ are:

$$\sin\left(\frac{\pi}{L}(2m+1)x^*\right) = \sin\left(\frac{\pi}{2}(2m+1)\right) = \sin\left(m\pi + \frac{\pi}{2}\right) = (-1)^m\,.$$

Using both formulas in **a)** and **b)** we obtain:

$$\frac{L^2}{4} \overset{\textbf{b)}}{=} u(x^*, t^*) \overset{\textbf{a)}}{=} \frac{8L^2}{\pi^3}\sum_{m=0}^{+\infty} \frac{(-1)^m}{(2m+1)^3}$$

$$\Longrightarrow \quad \sum_{m=0}^{+\infty} \frac{(-1)^m}{(2m+1)^3} = \frac{\pi^3}{32}\,.$$

5. Fourier Integral (*8 Points*)

Let

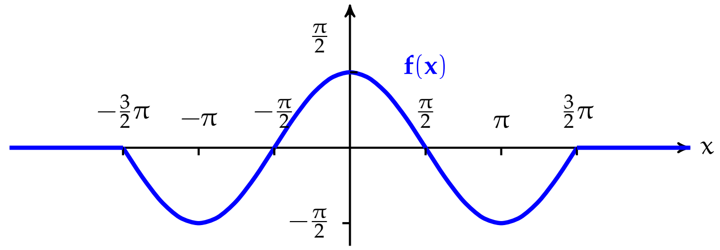
$$f(x) = \begin{cases} \frac{\pi}{2}\cos(x)\,, & x \in \left[-\frac{3}{2}\pi, \frac{3}{2}\pi\right] \\ 0\,. & \text{otherwise} \end{cases}$$

Sketch the graph of this function, and prove that for every $x \in \mathbb{R}$

$$\int_0^{+\infty} \frac{\cos\left(\frac{3}{2}\pi\omega\right)\cos(\omega x)}{\omega^2 - 1}\, d\omega = f(x).$$

Solution:

The graph of this function is



The function is continuous, this is clear both graphically and from the fact that it is defined by cases using continuous functions that have the same values on the overlaps. Therefore the function is equal to its Fourier integral on each $x \in \mathbb{R}$. In other words to prove that

$$\int_0^{+\infty} \frac{\cos\left(\frac{3}{2}\omega\pi\right)\cos(\omega x)}{\omega^2 - 1}\, d\omega = f(x)$$

we just need to show that the left-hand side is the Fourier integral of $f(x)$.
The function is even, therefore $B(\omega) = 0$, while

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\cos(\omega v)\, dv = \frac{2}{\pi}\int_0^{+\infty} f(v)\cos(\omega v)\, dv = \frac{\cancel{2}}{\cancel{\pi}}\int_0^{\frac{3}{2}\pi} \frac{\cancel{\pi}}{\cancel{2}}\cos(v)\cos(\omega v)\, dv =$$

$$= \frac{1}{2}\int_0^{\frac{3}{2}\pi} \left(\cos((1-\omega)v) + \cos((1+\omega)v)\right)\, dv = \frac{1}{2}\cdot\left(\frac{\sin((1-\omega)v)}{1-\omega} + \frac{\sin((1+\omega)v)}{1+\omega}\right)\Bigg|_0^{\frac{3}{2}\pi} =$$

$$= \frac{1}{2}\left(\frac{\sin\left(\frac{3}{2}\pi - \frac{3}{2}\omega\pi\right)}{1-\omega} + \frac{\sin\left(\frac{3}{2}\pi + \frac{3}{2}\omega\pi\right)}{1+\omega}\right) \overset{(*)}{=} \frac{1}{2}\left(\frac{\sin\left(\frac{3}{2}\pi\right)\cos(-\frac{3}{2}\omega\pi)}{1-\omega} + \frac{\sin\left(\frac{3}{2}\pi\right)\cos(\frac{3}{2}\omega\pi)}{1+\omega}\right) =$$

$$= \frac{1}{2}\left(\frac{-\cos\left(-\frac{3}{2}\omega\pi\right)}{1-\omega} - \frac{\cos\left(\frac{3}{2}\omega\pi\right)}{1+\omega}\right) = \frac{1}{2}\left(-\frac{\cos\left(\frac{3}{2}\omega\pi\right)}{1-\omega} - \frac{\cos\left(\frac{3}{2}\omega\pi\right)}{1+\omega}\right) =$$

$$= -\frac{1}{2}\cos\left(\frac{3}{2}\omega\pi\right)\cdot\left(\frac{1}{1-\omega} + \frac{1}{1+\omega}\right) = -\frac{1}{\cancel{2}}\cos\left(\frac{3}{2}\pi\omega\right)\cdot\frac{\cancel{2}}{1-\omega^2} = \frac{\cos\left(\frac{3}{2}\omega\pi\right)}{\omega^2 - 1}.$$

which completes the proof (in the equal $\overset{(*)}{=}$ we used the addition formula for the sine $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ together with the fact that $\cos\left(\pm\frac{3}{2}\pi\right) = 0$).