D-MAVT
D-MATL

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Analysis III
Summer Exam Solutions 2022

Exercise	1	2	3	4	5	6	Total
Value	8	10	8	12	15	15	68

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1. Periodicity and Even/Odd functions (*8 Points*)

Definition : A function is

- *even* if $f(x) = f(-x)$
- *odd* if $f(x) = -f(-x)$

Determine which of the following functions are even, odd, or neither. And determine which of the following functions is periodic and which is not. For the periodic ones, determine their fundamental period[1].
Write the answer in the box.

(e.g. $\sin(x)$ is a periodic function of period 2π and it's an odd function.)

a) $\cos(\frac{2\pi x}{L})$, where $L > 0$ is a constant.

b) $\sin(2x) + x^3$

c) $\sin(x^2)$

d) $4\cos(3x) + 16\sin(4x)$

[*Hint: Recall that every periodic, continuous function is bounded, and that every periodic, differentiable functions has periodic derivative.*]

Solution:

a) Periodic of fundamental period L and even because $\cos(\frac{-2\pi x}{L}) = \cos(\frac{2\pi x}{L})$.

b) Not periodic because of the x^3 and odd.

$$-\sin(-2x) - (-x)^3 = \sin(2x) - (-1)^3 x^3 = \sin(2x) + x^3$$

[1] A periodic function of period $P > 0$ is a function f such that $f(x + P) = f(x)$ for all $x \in \mathbb{R}$. The *fundamental period* of a periodic function is the smallest period P.

c) Not periodic because the derivative of $\sin(x^2)$ is not bounded.

$$\frac{d}{dx}(\sin(x^2)) = 2x\cos(2x).$$

Even because $\sin((-x)^2) = \sin(x^2)$

d) It is periodic of fundamental period 2π.
If $f(x)$ is periodic of period P_1 and $g(x)$ is periodic of period P_2, then their sum $f(x) + g(x)$ is periodic of period the least common multiple

$$P = \mathrm{LCM}(P_1, P_2)$$

of the two periods[2]. In this case $4\cos(3x)$ is periodic of fundamental period $2\pi/3$ while $16\sin(4x)$ is periodic of fundamental period $2\pi/4$, therefore their sum is periodic of period

$$P = \mathrm{LCM}\left(\frac{2\pi}{3}, \frac{\pi}{2}\right) = \mathrm{LCM}\,(4,3)\cdot\frac{\pi}{6} = \frac{12}{6}\pi = 2\pi\,.$$

It is easy to see that no smaller number is a period.

The function is neither even nor odd because cosine is an even function and sine is an odd function.

[2] By the *least common muliple* of two real numbers we mean the smallest number P such that there are positive *integer* numbers k_1, k_2 such that $P = k_1P_1 = k_2P_2$. In the case that there is no such number, we define it to be $+\infty$ and the consequence is that the function is not periodic.

2. Laplace Transform (*10 Points*)

Find the solution $f(t)$ of the following initial value problem:

$$\begin{cases} f''(t) - a^2 f(t) = a, \quad t > 0, \\ f(0) = 2, \quad f'(0) = a, \end{cases}$$

where $a > 0$ is a positive constant.

[*Hint:* $\mathcal{L}^{-1}\left(\frac{a}{s(s^2-a^2)}\right) = \mathcal{L}^{-1}\Big(\mathcal{L}(1)\mathcal{L}(\sinh(at))\Big)$ and then use the convolution for Laplace transform.]

Solution:

We apply the Laplace transform to the ODE in the initial value problem. We denote by $F = \mathcal{L}(f)$ the Laplace transform of the function f, and we denote the variable in the new domain by s as usual (so $F = F(s)$).

The first term to transform is the second derivative f'', for which we use the formula:

$$\mathcal{L}(f'') = s^2 F - sf(0) - f'(0) = s^2 F - 2s - a.$$

The second term gives

$$\mathcal{L}(-a^2 f) = -a^2 F.$$

Using the formula 1) in the Laplace transform table we find the right hand side

$$\mathcal{L}(a) = \frac{a}{s}.$$

In conclusion the ODE becomes the following algebraic equation:

$$s^2 F - 2s - a - a^2 F = \frac{a}{s} \qquad \Longrightarrow \qquad \boxed{F = \frac{a}{s(s^2-a^2)} + \frac{2s}{s^2-a^2} + \frac{a}{s^2-a^2}.}$$

The last step is to take the inverse Laplace transform of F. For the first term we use the hint and the convolution property (Property 7 on page 17 in the Lecture Notes):

