D-MAVT
D-MATL

Prof. Dr. F. Da Lio
ETH Zürich
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Analysis III
Exam Solutions

(e-mail to stefano.dalesio@math.ethz.ch)

Exercise	1	2	3	4	5	Total
Value	8	8	10	12	13	51

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1. Classification of PDEs (*8 Points*)

Consider the following second order PDEs (in what follows, $u = u(x, y)$ is a function of two variables x and y):

a) $\frac{1}{c^2}u_{xx} - u_{yy} + \frac{m^2c^2}{\hbar^2}u = 0$, where $m, c, \hbar > 0$ are positive constants.

b) $u_{xx} + u_{yy} + k^2u = 0$, where $k > 0$ is a positive constant.

c) $u_{xx} + xu_{yy} = 0$.

d) $yu_{xx} + 2x^{\frac{3}{2}}u_{xy} + u_{yy} = u_x + u_y + u$.

e) $u_{xx} + 2\cos(x)u_{xy} + yu_{yy} = e^{xy}$.

Classify each of them: hyperbolic, parabolic, elliptic, mixed type?
In the case of a mixed type equation, draw the regions in the plane $(x, y) \in \mathbb{R}^2$ in which it is hyperbolic, parabolic, elliptic.

Solution:

A general second order, linear, PDE has the form:

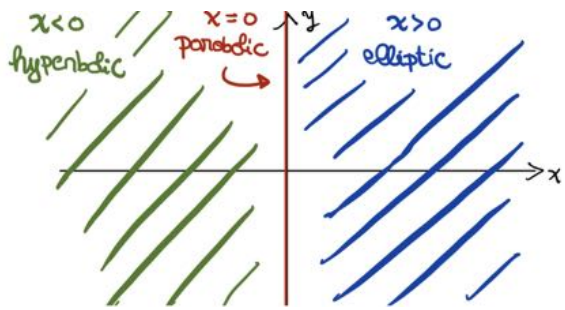
$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y),$$

where A, B, C can be themselves functions of the variables (x, y). The PDE is called hyperbolic, parabolic or elliptic, if the coefficient $AC - B^2$ is, respectively, smaller, equal or greater than zero. When the sign of the coefficient is not constant the equation is of mixed type, and we can determine in which regions of the plane it is hyperbolic, parabolic or elliptic.

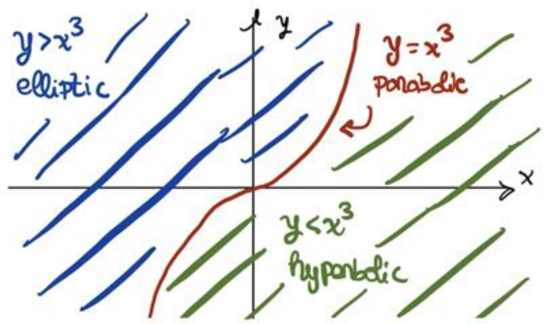
a) $AC - B^2 = -\frac{1}{c^2} < 0 \quad \Longrightarrow \quad$ hyperbolic

b) $AC - B^2 = 1 > 0 \quad \Longrightarrow \quad$ elliptic

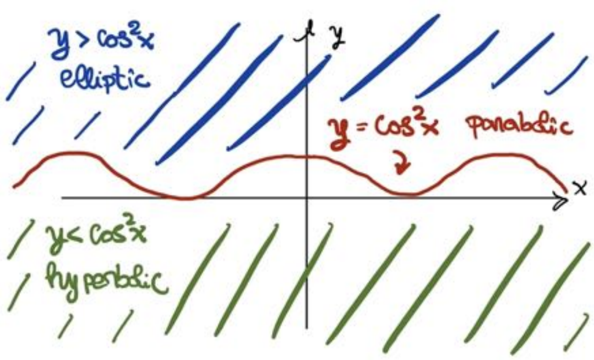
c) $AC - B^2 = x$ which changes sign, so the PDE is of mixed type. The regions in the plane in which the PDE is hyperbolic, parabolic and elliptic, are:



d) $AC - B^2 = y - x^3$ which changes sign, so the PDE is of mixed type, with the following regions:



e) $AC - B^2 = y - \cos^2(x)$ which also changes sign, so the PDE is of mixed type, with the following regions:



