Problems and suggested solutions

Laplace Transforms: $\left(F = \mathcal{L}(f) \right)$ (u = Heaviside function, δ = Dirac's delta function)

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geq 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)g(t-a)$	$\mathcal{L}(g)e^{-as}$
12)	$\delta(t-a)$	e^{-as}

Fourier transforms:

	$f(x)$	$\widehat{f}(\omega)$
1)	e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{\frac{-\omega^2}{4a}}$

	$f(x)$	$\widehat{f}(\omega)$
2)	$\begin{cases} e^{-ax}, & x \geq 0, \\ 0, & x < 0. \end{cases}$	$\frac{1}{\sqrt{2\pi}(a+i\omega)}$

	$f(x)$	$\widehat{f}(\omega)$
3)	$\begin{cases} 1, & \lvert x\rvert < 1, \\ 0, & \lvert x\rvert > 1. \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(\omega)}{\omega}$

Indefinite Integrals: $\left(n \in \mathbb{Z}_{\geq 1}\right)$

1) $$\int x \cos\left(\frac{n\pi}{L}x\right)\, dx = \frac{\cos\left(\frac{n\pi}{L}x\right) + \left(\frac{n\pi}{L}\right) x \sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2} \quad \text{(+constant)}$$

2) $$\int x^2 \cos\left(\frac{n\pi}{L}x\right)\, dx = \frac{\left(\left(\frac{n\pi}{L}\right)^2 x^2 - 2\right) \sin\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right) x \cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3} \quad \text{(+constant)}$$

3) $$\int x \sin\left(\frac{n\pi}{L}x\right)\, dx = \frac{\sin\left(\frac{n\pi}{L}x\right) - \left(\frac{n\pi}{L}\right) x \cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2} \quad \text{(+constant)}$$

4) $$\int x^2 \sin\left(\frac{n\pi}{L}x\right)\, dx = \frac{\left(2 - \left(\frac{n\pi}{L}\right)^2 x^2\right) \cos\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right) x \sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3} \quad \text{(+constant)}$$

5) $$\int \frac{1}{1+x^2}\, dx = \arctan(x) \quad \text{(+constant)}$$

You can use these formulas without justification.

Question 1

1.MC1 [3 Points] Consider the following initial value problem:

$$\begin{cases} f''(t) = 3 - \delta(t-\pi), \quad t > 0, \\ f(0) = a, \quad f'(0) = b, \end{cases}$$

where $a, b > 0$ are positive constants, and δ is the Dirac delta. Find the Laplace transform $\mathcal{L}(f) = F$ of the function f.

(A) $F(s) = \frac{3}{s^3} - \frac{e^{-\pi s}}{s^2} + \frac{b}{s} + \frac{a}{s^2}$.

(B) $F(s) = \frac{3}{s^3} - \frac{e^{-\pi s}}{s^2} + \frac{a}{s} + \frac{b}{s^2}$.

(C) $F(s) = \frac{3}{s^2} - \frac{e^{\pi s}}{s} + \frac{a}{s} + \frac{b}{s^2}$.

(D) $F(s) = \frac{3}{s^4} - \frac{e^{-\pi s}}{s^2} + \frac{a}{s^2} + \frac{b}{s^2}$.

Solution:

(B) The solution is $F(s) = \frac{3}{s^3} - \frac{e^{-\pi s}}{s^2} + \frac{a}{s} + \frac{b}{s^2}$.

We apply the Laplace transform to the ODE in the initial value problem. We denote by $F = \mathcal{L}(f)$ the Laplace transform of the function f, and we denote the variable in the new domain by s as usual (so $F = F(s)$).

