Problems and suggested solution

Laplace Transforms: ($F = \mathcal{L}(f)$) (u = Heaviside function, δ = Dirac's delta function)

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geq 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)g(t-a)$	$\mathcal{L}(g)e^{-as}$
12)	$\delta(t-a)$	e^{-as}

Fourier transforms:

	$f(x)$	$\widehat{f}(\omega)$
1)	e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{\frac{-\omega^2}{4a}}$

	$f(x)$	$\widehat{f}(\omega)$
2)	$\begin{cases} e^{-ax}, & x \geq 0, \\ 0, & x < 0. \end{cases}$	$\frac{1}{\sqrt{2\pi}(a+i\omega)}$

	$f(x)$	$\widehat{f}(\omega)$
3)	$\begin{cases} 1, & \lvert x\rvert < 1, \\ 0, & \lvert x\rvert > 1. \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(\omega)}{\omega}$

Indefinite Integrals: $\left(n \in \mathbb{Z}_{\geq 1}\right)$

1)	$\displaystyle\int x\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{\cos\left(\frac{n\pi}{L}x\right) + \left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}$ (+constant)
2)	$\displaystyle\int x^2\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{\left(\left(\frac{n\pi}{L}\right)^2x^2-2\right)\sin\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right)x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3}$ (+constant)
3)	$\displaystyle\int x\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{\sin\left(\frac{n\pi}{L}x\right) - \left(\frac{n\pi}{L}\right)x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}$ (+constant)
4)	$\displaystyle\int x^2\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{\left(2-\left(\frac{n\pi}{L}\right)^2x^2\right)\cos\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3}$ (+constant)
5)	$\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x)$ (+constant)

You can use these formulas without justification.

Question 1

1.MC1 [3 Points] Let f be a solution of the following ordinary differential equation (ODE),

$$\begin{cases} \frac{d^4 f(t)}{dt^4} = u(t-1), \\ f(0) = f'(0) = 0, \\ f''(0) = 1, \\ f'''(0) = 0, \end{cases}$$

where u is the Heaviside function. Find the Laplace transform $\mathcal{L}(f) = F$ of the function f.

(A) $F(s) = \frac{1}{s^4} + \frac{e^{-s}}{s^5}$.

(B) $F(s) = \frac{1}{s^3} + \frac{e^{-s}}{s^5}$.

(C) $F(s) = \frac{1}{s^3} + \frac{e^{-s}}{s^3}$.

(D) $F(s) = \frac{1}{s^2} + \frac{e^{-s}}{s^4}$.

Solution:

The solution is $F(s) = \frac{1}{s^3} + \frac{e^{-s}}{s^5}$.
We denote by $F(s)$ the Laplace transform of $f(t)$, and use the formula

$$\mathcal{L}\left(\frac{d^4 f}{dt^4}\right)(s) = s^4 F(s) - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)$$

which is simplified in our case because $f(0) = f'(0) = f'''(0) = 0$. We then get from the Laplace transform of both terms of the differential equation

$$s^4 F(s) - s = \frac{e^{-s}}{s} \quad \Leftrightarrow \quad F(s) = \frac{1}{s^3} + \frac{e^{-s}}{s^5}.$$

1.MC2 [3 Points] Find the inverse Laplace transform of

$$F(s) = \frac{s+4}{s^2-16} + \frac{1}{s^2+9}.$$

(A) $f(t) = e^{-4t} - \frac{1}{3}\sin(3t)$.

(B) $f(t) = e^{4t} + \frac{1}{3}\cos(3t)$.

(C) $f(t) = e^{4t} + \frac{1}{3}\sin(3t)$.

(D) $f(t) = e^{4t} + \frac{1}{3}\sin(9t)$.

Solution:

The solution is $f(t) = e^{4t} + \frac{1}{3}\sin(3t)$.

For the first term we have,

$$\frac{s+4}{s^2-16} = \frac{1}{s-4} \implies \mathcal{L}^{-1}\left(\frac{s+4}{s^2-16}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-4}\right) = e^{4t}.$$

And for the second term we have

$$\frac{1}{s^2+9} = \frac{1}{3}\frac{3}{s^2+3^2} \implies \mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{3}{s^2+3^2}\right) = \frac{1}{3}\sin(3t).$$

Hence, the solution is given by

$$f(t) = \mathcal{L}^{-1}\left(\frac{s+4}{s^2-16} + \frac{1}{s^2+9}\right) = e^{4t} + \frac{1}{3}\sin(3t).$$

1.MC3 [3 Points] Solve the following integral equation using the Fourier transform

$$\int_{-\infty}^{\infty} \frac{df}{dx}(x-y)g(y)\ dy = e^{-4x^2}.$$

where

$$g(y) = \begin{cases} 1, & |y| < 1, \\ 0, & |y| > 1. \end{cases}$$

(A) $f(x) = \frac{1}{8i\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\frac{-\omega^2}{8}} \frac{1}{\sin(\omega)} e^{i\omega x}\ d\omega$.