2. Laplace Transform (*8 Points*)

Find the solution $f = f(t)$ of the following initial value problem:

$$\begin{cases} f''(t) + \omega^2 f(t) = \omega\, \delta(t-a), & t > 0 \\ f(0) = 1, \quad f'(0) = \omega, \end{cases}$$

where $\omega, a > 0$ are positive constants.

Solution:

We apply the Laplace transform to the ODE in the initial value problem. We denote by $F = \mathcal{L}(f)$ the Laplace transform of the function f, and we denote the variable in the new domain by s as usual (so $F = F(s)$).

The first term to transform is the second derivative f'', for which we use the formula:

$$\mathcal{L}(f'') = s^2F - sf(0) - f'(0) = s^2F - s - \omega\,.$$

Then we have $\mathcal{L}(\omega^2 f) = \omega^2 F$ (by linearity) and finally the term in the right-hand side becomes:

$$\mathcal{L}(\omega\, \delta(t-a)) = \omega \mathcal{L}(\delta(t-a)) = \omega e^{-as}\,.$$

In conclusion the ODE becomes the following algebraic equation:

$$s^2F - s - \omega + \omega^2 F = \omega e^{-as} \qquad \Longrightarrow \qquad F = \frac{s}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2} + e^{-as} \cdot \frac{\omega}{s^2+\omega^2}$$

We recognise the first and second term as Laplace transforms of cosine and sine, respectively, while for the third term we can use the t-shifting property, and obtain, applying the inverse Laplace transform:

$$\boxed{f = f(t) = \mathcal{L}^{-1}(F) = \cos(\omega t) + \sin(\omega t) + u(t-a)\sin\left(\omega(t-a)\right)\,.}$$

3. **Fourier Integral** (*10 Points*)

Compute the Fourier integral of the function $f(x) = e^{-\pi|x|}$ and use it to compute the values of the following integral:

$$\int_{\mathbb{R}} \frac{\cos(\omega x)}{\omega^2 + \pi^2} \, d\omega$$

(for each $x \in \mathbb{R}$).

Solution:

The function $f(x) = e^{-\pi|x|}$ is an even and continuous function, so its Fourier integral contains only the cosine term and it is equal to the function on each point:

$$\int_0^{+\infty} A(\omega) \cos(\omega x) \, d\omega = e^{-\pi|x|}, \qquad \forall x \in \mathbb{R}. \tag{1}$$

We compute the coefficient $A(\omega)$:

$$A(\omega) = \frac{1}{\pi} \int_{\mathbb{R}} f(v) \cos(\omega v) \, dv = \frac{2}{\pi} \int_0^{+\infty} e^{-\pi v} \cos(\omega v) \, dv =$$

$$\frac{2}{\pi} \left[\frac{e^{-\pi v}(\omega \sin(\omega v) - \pi \cos(\omega v))}{\omega^2 + \pi^2} \right] \Bigg|_{v=0}^{v=+\infty} = \frac{2}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{\omega^2 + \pi^2} = \frac{2}{\omega^2 + \pi^2}$$

When we insert this result in (1) we obtain that for each $x \in \mathbb{R}$:

$$2 \int_0^{+\infty} \frac{\cos(\omega x)}{\omega^2 + \pi^2} \, d\omega = e^{-\pi|x|}.$$

One last observation is that the function in the integral on the left-hand side is an even function of ω, therefore the left-hand side is actually equal to the integral over all $\omega \in \mathbb{R}$, which is what we need to compute:

$$\boxed{\int_{\mathbb{R}} \frac{\cos(\omega x)}{\omega^2 + \pi^2} \, d\omega = e^{-\pi|x|}, \qquad \forall x \in \mathbb{R}.}$$

See the next page!