The first term to transform is the second derivative f'', for which we use the formula:

$$\mathcal{L}(f'') = s^2 F - sf(0) - f'(0) = s^2 F - sa - b.$$

The term in the right hand side becomes by linearity and using the table of the Laplace transform above,

$$\begin{aligned} \mathcal{L}(3 - \delta(t-\pi)) &= \mathcal{L}(3) - \mathcal{L}(\delta(t-\pi)) \\ &= \frac{3}{s} - e^{-\pi s}. \end{aligned}$$

In conclusion the ODE becomes the following algebraic equation:

$$s^2 F - sa - b = \frac{3}{s} - e^{-\pi s}.$$

Therefore,

$$F(s) = \frac{3}{s^3} - \frac{e^{-\pi s}}{s^2} + \frac{a}{s} + \frac{b}{s^2}.$$

1.MC2 [3 Points] Find the inverse Laplace transform of

$$F(s) = \frac{s+2}{s^2-4} + \frac{2s}{s^2-16}.$$

(A) $f(t) = e^{4t} + 2\cosh(4t)$.

(B) $f(t) = e^{2t} + 2\cos(4t)$.

(C) $f(t) = e^{-2t} + 2\sinh(4t)$.

(D) $f(t) = e^{2t} + 2\cosh(4t)$.

Solution:

(D) The solution is $f(t) = e^{2t} + 2\cosh(4t)$.

For the first term we have,

$$\frac{s+2}{s^2-4} = \frac{1}{s-2} \implies \mathcal{L}^{-1}\left(\frac{s+2}{s^2-4}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}.$$

And for the second term we have

$$\frac{2s}{s^2-16} = 2\frac{s}{s^2-4^2} \implies \mathcal{L}^{-1}\left(\frac{2s}{s^2-16}\right) = 2\mathcal{L}^{-1}\left(\frac{s}{s^2-4^2}\right) = 2\cosh(4t).$$

Hence, the solution is given by

$$f(t) = \mathcal{L}^{-1}\left(\frac{s+2}{s^2-4} + \frac{2s}{s^2-16}\right) = e^{2t} + 2\cosh(4t).$$

1.MC3 [3 Points] Let f be a continuous function such that $\lim_{x\to\infty} f(x) = 0$. Solve the following differential equation using the Fourier transform

$$f''(x) = \int_{-\infty}^{\infty} g(y) e^{-\pi(x-y)^2}\, dy + f(x).$$

where g is a given function.

(A) $f(x) = \int_{-\infty}^{\infty} \frac{1}{-1-\omega^2} \widehat{g}(\omega) e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega$.

(B) $f(x) = \int_{-\infty}^{\infty} \frac{1}{1-\omega^2} \widehat{g}(\omega) e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega$.

(C) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{-1-\omega^2} \widehat{g}(\omega) e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega$.

(D) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{-1+\omega^2} \widehat{g}(\omega) e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega$.

Solution:

(C) The solution is $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{-1-\omega^2} \widehat{g}(\omega) e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega$.

We take the Fourier transform on both sides of the equation. (We use the property of the convolution, derivative and Fourier transform of a Gaussian.)

$$-\omega^2 \widehat{f}(\omega) = \sqrt{2\pi}\widehat{g}(\omega) \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4\pi}} + \widehat{f}(\omega).$$

We can rewrite this equality as follow

$$\widehat{f}(\omega) = \frac{1}{-1-\omega^2} \widehat{g}(\omega) e^{-\frac{\omega^2}{4\pi}}.$$

Finally, we take the inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{-1-\omega^2} \widehat{g}(\omega) e^{-\frac{\omega^2}{4\pi}} e^{i\omega x} \, d\omega.$$

1.MC4 [3 Points] Determine if the following function is even, odd, or neither and if it is periodic or not.

$$\cos(8x) + 2x^2 - x^3.$$

(A) The function is even and not periodic.

(B) The function is odd and not periodic.

(C) The function is not even, odd or periodic.

(D) The function is neither even nor odd but is periodic.

Solution:

(C) The function is not even, odd or periodic.

Not periodic because of the $2x^2 - x^3$ and not even or odd because

$$\cos(8(-x)) + 2(-x)^2 - (-x)^3 = \cos(8x) + x^2 + x^3.$$

1.MC5 [3 Points] Let f be a 2π periodic continuous function such that $f(0) = \frac{1}{4}\coth(2\pi)$, where coth is the hyperbolic cotangent function. The complex Fourier series of f on the interval $[-\pi, \pi]$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi(4+n^2)} e^{inx}.$$

Find the value of the numerical series

$$\sum_{n=1}^{\infty} \frac{1}{(4+n^2)}.$$

(A) $\sum_{n=1}^{\infty} \frac{1}{(4+n^2)} = \frac{1}{4}\pi\coth(2\pi) - \frac{1}{8}$.