(B) $f(x) = \frac{1}{8i} \int_{-\infty}^{\infty} e^{\frac{-\omega^2}{8}} \sin(\omega) e^{i\omega x}\ d\omega$.

(C) $f(x) = \frac{1}{8i\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\frac{-\omega^2}{16}} \frac{1}{\sin(\omega)} e^{i\omega x}\ d\omega$.

(D) $f(x) = \frac{1}{8i} \int_{-\infty}^{\infty} e^{\frac{-\omega^2}{16}} \frac{\omega}{\sin(\omega)} e^{i\omega x}\ d\omega$.

Solution:

The solution is $f(x) = \frac{1}{8i\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\frac{-\omega^2}{16}} \frac{1}{\sin(\omega)} e^{i\omega x}\ d\omega$.

We take the Fourier transform on both sides of the equation. (We use the property of the convolution and derivative.)

$$i\omega\sqrt{2\pi}\widehat{f}(\omega)\widehat{g}(\omega) = \frac{1}{\sqrt{8}} e^{\frac{-\omega^2}{16}}.$$

We use the Fourier transform of g given in the table.

$$
\begin{aligned}
& i\omega\sqrt{2\pi}\widehat{f}(\omega)\sqrt{\frac{2}{\pi}}\frac{\sin(\omega)}{\omega} = \frac{1}{\sqrt{8}}e^{\frac{-\omega^2}{16}}. \\
& \Longleftrightarrow \\
& \widehat{f}(\omega) = \frac{1}{2i\sqrt{8}}e^{\frac{-\omega^2}{16}}\frac{1}{\sin(\omega)}.
\end{aligned}
$$

Finally, we take the inverse Fourier transform

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{2i\sqrt{8}}e^{\frac{-\omega^2}{16}}\frac{1}{\sin(\omega)}e^{i\omega x}\ d\omega \\
&= \frac{1}{8i\sqrt{\pi}}\int_{-\infty}^{\infty}e^{\frac{-\omega^2}{16}}\frac{1}{\sin(\omega)}e^{i\omega x}\ d\omega.
\end{aligned}
$$

1.MC4 [**1 Point**] Determine if the following function is even, odd, or neither and if it is periodic or not.

$$\cos(2\pi x) + 3x^4$$

(A) The function is odd and not periodic.

(B) The function is even and periodic.

(C) The function is even and not periodic.

(D) The function is odd and periodic.

Solution:

The function is even and not periodic.

Not periodic because of the $3x^4$ and even because

$$\cos(-2\pi x) + 3(-x)^4 = \cos(2\pi x) + 3x^4.$$

1.MC5 [**3 Points**] Let f be a 2π periodic continuous function such that $f(0) = \frac{1}{\pi^2}$ and its Fourier series on the interval $[-\pi, \pi]$ is given by

$$f(x) = \frac{1}{2\pi^2} + \sum_{n=1}^{\infty} \frac{45}{\pi^6 n^4} \cos(nx).$$

Find the value of the numerical series

$$\sum_{n=1}^{\infty} \frac{3}{\pi^2 n^4}.$$

(A) $\sum_{n=1}^{\infty} \frac{3}{\pi^2 n^4} = \frac{\pi^2}{30}$.

(B) $\sum_{n=1}^{\infty} \frac{3}{\pi^2 n^4} = \frac{\pi^4}{90}$.

(C) $\sum_{n=1}^{\infty} \frac{3}{\pi^2 n^4} = \frac{\pi^4}{30}$.

(D) $\sum_{n=1}^{\infty} \frac{3}{\pi^2 n^4} = \frac{\pi^2}{10}$.

Solution:

The solution is $\sum_{n=1}^{\infty} \frac{3}{\pi^2 n^4} = \frac{\pi^2}{30}$.