4. Laplace Equation on a rectangle (*12 Points*)

Find the solution od the following Laplace equation on the rectangle

$$R = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leqslant x \leqslant 1,\, 0 \leqslant y \leqslant 1\}$$

$$\begin{cases} u_{xx} + u_{yy} = 0\,, & (x,y) \in R \\ u(x,0) = u(x,1) = 0\,, & 0 \leqslant x \leqslant 1 \\ u(0,y) = 0\,, & 0 \leqslant y \leqslant 1 \\ u(1,y) = \sin\left(\pi(1-y)\right)\,. & 0 \leqslant y \leqslant 1 \end{cases}$$

You can manipulate appropriately any formula that can be useful from the lecture notes (or, alternatively, solve it via separation of variables from scratch).

Solution 1 (symmetry along the $x = y$ axis):

To solve it we use appropriately a formula learnt in the lecture notes for a similar problem: the Laplace equation on a rectangle with only nonzero boundary function the one on the side of the rectangle parallel to the x axis (while here is the one parallel to the y axis).

More precisely, let $R' = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b\}$ be a rectangle of sides lengths (a,b) and consider the problem:

$$\begin{cases} v_{xx} + v_{yy} = 0\,, & (x,y) \in R' \\ v(0,y) = v(a,y) = 0\,, & 0 \leqslant y \leqslant b \\ v(x,0) = 0\,, & 0 \leqslant x \leqslant a \\ v(x,b) = f(x)\,. & 0 \leqslant x \leqslant a \end{cases} \tag{2}$$

where $f(x)$ is an arbitrary function with $f(0) = f(a) = 0$. By applying a symmetry along the $x = y$ axis (that is: exchanging x and y) we observe that $v(x,y)$ solves (2) if and only if $u(x,y) := v(y,x)$ solves the problem:

$$\begin{cases} u_{xx} + u_{yy} = 0\,, & (x,y) \in R \\ u(x,0) = u(x,a) = 0\,, & 0 \leqslant x \leqslant b \\ u(0,y) = 0\,, & 0 \leqslant y \leqslant a \\ u(b,y) = f(y)\,. & 0 \leqslant y \leqslant a \end{cases}$$

on the mirrored rectangle $R = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leqslant x \leqslant b, 0 \leqslant y \leqslant a\}$. From the lecture notes we know that the general solution to (2) is

$$v(x,y) = \sum_{n=1}^{+\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right)$$

(with coefficient A_n determined by imposing the only nontrivial boundary condition with the function $f(x)$, but for the moment we leave it). In our problem we substitute the values $a = b = 1$, and apply the reflection along the $x = y$ axis to obtain the general solution

$$u(x,y) = v(y,x) = \sum_{n=1}^{+\infty} A_n \sin\left(n\pi y\right) \sinh\left(n\pi x\right)\,.$$

Now we find the coefficient A_n by imposing the boundary condition. It is useful to observe that the boundary function is nothing but just $f(y) = \sin(\pi(1-y)) = \sin(\pi y)$, so:

$$u(1,y) = \sum_{n=1}^{+\infty} A_n \sin\left(n\pi y\right) \sinh\left(n\pi\right) = \sin(\pi y)\,.$$

By uniqueness of the Fourier series representation of a function we obtain that the only nonzero coefficient is:

$$A_1 = \frac{1}{\sinh(\pi)},$$

so finally the solution is:

$$\boxed{u(x, y) = \frac{\sinh(\pi x)\sin(\pi y)}{\sinh(\pi)}}$$

Solution 2 (separation of variables from scratch):