(B) $\sum_{n=1}^{\infty} \frac{1}{(4+n^2)} = \frac{1}{2}\pi\coth(2\pi) - \frac{1}{4}$.

(C) $\sum_{n=1}^{\infty} \frac{1}{(4+n^2)} = \frac{1}{2}\pi\coth(2\pi) + \frac{1}{4}$.

(D) $\sum_{n=1}^{\infty} \frac{1}{(4+n^2)} = \frac{1}{4}\pi\coth(2\pi) + \frac{1}{8}$.

Solution:

(A) The solution is $\frac{1}{4}\pi\coth(2\pi) - \frac{1}{8}$.

We have

$$f(0) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi(4+n^2)} e^{in0} = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi(4+n^2)}.$$

Since the sum is converging we can rearrange the terms,

$$\sum_{n=-\infty}^{\infty} \frac{1}{2\pi(4+n^2)} = \frac{1}{8\pi} + 2\sum_{n=1}^{\infty} \frac{1}{2\pi(4+n^2)}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{2\pi(4+n^2)} = \frac{1}{2}f(0) - \frac{1}{16\pi}.$$

We can rewrite this as follow

$$\sum_{n=1}^{\infty} \frac{1}{(4+n^2)} = \pi f(0) - \frac{1}{8}.$$

We know that $f(0) = \frac{1}{4}\coth(2\pi)$ therefore,

$$\sum_{n=1}^{\infty} \frac{1}{(4+n^2)} = \frac{1}{4}\pi\coth(2\pi) - \frac{1}{8}.$$

1.MC6 [3 Points] Consider the following PDE (partial differential equation) for the function $u = u(x,y)$:

$$u_{xx}u_{xy} - \pi u_{xx} = 4u_{yy} - u_{xxy}.$$

Is this PDE linear ? Homogeneous? And what is the order of the PDE ?

(A) The PDE is not linear, not homogeneous and of order four.

(B) The PDE is not linear, not homogeneous and of order three.

(C) The PDE is not linear, homogeneous and of order four.

(D) The PDE is linear, homogeneous and of order three.

Solution:

(B) The PDE is not linear, not homogeneous and of order three.

It is not linear because there is the multiplication, $u_{xx}u_{xy}$. It is not homogeneous because of $u_{xx}u_{xy}$. And finally, the highest derivative is 3 in the term, u_{xxy}.

1.MC7 [3 Points] Wave equation with D'Alembert solution.

Consider the following wave equation with $c = 1$,

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

where

$$f(x) = \begin{cases} e^x + 4x^2 - 2, & x \in (0,2)\,, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^2, & x \in (0,2)\,, \\ 0, & \text{otherwise.} \end{cases}$$

Find the solution at position $x = 1$ and time $t = 10$, i.e. $u(1,10)$.

(A) $u(1,10) = \frac{1030}{3}$.
(B) $u(1,10) = \frac{2060}{3}$.
(C) $u(1,10) = \frac{2}{3}$.
(D) $u(1,10) = \frac{4}{3}$.

Solution:

(D)The solution is $u(1,10) = \frac{4}{3}$.

D'Alembert's formula for the solution of the wave equation is:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

With our given initial conditions, $c = 1$, $x = 1$ and $t = 10$, we get

$$\begin{aligned} u(1,10) &= \frac{1}{2}\left(f(11) + f(-9)\right) + \frac{1}{2}\int_{-9}^{11} g(s)\,ds \\ &= \frac{1}{2}\int_0^2 s^2\,ds = \frac{1}{2}\frac{1}{3}s^3\Big|_0^2 = \frac{4}{3}. \end{aligned}$$

Hence, the solution is

$$u(1,10) = \frac{4}{3}.$$

1.MC8 [3 Points] Let $u = u(x,y)$ be a harmonic function in a region $\mathcal{D} \subset \mathbb{R}^2$. The disk of radius 9 centred at 0, denoted by D_9, is contained in the region $\mathcal{D}$, i.e. $D_9 \subset \mathcal{D}$.