We know that $f(0) = \frac{1}{\pi^2}$ therefore

$$\frac{1}{\pi^2} = f(0) = \frac{1}{2\pi^2} + \sum_{n=1}^{\infty} \frac{45}{\pi^6 n^4} \cos(n0) = \frac{1}{2\pi^2} + \frac{15}{\pi^4} \sum_{n=1}^{\infty} \frac{3}{\pi^2 n^4}.$$

Hence,

$$\frac{15}{\pi^4} \sum_{n=1}^{\infty} \frac{3}{\pi^2 n^4} = \frac{1}{\pi^2} - \frac{1}{2\pi^2} \iff \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{30}.$$

1.MC6 [3 Points] Consider the following PDE (partial differential equation) for the function $u = u(t, x, y)$:

$$u_t u_{xyy} + 4u_{xx} = 4u_{yy} - 3x^2 + y.$$

Is this PDE linear ? Homogeneous? And what is the order of the PDE ?

(A) It is a non linear and homogeneous third order PDE.

(B) It is a linear and homogeneous fourth order PDE.

(C) It is a non linear and non homogeneous fourth order PDE.

(D) It is a non linear and non homogeneous third order PDE.

Solution:

It is a non linear and non homogeneous third order PDE.

It's nonlinear because there is the multiplication, $u_t u_{xyy}$. It is non homogeneous because of the term $-3x^2 + y$. And finally, the highest derivative is 3 in the term, u_{xyy}.

1.MC7 [3 Points] Wave equation with D'Alembert solution.

Consider the following wave equation with $c = 2\pi$,

$$\begin{cases} u_{tt} = 4\pi^2 u_{xx}, & x \in \mathbb{R},\, t \geq 0, \\ u(x,0) = \sin(x), & x \in \mathbb{R}, \\ u_t(x,0) = x\cos(x), & x \in \mathbb{R}. \end{cases}$$

Find the solution at time $t = 1$, i.e. $u(x,1)$.

(A) $u(x,1) = \sin(x) - \cos(x)$.
(B) $u(x,1) = 2\sin(x)$.
(C) $u(x,1) = \frac{1}{2}\left(\sin(x+2\pi) + \sin(x-2\pi)\right)$.
(D) $u(x,1) = \sin(x) - x^2\cos(x)$.

Solution:

The solution is $u(x,1) = 2\sin(x)$.

D'Alembert's formula for the solution of the wave equation is:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

With our given initial conditions, $c = 2\pi$ and $t = 1$, we get

$$\begin{aligned} u(x,1) &= \frac{1}{2}\left(\sin(x+2\pi) + \sin(x-2\pi)\right) + \frac{1}{4\pi}\int_{x-2\pi}^{x+2\pi} s\cos(s)ds \\ &= \frac{1}{2}\left(\sin(x) + \sin(x)\right) + \frac{1}{4\pi}(\cos(s) + s\sin(s))\Bigg|_{x-2\pi}^{x+2\pi} \\ &= \sin(x) + \frac{1}{4\pi}(\cos(x+2\pi) + (x+2\pi)\sin(x+2\pi)) \\ &\quad - \frac{1}{4\pi}(\cos(x-2\pi) + (x-2\pi)\sin(x-2\pi)) \\ &= \sin(x) + \frac{1}{4\pi}(\cos(x) + (x+2\pi)\sin(x)) \\ &\quad - \frac{1}{4\pi}(\cos(x) + (x-2\pi)\sin(x)) \\ &= \sin(x) + \sin(x) \\ &= 2\sin(x). \end{aligned}$$

(We used that $\sin(x \pm 2\pi) = \sin(x)$ and $\cos(x \pm 2\pi) = \cos(x)$.) Hence, the solution is

$$u(x,1) = 2\sin(x).$$

1.MC8 [3 Points] Let $u = u(x,y)$ be a harmonic function in D_4 the disk of radius 4 centred at 0. (We denote by ∂D_4 the boundary of the disk.)

The maximum value of u is at $(x,y) = (0,0)$, i.e. $\max_{D_4} u(x,y) = u(0,0)$.

Which of the following statements is true?

(A) u is not constant in D_4.

(B) $u(0,0) < u(x,y)\ \forall (x,y) \in \partial D_4$.

(C) $u(0,0) > u(x,y)\ \forall (x,y) \in D_4$.

(D) We have $u(1,0) = u(0,1) = u(-1,0) = u(0,-1)$.

Solution:

The solution is: We have $u(1,0) = u(0,1) = u(-1,0) = u(0,-1)$.

u is harmonic in D_4 and u takes its maximum values on the interior of the disk D_4. Therefore by the maximum principle Theorem (see page 78 in the Lecture Notes), u is constant in D_4. If u is constant then $u(1,0) = u(0,1) = u(-1,0) = u(0,-1)$ because all these points are in the disk D_4.