We search for particular solutions of the PDE with separated variables $u(x, y) = F(x)G(y)$, for which the PDE becomes:

$$F''G + FG'' = 0 \quad \Leftrightarrow \quad \frac{F''}{F} = -\frac{G''}{G} = k,$$

for some fixed $k \in \mathbb{R}$. The first two boundary conditions $u(x, 0) = u(x, 1) = 0$ translate into $G(0) = G(1) = 0$, so that:

$$\begin{cases} G'' = -kG\,, \\ G(0) = G(1) = 0\,, \end{cases} \tag{3}$$

while the boundary condition $u(0, y) = 0$ becomes $F(0) = 0$, so that

$$\begin{cases} F'' = kF\,, \\ F(0) = 0\,. \end{cases} \tag{4}$$

We ignore the last boundary condition for the moment, as we are going to use it only at the end. First we want to solve the system (3), which has nontrivial solutions only for $k > 0$. For positive values of k the differential equation has general solution:

$$G(y) = A\cos(\sqrt{k}y) + B\sin(\sqrt{k}y)\,.$$

By imposing $G(0) = 0$ we obtain $A = 0$, while imposing $G(1) = 0$ we obtain that $\sqrt{k} = n\pi$ for some integer number $n \geqslant 1$. In conclusion we have one solution for each $n \geqslant 1$, with constant $k = n^2\pi^2$, of the form:

$$G_n(y) = B_n \sin(n\pi y)\,.$$

The corresponding differential equation for F is $F'' = n^2\pi^2 F$, which has general solution:

$$F_n(x) = A_n^* e^{n\pi x} + B_n^* e^{-n\pi x}\,.$$

By imposing $F(0) = 0$ we obtain $B_n^* = -A_n^*$, so that $F_n(x) = 2A_n^* \sinh(n\pi x)$. Putting this together with G_n and renaming the constants we obtain a solution of the Laplace equation on this rectangle, for each n:

$$u_n(x, y) = A_n \sinh(n\pi x)\sin(n\pi y)\,,$$

and by the superposition principle a general solution of the form:

$$u(x, y) = \sum_{n=1}^{+\infty} A_n \sinh(n\pi x)\sin(n\pi y)\,.$$

Finally we obtain the coefficients A_n by imposing the last boundary condition that we did not yet consider:

$$u(1,y) = \sum_{n=1}^{+\infty} A_n \sinh(n\pi)\sin(n\pi y) = \sin(\pi(1-y)) \quad \Longrightarrow$$
$$\Longrightarrow \quad \begin{cases} A_1 = 1/\sinh(\pi), \\ A_n = 0, \quad n \geqslant 2 \end{cases}$$

which yields to the solution

$$\boxed{u(x,y) = \frac{\sinh(\pi x)\sin(\pi y)}{\sinh(\pi)}}$$

Please turn!

5. Wave Equation (*13 Points*)

Consider the following 1-dimensional wave equation on the interval $[0, L]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}\,, & x \in [0, L],\ t \geqslant 0 \\ u(0,t) = u(L,t) = 0\,, & t \geqslant 0 \\ u(x,0) = 0\,, & 0 \leqslant x \leqslant L \\ u_t(x,0) = x\,, & 0 \leqslant x \leqslant L \end{cases}$$

a) (*6 Points*) Find the solution in Fourier series. You can use the formula from the lecture notes.

Solution:

The general solution (via Fourier series) of the wave equation on the interval $[0, L]$ with initial data $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ is:

$$u(x,t) = \sum_{n=1}^{+\infty} [B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)] \sin\left(\frac{n\pi}{L} x\right)\,,$$

where:

$$\lambda_n = \frac{cn\pi}{L}\,,$$
$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx\,,$$
$$B_n^* = \frac{1}{\lambda_n}\frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx\,,$$