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{a}{s(s^2-a^2)}\right) &= \mathcal{L}^{-1}\Big(\mathcal{L}(1)\mathcal{L}(\sinh(at))\Big) \\ &= \int_0^t 1 \cdot \sinh(at')\, dt' \\ &= \frac{\cosh(at)}{a} - \frac{\cosh(0)}{a} \\ &= \frac{\cosh(at)}{a} - \frac{1}{a}. \end{aligned}$$

For the second and third term, we use the formula 9) and 10) in the Laplace transform table. Therefore we have,

$$\begin{aligned}
f(t) = \mathcal{L}^{-1}(F) &= \mathcal{L}^{-1}\left(\frac{a}{s(s^2-a^2)} + \frac{2s}{s^2-a^2} + \frac{a}{s^2-a^2}\right) \\
&= \mathcal{L}^{-1}\left(\frac{a}{s(s^2-a^2)}\right) + 2\mathcal{L}^{-1}\left(\frac{s}{s^2-a^2}\right) + \mathcal{L}^{-1}\left(\frac{a}{s^2-a^2}\right) \\
&= \frac{\cosh(at)}{a} - \frac{1}{a} + 2\cosh(at) + \sinh(at) \\
&= (2+\frac{1}{a})\cosh(at) + \sinh(at) - \frac{1}{a}.
\end{aligned}$$

Hence,

$$\boxed{f(t) = (2+\frac{1}{a})\cosh(at) + \sinh(at) - \frac{1}{a}.}$$

See the next page!

3. Fourier transform (*8 Points*)

Compute the Fourier transform of the following function (you don't need to compute the case $w = 0$):

$$f(x) = \begin{cases} \sqrt{2\pi}(1+x), & 0 \le x \le \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that the Fourier transform of f is given by

$$\mathcal{F}(f)(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx.$$

Solution:

$$\begin{aligned}
\mathcal{F}(f)(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\pi} \sqrt{2\pi}(1+x) e^{-iwx}\, dx \\
&= \int_0^\pi e^{-iwx}\, dx + \int_0^\pi x e^{-iwx}\, dx \\
&= \left.\frac{e^{-iwx}}{-iw}\right|_0^\pi + \left[x \frac{e^{-iwx}}{-iw}\right]_0^\pi - \int_0^\pi \frac{e^{-iwx}}{-iw}\, dx \\
&= \frac{1}{-iw}(e^{-iw\pi} - 1) + \pi \frac{\mathrm{e}^{-iw\pi}}{-iw} - \left.\frac{e^{-iwx}}{(-iw)^2}\right|_0^\pi \\
&= \frac{1}{-iw}(e^{-iw\pi} - 1) + \pi \frac{\mathrm{e}^{-iw\pi}}{-iw} + \frac{e^{-iw\pi}}{w^2} - \frac{1}{w^2} \\
&= \left(\frac{1}{-iw} + \frac{\pi}{-iw} + \frac{1}{w^2}\right) e^{-iw\pi} + \frac{1}{iw} - \frac{1}{w^2} \\
&= \left(\frac{i}{w} + \frac{i\pi}{w} + \frac{1}{w^2}\right) e^{-iw\pi} - \frac{i}{w} - \frac{1}{w^2}
\end{aligned}$$

4. **Wave Equation with D'Alembert solution** (*12 Points*)

Let $c > 0$. Consider the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = \frac{1}{c}\arctan(x), & x \in \mathbb{R} \\ u_t(x,0) = \frac{1}{1+x^2}, & x \in \mathbb{R}. \end{cases}$$

Find the solution $u(x,t)$. You may use D'Alembert formula.
[Simplify the expression as much as possible: no unsolved integrals].

Solution:

D'Alembert's formula for the solution of the wave equation is:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

With our given initial conditions we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\frac{1}{c}\arctan(x+ct) + \frac{1}{c}\arctan(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{1}{1+s^2} ds \\ &= \frac{1}{2c}\left(\arctan(x+ct) + \arctan(x-ct)\right) + \frac{1}{2c}\arctan(s)\Big|_{x-ct}^{x+ct} \\ &= \frac{1}{2c}\left(\arctan(x+ct) + \arctan(x-ct)\right) + \frac{1}{2c}\left(\arctan(x+ct) - \arctan(x-ct)\right) \\ &= \frac{1}{c}\arctan(x+ct) \end{aligned}$$

Hence, the solution is

$$\boxed{u(x,t) = \frac{1}{c}\arctan(x+ct).}$$

5. Wave Equation with inhomogeneous boundary conditions (*15 Points*)

Find the solution of the following wave equation (**with inhomogeneous boundary conditions**) on the interval $[0, \pi]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t \geq 0,\ x \in [0, \pi] \\ u(0,t) = 3\pi^2, & t \geq 0 \\ u(\pi, t) = 7\pi, & t \geq 0 \\ u(x, 0) = 2\sin(5x) + \sin(4x) + (7 - 3\pi)x + 3\pi^2, & x \in [0, \pi] \\ u_t(x, 0) = 0. & x \in [0, \pi] \end{cases} \tag{1}$$

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w''(x) = 0$, $w(0) = 3\pi^2$, and $w(\pi) = 7\pi$.