1.MC9 [3 Points] Consider the Neumann problem for the following PDE,

$$\begin{cases} \nabla^2 u = f, & \text{in } D_2, \\ \frac{\partial u}{\partial n} = g, & \text{on } \partial D_2, \end{cases}$$

with D_2 the disk of radius 2 centred at 0 and f and g are two given functions such that

$$\int_{D_2} f(x)\ dx = 2, \qquad \text{and} \qquad \int_{\partial D_2} g(x)\ dx = 2.$$

Which of the following is true:

(A) There are infinitely many solutions.

(B) There is no solution.

(C) There are two solutions.

(D) We cannot conclude that (A), (B), or (C) are true.

Solution:

We cannot conclude that (A), (B), or (C) are true.

Indeed, let's assume that $u = u(x)$ is a solution of the PDE. (With $x \in D_1$.) Then we integrate the PDE on D_2 and use the divergence Theorem.

$$\begin{aligned} \int_{D_2} \nabla^2 u(x)\ dx = \int_{D_2} f(x)\ dx &\iff \int_{D_2} \operatorname{div}(\nabla u(x))\ dx = \int_{D_2} f(x)\ dx \\ \iff \int_{\partial D_2} \nabla u(x) \cdot n\ dx = \int_{D_2} f(x)\ dx &\iff \int_{\partial D_2} \frac{\partial u}{\partial n}\ dx = \int_{D_2} f(x)\ dx \\ \iff \int_{\partial D_2} g(x)\ dx = \int_{D_2} f(x)\ dx &\iff 2 = 2. \end{aligned}$$

Therefore, we can not conclude anything.

Question 2

2.Q1 [15 Points] Separation of variables for the Heat equation

Consider the following time-dependent version of the Heat equation on the interval $[0, \pi]$. We also impose boundary conditions and we look for a solution $u = u(x, t)$ such that:

$$\begin{cases} u_t = (t + t^3)u_{xx}, & x \in [0, \pi], t \in [0, +\infty), \\ u(0, t) = 0, & t \in [0, +\infty), \\ u(\pi, t) = 0, & t \in [0, +\infty), \\ u(x, 0) = f(x), & x \in [0, \pi], \end{cases}$$

where f is given by

$$f(x) = \begin{cases} x & \text{if} \quad x \in [0, \pi), \\ 0 & \text{if} \quad x = \pi. \end{cases}$$

Find the solution $u(x, t)$ using separation of variable. Proceed as in the lecture and adapt the steps if necessary.

Solution:

We use separation of variable $u(x, t) = F(x)G(t)$. The differential equation becomes:

$$F(x)\dot{G}(t) = (t + t^3)F''(x)G(t),$$

which is convenient to rewrite as

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{(t + t^3)G(t)}$$

because it becomes clear that we are comparing a function of x with a function of t, and the only way that this equality might be true is that both these functions are equal and constant:

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{(t + t^3)G(t)} = k, \qquad k \in \mathbb{R}.$$

The boundary conditions are

$$u(0, t) = F(0)G(t) = 0 \quad \text{and} \quad u(\pi, t) = F(\pi)G(t) = 0 \quad \forall t \in [0, +\infty)$$

which in order to be true, excluding the trivial solution $G(t) \equiv 0$, become:

$$F(0) = F(\pi) = 0.$$

In other words the initial PDE with boundary conditions becomes the system of coupled equations

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(\pi) = 0, \end{cases} \qquad \text{and} \quad \dot{G}(t) = k(t + t^3)G(t).$$

We first solve the system for $F(x)$, distinguishing the cases of k positive, zero, or negative. For $k > 0$ the general solution of the ODE is

$$F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x},$$

which is, however, not compatible with the boundary conditions, in the sense that the only solution of this form satisfying the boundary conditions is the trivial solution: $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_1 + C_2 \quad \Leftrightarrow \quad C_2 = -C_1 \quad \implies F(x) = C_1 \left(e^{\sqrt{k}x} - e^{-\sqrt{k}x} \right)$$

but then imposing the other condition:

$$0 = F(\pi) = C_1 \left(e^{\sqrt{k}\pi} - e^{-\sqrt{k}\pi} \right) \quad \Leftrightarrow \quad \begin{array}{l} \text{either } C_1 = 0 \\ \text{or } e^{2\sqrt{k}\pi} = 1 \end{array}$$

which implies $C_1 = 0$ (and consequently $C_2 = -C_1 = 0$) because $2\sqrt{k}\pi \neq 0$ and therefore its exponential is not 1.

