D-MAVT
D-MATL

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Analysis III
Mock Exam Solutions

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1. Classification of PDEs

Consider the following PDEs (in what follows, $u = u(x, y)$ is a function of two variables x and y). Classify each of them: hyperbolic, parabolic, elliptic, mixed type.

a) $u_{xx} + u_{yy} + k^2 u = 0$, where $k > 0$ is a positive constant.

b) $y u_{xx} + 2x^{\frac{3}{2}} u_{xy} + u_{yy} = u_x + u_y + u$.

c) $u_{xx} + 2\cos(x) u_{xy} + y u_{yy} = e^{xy}$.

Solution:

A general second order, linear, PDE has the form:

$$A u_{xx} + 2B u_{xy} + C u_{yy} = F(x, y, u, u_x, u_y)\,,$$

where A, B, C can be themselves functions of the variables (x, y). The PDE is called hyperbolic, parabolic or elliptic, if the coefficient $AC - B^2$ is, respectively, smaller, equal or greater than zero. When the sign of the coefficient is not constant the equation is of mixed type.

a) $AC - B^2 = 1 > 0 \quad \Longrightarrow \quad$ elliptic.
b) $AC - B^2 = y - x^3$ which changes sign, so the PDE is of mixed type.
c) $AC - B^2 = y - \cos^2(x)$ which also changes sign, so the PDE is of mixed type,

Periodicity

Determine which of the following functions is periodic and which is not. For the periodic ones, determine their fundamental period[1] if it exists.

d) $4\cosh(x)$

Solution:

It is not periodic.
Explanation: The function is continuous and not bounded, in fact:

$$\lim_{|x| \to +\infty} \cosh(x) = +\infty\,,$$

therefore it is not periodic.

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[1] A periodic function of period $P > 0$ is a function f such that $f(x + P) = f(x)$ for all $x \in \mathbb{R}$. The *fundamental period* of a periodic function is the smallest period P.

e) $\cos(x^3)$

Solution:

It is not periodic.
Explanation: The function is differentiable and its derivative is $-3x^2 \sin(x^3)$. If $\cos(x^3)$ were periodic, then also its derivative would be. However, the derivative is not periodic because it is continuous and not bounded (for example, along the sequence $x_n = \sqrt[3]{2n\pi + \pi/2}$ it assumes increasingly bigger values with limit $+\infty$).

f) $\cos(15x) + 3\sin(6x)$

Solution:

It is periodic of fundamental period $2\pi/3$.
Explanation: If $f(x)$ is periodic of period P_1 and $g(x)$ is periodic of period P_2, then their sum $f(x) + g(x)$ is periodic of period the least common multiple

$$P = \mathrm{LCM}(P_1, P_2)$$

of the two periods[2]. In this case $\cos(15x)$ is periodic of fundamental period $2\pi/15$ while $3\sin(6x)$ is periodic of fundamental period $\pi/3$, therefore their sum is periodic of period

$$P = \mathrm{LCM}\left(\frac{2\pi}{15}, \frac{\pi}{3}\right) = \mathrm{LCM}\,(2, 5) \cdot \frac{\pi}{15} = \frac{10}{15}\pi = \frac{2}{3}\pi\,.$$

It is easy to see that no smaller number is a period.

2. Laplace Transform

Find the solution $f = f(t)$ of the following initial value problem:

$$\begin{cases} f''(t) + \omega^2 f(t) = \omega\,\delta(t-a), & t > 0 \\ f(0) = 1, \quad f'(0) = \omega, \end{cases}$$

where $\omega, a > 0$ are positive constants.

Solution:

We apply the Laplace transform to the ODE in the initial value problem. We denote by $F = \mathcal{L}(f)$ the Laplace transform of the function f, and we denote the variable in the new domain by s as usual (so $F = F(s)$).

The first term to transform is the second derivative f'', for which we use the formula:

$$\mathcal{L}(f'') = s^2 F - s f(0) - f'(0) = s^2 F - s - \omega\,.$$

[2] By the *least common muliple* of two real numbers we mean the smallest number P such that there are positive *integer* numbers k_1, k_2 such that $P = k_1 P_1 = k_2 P_2$. In the case that there is no such number, we define it to be $+\infty$ and the consequence is that the function is not periodic.