The maximum value of u in $\mathcal{D}$ is at $(x,y) = (2,2)$, i.e. $\max_{\mathcal{D}} u(x,y) = u(2,2)$.

Which of the following statements is true?

(A) u is not constant in $\mathcal{D}$.
(B) u is constant in D_9 but not in $\mathcal{D}$.
(C) u is constant in $\mathcal{D}$.
(D) We cannot conclude that (A), (B) and (C) are true.

Solution:

(C) The solution is: u is constant in $\mathcal{D}$.

u is harmonic in $\mathcal{D}$ and since $D_9 \subset \mathcal{D}$ the point $(2,2) \in \mathcal{D}$. Therefore the maximum is attained on the interior of the region $\mathcal{D}$. Hence u must be constant.

1.MC9 [3 Points] Consider the following Neumann problem (Laplace equation with fixed normal derivative on the boundary):

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ \partial_n u(R,\theta) = \theta(2\pi - \theta), & 0 \leq \theta \leq 2\pi \Big(\text{ parametrising } \partial D_R \Big) \end{cases}$$

with D_R the disk center in the origin and radius R and ∂D_R is the boundary of D_R.

Which of the following is true:

(A) There is no solution.

(B) There are two solutions.

(C) There are infinitely many solutions.

(D) We cannot conclude that (A), (B), or (C) are true.

Solution:

(A) There is no solution.

Let $A \subset \mathbb{R}^2$ be a (regular) region of the plane and the curve $\gamma = \partial A$ its boundary. As explained in the lecture notes, if u solves the Neumann problem on A

$$\begin{cases} \nabla^2 u = 0, & \text{in } A \\ \partial_n u = g, & \text{on } \gamma \end{cases}$$

then the integral of g on the boundary must vanish because of the divergence theorem

$$\int_\gamma g\, d\gamma = \int_\gamma \Big(\partial_n u\Big)\, d\gamma = \int_\gamma \Big(\nabla u \cdot n\Big)\, d\gamma = \int_A div(\nabla u)\, dA = \int_A \Big(\nabla^2 u\Big)\, dA = \int_A 0\, dA = 0.$$

In our case the region is a disk $A = D_R$ and the integral on the boundary is

$$\begin{aligned} \int_\gamma g\, d\gamma &= \int_0^{2\pi} \theta(2\pi - \theta)\, d\theta = \int_0^{2\pi} \Big(-\theta^2 + 2\pi\theta\Big)\, d\theta \\ &= \Big(-\frac{1}{3}\theta^3 + \pi\theta^2\Big)\Bigg|_0^{2\pi} = \Big(-\frac{8}{3}\pi^3 + 2\pi^3\Big) \neq 0. \end{aligned}$$

This means that the problem is ill-posed and there is no solution.

Question 2

2.Q1 [15 Points] Separation of variables for the Heat equation

Consider the following time-dependent version of the Heat equation on the interval $[0,1]$. We also impose boundary conditions and we look for a solution $u = u(x,t)$ such that:

$$\begin{cases} u_t(x,t) = \cos(t) u_{xx}(x,t), & x \in [0,1], t \in [0,+\infty), \\ u(0,t) = 0, & t \in [0,+\infty), \\ u(1,t) = 0, & t \in [0,+\infty), \\ u(x,0) = f(x), & x \in [0,1], \end{cases}$$

where f is a given function. The **complex Fourier series** of the periodic odd extension of f on $(-1,1)$ is given by

$$f(x) := \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} i \frac{(-1)^n}{n\pi} e^{in\pi x}.$$

Find the solution $u(x,t)$ using separation of variables. Proceed as in the lecture and adapt the steps if necessary.

Hint: It might be useful to use the relation between the complex and real Fourier coefficients.