For $k = 0$ the general solution is $F(x) = C_1 x + C_2$ which is also not compatible with boundary conditions unless $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_2 \quad \implies F(x) = C_1 x$$

and then

$$0 = F(\pi) = C_1 \pi \quad \Leftrightarrow \quad C_1 = 0.$$

It remains the case $k < 0$, in which its convenient to write it in the form $k = -p^2$ for positive real number p, and general solutions of $F'' = -p^2 F$ are:

$$F(x) = A\cos(px) + B\sin(px).$$

We impose the boundary conditions:

$$0 = F(0) = A \quad \implies F(x) = B\sin(px)$$

and

$$0 = F(\pi) = B\sin(p\pi) \quad \overset{(\text{if } B \neq 0)}{\Leftrightarrow} \quad p\pi = n\pi, \quad n \in \mathbb{Z}_{\geq 1}$$

Conclusion: we have a non-trivial solution for each $n \geq 1$, $k = k_n = -n^2$:

$$F_n(x) = B_n \sin(nx).$$

The corresponding equation for $G(t)$ is

$$\dot{G} = -(t + t^3) n^2 G$$

which has general solution

$$G_n(t) = C_n e^{-n^2 (\frac{t^2}{2} + \frac{t^4}{4})}.$$

The conclusion is that for every $n \geq 1$ we have a solution

$$u_n(x,t) = F_n(x)G_n(t) = A_n e^{-n^2(\frac{t^2}{2}+\frac{t^4}{4})} \sin(nx), \qquad \text{with} \quad A_n = B_n C_n.$$

Then by the Superposition Principle, the function

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2(\frac{t^2}{2}+\frac{t^4}{4})} \sin(nx)$$

is also a solution. By imposing the initial condition $u(x,0) = f(x)$, we have

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) = f(x).$$

Therefore,

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\ dx = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\ dx \\ &= \frac{2}{\pi}\frac{\sin(nx) - nx\cos(nx)}{n^2}\bigg|_0^{\pi} = \frac{2}{\pi}\frac{-n\pi(-1)^n}{n^2} \\ &= \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

Hence the final solution is given by,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} e^{-n^2(\frac{t^2}{2}+\frac{t^4}{4})} \sin(nx).$$

Question 3

3.Q1 [10 Points] Wave equation

Find the solution $u = u(x,t)$ of the 1-dimensional wave equation on the interval $[0, L]$ with the constant $c > 0$ and the following boundary and initial conditions:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 \leq x \leq L, t \geq 0, \\ u(0,t) = 0 = u(L,t), & t \geq 0, \\ u(x,0) = 4\sin\left(\frac{5\pi}{L}x\right), & 0 \leq x \leq L, \\ u_t(x,0) = \sin\left(\frac{2\pi}{L}x\right), & 0 \leq x \leq L. \end{cases}$$

You can use the general formula directly to obtain the solution. For this exercise, no points will be given for detailing all the steps of the separation of variable .

Solution:

The formula for the solution via Fourier series is:

$$u(x,t) = \sum_{n=1}^{+\infty} \left(B_n \cos\left(\frac{cn\pi}{L}t\right) + B_n^* \sin\left(\frac{cn\pi}{L}t\right)\right) \sin(\frac{n\pi}{L}x).$$

To find the coefficients B_n we impose the initial condition on $u(x,0)$:

$$u(x,0) = \sum_{n=1}^{+\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = 4\sin\left(\frac{5\pi}{L}x\right) \quad \implies \quad \begin{cases} B_n = 4 & \text{if } n = 5, \\ B_n = 0 & \text{otherwise.} \end{cases}$$

To find the coefficients B_n^* we impose the initial condition on $u_t(x,0)$:

$$u_t(x,0) = \sum_{n=1}^{+\infty} B_n^* \frac{cn\pi}{L} \sin\left(\frac{n\pi}{L}x\right) = \sin\left(\frac{2\pi}{L}x\right) \quad \implies \quad \begin{cases} B_n^* = \frac{L}{2c\pi} & \text{if } n = 2, \\ B_n^* = 0 & \text{otherwise.} \end{cases}$$

Therefore the final solution is given by

$$u(x,t) = \frac{L}{2c\pi} \sin\left(\frac{2c\pi}{L}t\right) \sin\left(\frac{2\pi}{L}x\right) + 4\cos\left(\frac{5c\pi}{L}t\right) \sin\left(\frac{5\pi}{L}x\right).$$