ETH Zürich Autumn 2021
Prof. Dr. A. Iozzi

Mock Exam Analysis III
D–MAVT, D–MATL

Surname:	
First Name:	
Student Card Nr.:	
Exam Nr.:	

This page contains the generalities of the student: surname, first name, student card (Legi) number and exam number. The exam number is a number that identifies uniquely the student.

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Signature

ETH Zürich
Prof. Dr. A. Iozzi

Autumn 2021

Mock Exam Analysis III
D–MAVT, D–MATL

Exam Nr.:	

Please do not fill!

Exercise	Value	Points	Control
1			
2			
3			
4			
5			
6			
Total			

Please do not fill!

Completeness	

Please turn!

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Before the exam:

- Turn off your mobile phone and place it inside your briefcase/backpack.
- Put your bags on the floor. No bags on the desk!
- Place your Student Card (Legi) on the desk.

During the exam, please:

- Start every exercise on a new piece of paper.
- Put your **exam number** on the top right corner of every page.
- Motivate your answers. Write down calculations and intermediate results.
- Provide at most **one** solution to each exercise.
- **Do not** write with pencils. Please avoid using **red** or **green** ink pens.

After the exam:

- Make sure that every solutions sheet has your exam number on it.
- Place back the exam sheets, together with your solutions, in the envelope.

Allowed aids:

- 20 pages (= 10 sheets) DIN A4 handwritten or typed personal summary.
- An English (or English-German) dictionary.
- **No** further aids are allowed. In particular neither communication devices, nor pocket calculators.

Good Luck!

Laplace Transforms: $\left(F = \mathcal{L}(f) \right)$

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geq 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)$	$\frac{1}{s}e^{-as}$
12)	$\delta(t-a)$	e^{-as}

(Γ = Gamma function, u = Heaviside function, δ = Delta function)

Indefinite Integrals: $\left(n \in \mathbb{Z}_{\geq 1} \right)$

1)	$\displaystyle\int x\cos(nx)\,dx = \frac{\cos(nx) + nx\sin(nx)}{n^2}$ (+ constant)
2)	$\displaystyle\int x^2\cos(nx)\,dx = \frac{(n^2x^2-2)\sin(nx) + 2nx\cos(nx)}{n^3}$ (+ constant)
3)	$\displaystyle\int x\sin(nx)\,dx = \frac{\sin(nx) - nx\cos(nx)}{n^2}$ (+ constant)
4)	$\displaystyle\int x^2\sin(nx)\,dx = \frac{(2-n^2x^2)\cos(nx) + 2nx\sin(nx)}{n^3}$ (+ constant)
5)	$\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x)$ (+ constant)

You can use these formulas without justification.

1. Classification of PDEs

Consider the following PDEs (in what follows, $u = u(x, y)$ is a function of two variables x and y). Classify each of them: hyperbolic, parabolic, elliptic, mixed type. **(Write the answer in the box)**

a) $u_{xx} + u_{yy} + k^2 u = 0$, where $k > 0$ is a positive constant.

b) $yu_{xx} + 2x^{\frac{3}{2}} u_{xy} + u_{yy} = u_x + u_y + u$.

c) $u_{xx} + 2\cos(x) u_{xy} + y u_{yy} = e^{xy}$.

Periodicity

Determine which of the following functions is periodic and which is not. For the periodic ones, determine their fundamental period[1]. **(Write the answer in the box)**

d) $4\cosh(x)$

e) $\cos(x^3)$

[1] A periodic function of period $P > 0$ is a function f such that $f(x + P) = f(x)$ for all $x \in \mathbb{R}$. The *fundamental period* of a periodic function is the smallest period P.

See the next page!

f) $\cos(15x) + 3\sin(6x)$

[*Hint:* *Recall that every periodic, continuous function is bounded, and that every periodic, differentiable functions has periodic derivative.*]

2. Laplace Transform

Find the solution $f(t)$ of the following initial value problem:

$$\begin{cases} f''(t) + \omega^2 f(t) = \omega\,\delta(t-a), & t > 0 \\ f(0) = 1, \quad f'(0) = \omega, \end{cases}$$

where $\omega, a > 0$ are positive constants.

Write the final answer in the box.

$f(t) =$

3. Fourier Integral

Compute the Fourier integral of the function $f(x) = \mathrm{e}^{-\pi|x|}$.

Write the final answer in the box.

$f(x) =$

4. Wave Equation with D'Alembert solution

Let $c > 0$. Consider the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = e^{-x^2}\sin^2(x) + x, & x \in \mathbb{R} \\ u_t(x,0) = xe^{-x^2}. & x \in \mathbb{R} \end{cases}$$

a) Find the solution $u(x,t)$. You may use D'Alembert formula.
[Simplify the expression as much as possible: no unsolved integrals].

b) For a fixed $a \in \mathbb{R}$, determine the asymptotic limit

$$\lim_{t\to+\infty} u(a,t).$$

5. Wave Equation with inhomogeneous boundary conditions

Find the solution of the following wave equation (**with inhomogeneous boundary conditions**) on the interval $[0,\pi]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t \geq 0,\ x \in [0,\pi], \\ u(0,t) = a, & t \geq 0, \\ u(\pi,t) = b, & t \geq 0, \\ u(x,0) = \frac{b-a}{\pi}x + a, & x \in [0,\pi], \\ u_t(x,0) = x, & x \in [0,\pi], \end{cases} \tag{1}$$

where $a, b > 0$ are positive constants. You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w''(x) = 0$, $w(0) = a$, and $w(\pi) = b$.