Solution:

We use separation of variable $u(x,t) = F(x)G(t)$. The differential equation becomes:

$$F(x)\dot{G}(t) = \cos(t) F''(x) G(t),$$

which is convenient to rewrite as

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{\cos(t) G(t)}$$

because it becomes clear that we are comparing a function of x with a function of t, and the only way that this equality might be true is that both these functions are equal and constant:

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{\cos(t) G(t)} = k, \qquad k \in \mathbb{R}.$$

The boundary conditions are

$$u(0,t) = F(0)G(t) = 0 \quad \text{and} \quad u(1,t) = F(1)G(t) = 0 \quad \forall t \in [0,+\infty)$$

which in order to be true, excluding the trivial solution $G(t) \equiv 0$, become:

$$F(0) = F(1) = 0.$$

In other words the initial PDE with boundary conditions becomes the system of coupled equations

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(1) = 0, \end{cases} \qquad \text{and} \quad \dot{G}(t) = k\cos(t)G(t).$$

We first solve the system for $F(x)$, distinguishing the cases of k positive, zero, or negative. For $k > 0$ the general solution of the ODE is

$$F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x},$$

which is, however, <u>not</u> compatible with the boundary conditions, in the sense that the only solution of this form satisfying the boundary conditions is the trivial solution: $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_1 + C_2 \quad \Leftrightarrow \quad C_2 = -C_1 \quad \implies F(x) = C_1\left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right)$$

but then imposing the other condition:

$$0 = F(1) = C_1\left(e^{\sqrt{k}} - e^{-\sqrt{k}}\right) \quad \Leftrightarrow \quad \begin{array}{l} \text{either } C_1 = 0 \\ \text{or } e^{2\sqrt{k}} = 1 \end{array}$$

which implies $C_1 = 0$ (and consequently $C_2 = -C_1 = 0$) because $2\sqrt{k} \neq 0$ and therefore its exponential is not 1.

For $k = 0$ the general solution is $F(x) = C_1 x + C_2$ which is also not compatible with boundary conditions unless $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_2 \quad \implies F(x) = C_1 x$$

and then

$$0 = F(1) = C_1 \quad \Leftrightarrow \quad C_1 = 0.$$

It remains the case $k < 0$, in which its convenient to write it in the form $k = -p^2$ for positive real number p, and general solutions of $F'' = -p^2 F$ are:

$$F(x) = A\cos(px) + B\sin(px).$$

We impose the boundary conditions:

$$0 = F(0) = A \quad \implies F(x) = B\sin(px)$$

and

$$0 = F(1) = B\sin(p) \quad \overset{(\text{if } B \neq 0)}{\Leftrightarrow} \quad p = n\pi, \quad n \in \mathbb{Z}_{\geq 1}$$

<u>Conclusion:</u> we have a non-trivial solution for each $n \geq 1$, $k = k_n = -n^2\pi^2$:

$$F_n(x) = B_n \sin(n\pi x).$$

The corresponding equation for $G(t)$ is

$$\dot{G}(t) = -\cos(t)n^2\pi^2 G(t)$$

which has general solution[a]

$$G_n(t) = C_n e^{-n^2\pi^2 \sin(t)}.$$

The conclusion is that for every $n \geq 1$ we have a solution

$$u_n(x,t) = F_n(x)G_n(t) = A_n e^{-n^2\pi^2 \sin(t)} \sin(n\pi x), \qquad \text{with} \quad A_n = B_n C_n.$$

Then by the Superposition Principle, the function

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 \sin(t)} \sin(n\pi x)$$

is also a solution. By imposing the initial condition $u(x,0) = f(x)$, we have

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = f(x).$$

Therefore,

$$\sum_{n=1}^{\infty} A_n \sin(n\pi x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} i\,\frac{(-1)^n}{n\pi} e^{in\pi x}.$$