Solution:

The only functions with second derivative zero are the linear functions

$$w(x) = \alpha x + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

Imposing the boundary conditions we find the right coefficients

$$\begin{cases} 3\pi^2 = w(0) = \alpha \cdot 0 + \beta \\ 7\pi = w(\pi) = \alpha \cdot \pi + \beta \end{cases} \quad \Leftrightarrow \quad \begin{cases} \alpha = \frac{7\pi - 3\pi^2}{\pi} \\ \beta = 3\pi^3 \end{cases} \quad \Leftrightarrow \quad \boxed{w(x) = (7 - 3\pi)x + 3\pi^2.}$$

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

Solution:

The PDE doesn't change because w is independent of time and has second derivative in x zero. The boundary conditions become homogeneous (that's why we chose this w)

$$v(0,t) = u(0,t) - w(0) = 3\pi^2 - 3\pi^2 = 0$$
$$v(\pi, t) = u(\pi, t) - w(\pi) = 7\pi - (7 - 3\pi)\pi - 3\pi^2 = 0.$$

The initial position of the wave changes in

$$\begin{aligned} v(x, 0) = u(x, 0) - w(x) &= 2\sin(5x) + \sin(4x) + (7 - 3\pi)x + 3\pi^2 - (7 - 3\pi)x - 3\pi^2 \\ &= 2\sin(5x) + \sin(4x), \end{aligned}$$

while the initial speed doesn't change (because, again, w is independent of time). Finally

$$\boxed{\begin{cases} v_{tt} = c^2 v_{xx}, & t \geq 0,\ x \in [0, \pi] \\ v(0,t) = v(\pi, t) = 0, & t \geq 0 \\ v(x, 0) = 2\sin(5x) + \sin(4x), & x \in [0, \pi] \\ v_t(x, 0) = 0. & x \in [0, \pi] \end{cases}}$$

c) (i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

Solution:

This is a standard homogeneous wave equation with homogeneous boundary conditions. The formula from the script is

$$v(x,t) = \sum_{n=1}^{+\infty} \big(B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)\big) \sin\left(\frac{n\pi}{L}x\right), \quad \lambda_n = \frac{cn\pi}{L}$$
$$\stackrel{(L=\pi)}{=} \sum_{n=1}^{+\infty} \Big(B_n \cos(cnt) + B_n^* \sin(cnt)\Big) \sin(nx).$$

The coefficients $B_n^* = 0$, because the initial speed is zero, while the coefficients B_n are the Fourier coefficients of the odd, 2π-periodic extension of the initial position datum $v(x,0) = 2\sin(5x) + \sin(4x)$, that is:

$$\sum_{n=1}^{+\infty} B_n \sin(nx) = 2\sin(5x) + \sin(4x).$$

By identifying the term we have, $B_4 = 1$, $B_5 = 2$ and $B_n = 0$ otherwise. Finally we get the following solution

$$\boxed{v(x,t) = \cos(4ct)\sin(4x) + 2\cos(5ct)\sin(5x).}$$

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (1).

Solution:

The solution $u(x,t)$ of the inhomogeneous problem is

$$\boxed{u(x,t) = \cos(4ct)\sin(4x) + 2\cos(5ct)\sin(5x) + (7-3\pi)x + 3\pi^2.}$$

6. Separation of variables for the Heat equation (*15 Points*)

Consider the following time-dependent version of the heat equation on the interval $[0, L]$. We also impose boundary conditions and we look for a solution $u = u(x, t)$ such that:

$$\begin{cases} u_t = (1+t)u_{xx}, & x \in [0, L], t \in [0, +\infty), \\ u(0, t) = 0, & t \in [0, +\infty), \\ u(L, t) = 0, & t \in [0, +\infty), \\ u(x, 0) = f(x), & x \in [0, L]. \end{cases}$$

Where f is a given function. The Fourier series of the $2L$ periodic odd extension of f is given by

$$f(x) := \sum_{n=1}^{\infty} \frac{\pi^2}{(8+n)^2} \sin\left(\frac{n\pi}{L}x\right).$$

Find the solution $u(x, t)$ using separation of variable. Proceed as in the lecture and adapt the steps if necessary.