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

c) (i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.
(ii) Write down explicitly the solution $u(x,t)$ of the original problem (1).

See the next page!

6. Separation of variable

Consider the following time-dependent version of the heat equation on the interval $[0, L]$. We also impose boundary conditions and we look for a solution $u = u(x, t)$ such that:

$$\begin{cases} u_t = t^3 u_{xx}, & x \in [0, L], t \in (0, +\infty), \\ u(0, t) = 0, & t \in [0, +\infty), \\ u(L, t) = 0, & t \in [0, +\infty), \\ u(x, 0) = \sin(\frac{3\pi x}{L}) + 2\sin(\frac{\pi x}{L}) & x \in [0, L]. \end{cases}$$

Find the solution $u(x, t)$ using separation of variable. Proceed as in the lecture and adapt the steps if necessary.

You have more exercises below.

The exam will have 6 exercises as above. Here you can find some additional exercises for you personal training.

7. Fourier Series

Compute the complex Fourier series of the function $f(x) = 5e^{i\frac{4\pi}{L}x} + x$ on the interval $[-L, L]$.

Write the final answer in the box.

$f(x) =$

8. Fourier transform

Compute the Fourier transform of the function $f(x) = e^{-ax}u(x-b)$, where $a, b > 0$ are positive constants and u is the Heaviside function.

Write the final answer in the box.

$\mathcal{F}(f)(w) =$

9. Laplace Equation on a rectangle

Find the solution of the following Laplace equation on the rectangle

$$R = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x \le 1,\, 0 \le y \le 1\}$$

$$\begin{cases} u_{xx} + u_{yy} = 0\,, & (x, y) \in R \\ u(x, 0) = u(x, 1) = 0\,, & 0 \le x \le 1 \\ u(0, y) = 0\,, & 0 \le y \le 1 \\ u(1, y) = \sin(\pi(1 - y))\,. & 0 \le y \le 1 \end{cases}$$

You can manipulate appropriately any formula that can be useful from the lecture notes (or, alternatively, solve it via separation of variables from scratch).

10. Heat Equation with inhomogeneous boundary conditions

Consider the following problem:

$$\begin{cases} u_t = c^2 u_{xx}, & x \in [0,\pi],\ t \geq 0 \\ u(0,t) = 2, & t \geq 0 \\ u(\pi,t) = 3, & t \geq 0 \\ u(x,0) = f(x), & x \in [0,\pi] \end{cases} \tag{2}$$

where

$$f(x) = \sin(x) - 3\sin(3x) + \frac{x}{\pi} + 2.$$

The boundary conditions are not homogeneous, therefore one cannot directly apply the formulas known. You should argue as follows:

a) Construct a function $w(x)$ with $w(0) = 2$, $w(\pi) = 3$ and $w'' = 0$.

b) Let u be a solution of the above problem (2). State the corresponding problem solved by the function $v(x,t) := u(x,t) - w(x)$.

c) Solve the problem for v using the formula of the lecture notes or using the method of separation of variables from scratch.

d) Find the solution u of the original problem (2) .

11. Laplace equation in an unbounded region

Find the general solution for the following problem:

$$\begin{cases} u_{xx} + u_{yy} = 0, & -\infty \leq x \leq \infty, 0 \leq y, \\ u(x,0) = f(x), & -\infty \leq x \leq \infty, \end{cases} \tag{3}$$

where $f(x)$ is any arbitrary function.

You must proceed as follows.

a) Show that you can transform the system (3) into

$$\begin{cases} -w^2\widehat{u}(w,y) + \frac{\partial^2}{\partial y^2}\widehat{u}(w,y) = 0, \\ \widehat{u}(w,0) = \widehat{f}(w). \end{cases} \tag{4}$$

Where $\widehat{u}(w,y)$ denotes the Fourier transform of $u(x,y)$ with respect to the x variable. That is:

$$\widehat{u}(w,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{-iwx}\, dx.$$

b) Show that $\widehat{u}(w,y) = \widehat{f}(w)e^{-|w|y}$ is a solution of the system (4).
(Where $|w|$ is the absolute value of w).

c) Find the solution of the system (3). [Simplify the expression as much as possible: **no more w in your final answer**. Use the properties of the Fourier transform].

[*Hint:* $\mathcal{F}^{-1}(e^{-|w|y}) = \frac{1}{\sqrt{2\pi}}\frac{2y}{y^2+x^2}$.]

[*Hint:* $\widehat{h}(w)\widehat{g}(w) = \frac{1}{\sqrt{2\pi}}\widehat{(h*g)}(w)$.]

12. Wave Equation

Consider the following 1-dimensional wave equation on the interval $[0, L]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}\,, & x \in [0,L],\ t \geq 0 \\ u(0,t) = u(L,t) = 0\,, & t \geq 0 \\ u(x,0) = 0\,, & 0 \leq x \leq L \\ u_t(x,0) = x\,, & 0 \leq x \leq L \end{cases}$$

a) Find the solution in Fourier series. You can use the formula from the lecture notes.

b) Remember that the solution can also be written as

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds\,,$$

where g^* is the odd, $2L$-periodic extension of the velocity initial datum $g = u_t(\cdot, 0)$. Use this formula to compute

$$u\left(\frac{L}{2}, \frac{3L}{2c}\right) = \ ?$$

c) Compare the result from **b)** with the formula from **a)** evaluated in the point $(x,t) = (L/2, 3L/2c)$ to find the value of the following numerical series:

$$\sum_{m=0}^{+\infty} \frac{1}{(2m+1)^2} = \ ?$$