The formula relating the real coefficients to the complex coefficients is

$$\begin{cases} a_0 = c_0 \\ a_n = c_n + c_{-n} \quad (n \geq 1) \\ b_n = i\,(c_n - c_{-n}) \end{cases}$$

and substituting we get

$$\begin{cases} a_0 = c_0 = 0 \\ a_n = c_n + c_{-n} = i\,\frac{(-1)^n}{n\pi} - i\,\frac{(-1)^n}{n\pi} = 0 \\ b_n = i(c_n - c_{-n}) = i\left(i\,\frac{(-1)^n}{n\pi} + i\,\frac{(-1)^n}{n\pi}\right) = (-1)^{n+1}\frac{2}{n\pi}. \end{cases}$$

Thus, we get

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} i\,\frac{(-1)^n}{n\pi} e^{in\pi x} = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n\pi} \sin(n\pi x).$$

This yield,

$$A_n = (-1)^{n+1}\frac{2}{n\pi}.$$

Hence the final solution is given by,

$$u(x,t)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n\pi}e^{-n^2\pi^2\sin(t)}\sin\left(n\pi x\right).$$

[a] A homogeneous linear differential equation of first order with variable coefficients is an equation of the form

$$y'(t)=a(t)y(t),$$

where $a=a(t)$ possibly varies with t. As it is easy to check, the general solution is given by

$$y(t)=ce^{A(t)},\quad c\in\mathbb{R},$$

where $A(t)$ is any primitive of $a(t)$:

$$A(t)=\int a(s)\,ds.$$

Question 3

3.Q1 [10 Points] PDE with Fourier transform

Solve the following partial differential equation on an infinite bar:

$$\begin{cases} u_t(x,t) = \frac{1}{2}u_{xx}(x,t) + u(x,t), & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = xe^{-\frac{1}{2}x^2}, & x \in \mathbb{R} \end{cases}$$

via the Fourier transform with respect to x. You must simplify your solution as much as possible, no unsolved integrals.
Hints: You can proceed as follow:

- First, transform the partial differential equation into a differential equation in time t using the Fourier transform. Use that, for $a > 0$,
$$\mathcal{F}\left(xe^{-ax^2}\right)(\omega) = \frac{-i\omega}{(2a)^{3/2}}e^{-\frac{\omega^2}{4a}}.$$
- Solve the solution of this ODE.
- Finally, take the inverse Fourier transform to find the solution $u(x,t)$. Use that, for $b > 0$,
$$\mathcal{F}^{-1}\left(-i\omega e^{-b\omega^2}\right)(x) = \frac{1}{(2b)^{3/2}}xe^{-\frac{x^2}{4b}}.$$

Solution:

Let us denote by $\widehat{u}(w,t) := \mathcal{F}\Big(u(\cdot,t)\Big)(w)$ the Fourier transform with respect to the space variable. Then, the PDE transforms into

$$\begin{cases} \widehat{u}_t = -\frac{1}{2}\omega^2\widehat{u} + \widehat{u} \\ \widehat{u}(\omega,0) = -i\omega e^{-\frac{1}{2}\omega^2} \end{cases}$$

This is nothing but an ODE for $\widehat{u}$ in the variable t. The general solution is simply given by

$$\widehat{u}(\omega,t) = \widehat{u}(\omega,0)e^{(-\frac{1}{2}\omega^2+1)t} = -i\omega e^{-\frac{1}{2}\omega^2}e^{(-\frac{1}{2}\omega^2+1)t} = -i\omega e^{-\frac{1}{2}(1+t)\omega^2}e^t.$$

To obtain the solution u we need to apply the Fourier inverse transform on the above equation. The left hand side is then equal to $u(x,t)$, whereas the right hand side can be taken care using the hint,

$$\mathcal{F}^{-1}(-i\omega e^{-b\omega^2}e^t) = e^t\mathcal{F}^{-1}(-i\omega e^{-b\omega^2}) = \frac{xe^t}{(2b)^{3/2}}e^{-\frac{x^2}{4b}},$$

and setting $b = \frac{1}{2}(1+t)$ in the above formula. Note that the term e^t can be taken out of the inverse Fourier transform because e^t does not depend on w. The solution is therefore given by

$$u(x,t) = \frac{xe^t}{(1+t)^{3/2}}e^{-\frac{1}{2(1+t)}x^2}.$$