ETH Zürich
Prof. Dr. F. Da Lio

Summer 2021
25 August

Exam Analysis III
D–MAVT, D–MATL

Exam Nr.:	

Please do not fill!

Exercise	Value	Points	Control
1	8		
2	8		
3	11		
4	14		
5	8		
Total	49		

Please do not fill!

Completeness	

Before the exam:

- Turn off your mobile phone and place it inside your briefcase/backpack.
- Put your bags on the floor. No bags on the desk!
- Place your Student Card (Legi) on the desk.

During the exam, please:

- Start every exercise on a new piece of paper.
- Put your **exam number** on the top right corner of every page.
- Motivate your answers. Write down calculations and intermediate results.
- Provide at most **one** solution to each exercise.
- **Do not** write with pencils. Please avoid using **red** or **green** ink pens.

After the exam:

- Make sure that every solutions sheet has your exam number on it.
- Place back the exam sheets, together with your solutions, in the envelope.

Allowed aids:

- 20 pages (= 10 sheets) DIN A4 handwritten or typed personal summary.
- An English (or English-German) dictionary.
- **No** further aids are allowed. In particular neither communication devices, nor pocket calculators.

Good Luck!

Laplace Transforms: $(F = \mathcal{L}(f))$

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	t^n, $n \in \mathbb{Z}_{\geqslant 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	t^a, $a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)$	$\frac{1}{s}e^{-as}$
12)	$\delta(t-a)$	e^{-as}

(Γ = Gamma function, u = Heaviside function, δ = Delta function)

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1. Periodicity (*8 Points*)

Determine which of the following functions is periodic and which is not. For the periodic ones, determine their fundamental period[1]. For the non-periodic ones, explain/prove why they are not periodic.

a) (*2 Points*) $|\sin(x+2)|$

b) (*2 Points*) $4\cosh(x)$

c) (*2 Points*) $\cos(x^3)$

d) (*2 Points*) $\cos(15x) + 3\sin(6x)$

[*Hint:* *Recall that every periodic, continuous function is bounded, and that every periodic, differentiable functions has periodic derivative.*]

[1] A periodic function of period $P > 0$ is a function f such that $f(x+P) = f(x)$ for all $x \in \mathbb{R}$. The *fundamental period* of a periodic function is the smallest period P.

2. Laplace Transform (*8 Points*)

Find the solution $y : [0, +\infty) \to \mathbb{R}$ of the following integral equation with initial condition:

$$\begin{cases} \displaystyle\int_0^t y'(\tau)(t^2 - 2t\tau + \tau^2)\, d\tau = t^3 \\ y(0) = 1 \end{cases}$$

using the Laplace transform.

[<u>*Hint 1:*</u> *Recognize* $t^2 - 2t\tau + \tau^2 = g(t-\tau)$ *for some function* g.
<u>*Hint 2:*</u> *The integral is then the convolution of* y' *and* g.]

3. Dirichlet problem (*11 Points*)

a) (*7 Points*) Consider the Dirichlet problem for the wave equation on the interval $[0, L]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}\,, & x \in [0, L],\ t \geqslant 0 \\ u(0,t) = u(L,t) = 0\,, & t \geqslant 0 \end{cases} \tag{1}$$

Use the method of sepation of variables, showing all the steps, until you find the general solution in Fourier series:

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos\left(\frac{cn\pi}{L} t\right) + B_n^* \sin\left(\frac{cn\pi}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

b) (*4 Points*) Find the coefficients B_n, B_n^* for the problem (1) with the following initial conditions:

$$\begin{cases} u(x,0) = 0, & x \in [0, L] \\ u_t(x,0) = g(x), & x \in [0, L] \end{cases}$$

where $g(x) = \sum\limits_{n=1}^{100} \frac{n^2}{1+n^2} \sin\left(\frac{n\pi}{L} x\right)$.

4. Wave Equation (*14 Points*)

Consider the following 1-dimensional wave equation on the interval $[0, L]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}\,, & x \in [0, L],\ t \geqslant 0 \\ u(0,t) = u(L,t) = 0\,, & t \geqslant 0 \\ u(x,0) = -x^2 + Lx\,, & x \in [0, L] \\ u_t(x,0) = 0\,, & x \in [0, L] \end{cases}$$

a) (*6 Points*) Find the solution in Fourier series. You can use the formula from the lecture notes.

b) (*4 Points*) Remember that the solution can also be written as

$$u(x,t) = \frac{1}{2}\big(f^*(x - ct) + f^*(x + ct)\big)\,,$$

where f^* is the odd, 2L-periodic extension of the initial datum $f = u(\cdot, 0)$. Use this formula to compute

$$u\left(\frac{L}{2}, \frac{2L}{c}\right) = \ ?$$

c) (*4 Points*) Compare the result from **b)** with the formula from **a)** evaluated in the point $(x,t) = (L/2, 2L/c)$ to find the value of the following numerical series:

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3} = \ ?$$

5. Fourier Integral (*8 Points*)

Let

$$f(x) = \begin{cases} \frac{\pi}{2}\cos(x)\,, & x \in \left[-\frac{3}{2}\pi, \frac{3}{2}\pi\right] \\ 0\,. & \text{otherwise} \end{cases}$$

Sketch the graph of this function, and prove that for every $x \in \mathbb{R}$

$$f(x) = \int_0^{\infty} \frac{\cos\left(\frac{3}{2}\pi\omega\right)\cos(\omega x)}{\omega^2 - 1}\, d\omega\,.$$