Solution:

We use separation of variable $u(x, t) = F(x)G(t)$. The differential equation becomes:

$$F(x)\dot{G}(t) = (1+t)F''(x)G(t),$$

which is convenient to rewrite as

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{(1+t)G(t)}$$

because it becomes clear that we are comparing a function of x with a function of t, and the only way that this equality might be true is that both these functions are equal and constant:

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{(1+t)G(t)} = k, \qquad k \in \mathbb{R}.$$

The boundary conditions are

$$u(0, t) = F(0)G(t) = 0 \quad \text{and} \quad u(L, t) = F(L)G(t) = 0 \quad \forall t \in [0, +\infty)$$

which in order to be true, excluding the trivial solution $G(t) \equiv 0$, become:

$$F(0) = F(L) = 0.$$

In other words the initial PDE with boundary conditions becomes the system of coupled equations

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(L) = 0, \end{cases} \qquad \text{and} \quad \dot{G}(t) = k(1+t)G(t).$$

We first solve the system for $F(x)$, distinguishing the cases of k positive, zero, or negative. For $k > 0$ the general solution of the ODE is

$$F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x},$$

which is, however, <u>not</u> compatible with the boundary conditions, in the sense that the only solution of this form satisfying the boundary conditions is the trivial solution: $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_1 + C_2 \quad \Leftrightarrow \quad C_2 = -C_1 \quad \Longrightarrow F(x) = C_1\left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right)$$

but then imposing the other condition:

$$0 = F(L) = C_1\left(e^{\sqrt{k}L} - e^{-\sqrt{k}L}\right) \quad \Leftrightarrow \quad \begin{array}{l}\text{either } C_1 = 0\\ \text{or } e^{2\sqrt{k}L} = 1\end{array}$$

which implies $C_1 = 0$ (and consequently $C_2 = -C_1 = 0$) because $2\sqrt{k}L \neq 0$ and therefore its exponential is not 1.

For $k = 0$ the general solution is $F(x) = C_1 x + C_2$ which is also not compatible with boundary conditions unless $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_2 \quad \Longrightarrow F(x) = C_1 x$$

and then

$$0 = F(L) = C_1 L \quad \Leftrightarrow \quad C_1 = 0.$$

It remains the case $k < 0$, in which its convenient to write it in the form $k = -p^2$ for positive real number p, and general solutions of $F'' = -p^2 F$ are:

$$F(x) = A\cos(px) + B\sin(px).$$

We impose the boundary conditions:

$$0 = F(0) = A \quad \Longrightarrow F(x) = B\sin(px)$$

and

$$0 = F(L) = B\sin(pL) \quad \overset{(\text{if } B \neq 0)}{\Leftrightarrow} \quad pL = n\pi, \quad n \in \mathbb{Z}_{\geq 1}$$

<u>Conclusion:</u> we have a nontrivial solution for each $n \geq 1$, $k = k_n = -\frac{n^2\pi^2}{L^2}$:

$$F_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right).$$

The corresponding equation for $G(t)$ is

$$\dot{G} = -(1+t)\frac{n^2\pi^2}{L^2}G$$

which has general solution

$$G_n(t) = C_n e^{-\frac{n^2\pi^2}{L^2}(t + \frac{t^2}{2})}.$$

The conclusion is that for every $n \geq 1$ we have a solution

$$u_n(x,t) = F_n(x)G_n(t) = A_n e^{-\frac{n^2\pi^2}{L^2}(t+\frac{t^2}{2})}\sin\left(\frac{n\pi}{L}x\right), \qquad \text{with} \quad A_n = B_n C_n.$$

 See the next page!

Then by the Superposition Principle, the function

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(x,t) = \sum_{n=1}^{+\infty} A_n \mathrm{e}^{-\frac{n^2\pi^2}{L^2}(t+\frac{t^2}{2})} \sin\left(\frac{n\pi}{L}x\right)$$

is also a solution. By imposing the initial condition $u(x,0) = f(x)$, we have

$$\sum_{n=1}^{+\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} \frac{\pi^2}{(8+n)^2} \; \sin\left(\frac{n\pi}{L}x\right).$$

Therefore,

$$A_n = \frac{\pi^2}{(8+n)^2}.$$

Hence the final solution is given by,

$$\boxed{u(x,t) = \sum_{n=1}^{+\infty} u_n(x,t) = \sum_{n=1}^{+\infty} \frac{\pi^2}{(8+n)^2} \mathrm{e}^{-\frac{n^2\pi^2}{L^2}(t+\frac{t^2}{2})} \sin\left(\frac{n\pi}{L}x\right)}$$