Here $f(x) = 0$, so $B_n \equiv 0$, while $g(x) = x$, so :

$$B_n^* = \frac{1}{\lambda_n}\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right) dx = \frac{1}{\lambda_n}\frac{2}{L}\int_0^L x \sin\left(\frac{n\pi}{L}x\right) dx =$$
$$= \frac{1}{\lambda_n}\frac{2}{L} \cdot \left[\frac{\sin\left(\frac{n\pi}{L}x\right) - \frac{n\pi}{L}x\cos\left(\frac{n\pi}{L}x\right)}{\frac{n^2\pi^2}{L^2}}\right]\Bigg|_{x=0}^{x=L} = \frac{1}{\lambda_n}\frac{2}{L}\cdot\left[-\frac{L^2}{n\pi}\cos(n\pi)\right] =$$
$$= \frac{\cancel{L}}{cn\pi}\cdot\frac{2}{\cancel{L}}\cdot\left[\frac{L^2}{n\pi}(-1)^{n+1}\right] = \frac{(-1)^{n+1}2L^2}{c\pi^2 n^2}\,.$$

So the solution is

$$\boxed{u(x,t) = \frac{2L^2}{c\pi^2}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^2}\sin\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)}$$

b) (*3 Points*) Remember that the solution can also be written as

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds\,,$$

where g^* is the odd, $2L$-periodic extension of the velocity initial datum g. Use this formula to compute

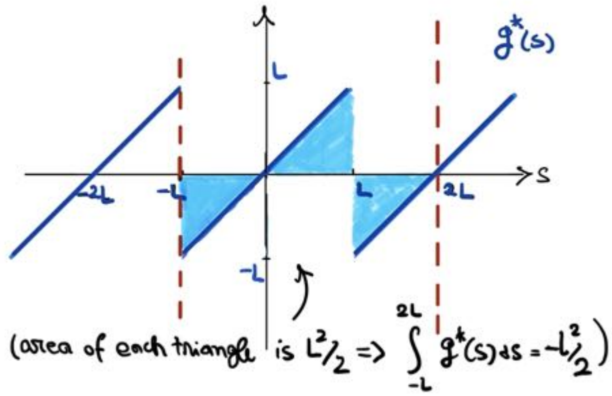
$$u\left(\frac{L}{2}, \frac{3L}{2c}\right) = \ ?$$

Solution:

We observe that for $x = L/2$ and $t = 3L/2c$ the interval over which we need to integrate is:

$$[x - ct, x + ct] = [-L, 2L]$$

The odd, 2L-periodic extension g^* of the initial datum is:



so the desired integral is :

$$\boxed{u\left(\frac{L}{2}, \frac{3L}{2c}\right) = \frac{1}{2c}\int_{-L}^{2L} g^*(s)\, ds = -L^2/4c\,.}$$

c) (*4 Points*) Compare the result from **b)** with the formula from **a)** evaluated in the point $(x, t) = (L/2, 3L/2c)$ to find the value of the following numerical series:

$$\sum_{m=0}^{+\infty} \frac{1}{(2m+1)^2} = \ ?$$

Solution:

Computing the same value $u(L/2, 3L/2c)$ from the formula obtained in **a)** we obtain:

$$u\left(\frac{L}{2}, \frac{3L}{2c}\right) = \frac{2L^2}{c\pi^2} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} \sin\left(\frac{3n\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right),$$

and we observe that

$$\sin\left(\frac{3n\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right) = \begin{cases} -1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

so we sum only over indices of the form $n = 2m + 1$, with $m = 0, 1, 2, \ldots$ (starting at zero because we start from $n = 1$), for which $(-1)^{n+1} = 1$, and we obtain:

$$-\frac{L^2}{4c} \overset{\mathbf{b)}}{=} u\left(\frac{L}{2}, \frac{3L}{2c}\right) = -\frac{2L^2}{c\pi^2} \sum_{m=0}^{+\infty} \frac{1}{(2m+1)^2},$$

Please turn!

from which the desired value of the numerical series follows:

$$\boxed{\sum_{m=0}^{+\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}\,.}$$