Then we have $\mathcal{L}(\omega^2 f) = \omega^2 F$ (by linearity) and finally the term in the right-hand side becomes:

$$\mathcal{L}(\omega\,\delta(t-a)) = \omega\mathcal{L}(\delta(t-a)) = \omega \mathrm{e}^{-as}\,.$$

In conclusion the ODE becomes the following algebraic equation:

$$s^2F - s - \omega + \omega^2 F = \omega \mathrm{e}^{-as} \qquad \Longrightarrow \qquad F = \frac{s}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2} + \mathrm{e}^{-as}\cdot\frac{\omega}{s^2+\omega^2}$$

We recognise the first and second term as Laplace transforms of cosine and sine, respectively, while for the third term we can use the t-shifting property, and obtain, applying the inverse Laplace transform:

$$\boxed{f(t) = \mathcal{L}^{-1}(F) = \cos(\omega t) + \sin(\omega t) + u(t-a)\sin\left(\omega(t-a)\right)\,.}$$

3. Fourier Integral

Compute the Fourier integral of the function $f(x) = \mathrm{e}^{-\pi|x|}$.

Solution:

The function $f(x) = \mathrm{e}^{-\pi|x|}$ is an even and continuous function, so its Fourier integral contains only the cosine term and it is equal to the function on each point:

$$\mathrm{e}^{-\pi|x|} = \int_0^{+\infty} A(\omega)\cos(\omega x)\,d\omega\,, \qquad \forall x\in\mathbb{R}\,. \tag{1}$$

We compute the coefficient $A(\omega)$:

$$A(\omega) = \frac{1}{\pi}\int_{\mathbb{R}} f(v)\cos(\omega v)\,dv = \frac{2}{\pi}\int_0^{+\infty} \mathrm{e}^{-\pi v}\cos(\omega v)\,dv =$$

$$\frac{2}{\pi}\left[\frac{\mathrm{e}^{-\pi v}(\omega\sin(\omega v) - \pi\cos(\omega v))}{\omega^2+\pi^2}\right]\Bigg|_{v=0}^{v=+\infty} = \frac{2}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{\omega^2+\pi^2} = \frac{2}{\omega^2+\pi^2}$$

When we insert this result in (1) we obtain that for each $x\in\mathbb{R}$:

$$\boxed{\mathrm{e}^{-\pi|x|} = 2\int_0^{+\infty}\frac{\cos(\omega x)}{\omega^2+\pi^2}\,d\omega\,.}$$

4. Wave Equation with D'Alembert solution

Let $c > 0$. Consider the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x\in\mathbb{R},\ t\geq 0 \\ u(x,0) = e^{-x^2}\sin^2(x) + x, & x\in\mathbb{R} \\ u_t(x,0) = xe^{-x^2}. & x\in\mathbb{R} \end{cases}$$

a) Find the solution $u(x,t)$. You may use D'Alembert formula.
[Simplify the expression as much as possible: no unsolved integrals].

Solution:

D'Alembert's formula for the solution of the wave equation is:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds.$$

With our given initial conditions we get

$$u(x,t) = \frac{1}{2}\left(e^{-(x+ct)^2}\sin^2(x+ct) + x + \cancel{ct} + e^{-(x-ct)^2}\sin^2(x-ct) + x - \cancel{ct}\right) +$$
$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} se^{-s^2} ds =$$
$$= \frac{1}{2}\left(e^{-(x+ct)^2}\sin^2(x+ct) + e^{-(x-ct)^2}\sin^2(x-ct) + 2x\right) +$$
$$+ \frac{1}{2c}\left(-\frac{1}{2}e^{-s^2}\right)\Bigg|_{x-ct}^{x+ct} =$$
$$= \boxed{\frac{1}{2}\left(e^{-(x+ct)^2}\sin^2(x+ct) + e^{-(x-ct)^2}\sin^2(x-ct) + 2x\right) - \frac{1}{4c}\left(e^{-(x+ct)^2} - e^{-(x-ct)^2}\right).}$$

b) For a fixed $a \in \mathbb{R}$, determine the asymptotic limit

$$\lim_{t\to+\infty} u(a,t).$$

Solution:

Let's observe first that, for a fixed $a \in \mathbb{R}$,

$$\lim_{t\to+\infty} e^{-(a\pm ct)^2} = 0,$$

while clearly the terms $\sin^2(a \pm ct)$ are bounded (by 1).
Therefore, between the 5 addends we have in the solution, only the third will contibute to the limit -with limit a- and we get

$$\boxed{\lim_{t\to+\infty} u(a,t) = a.}$$

5. Wave Equation with inhomogeneous boundary conditions

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Find the solution of the following wave equation (**with inhomogeneous boundary conditions**) on the interval $[0,\pi]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t \geq 0,\ x \in [0,\pi], \\ u(0,t) = a, & t \geq 0, \\ u(\pi,t) = b, & t \geq 0, \\ u(x,0) = \frac{b-a}{\pi}x + a, & x \in [0,\pi], \\ u_t(x,0) = x, & x \in [0,\pi], \end{cases} \tag{2}$$

where $a, b > 0$ are positive constants. You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w'' = 0$, $w(0) = a$, and $w(\pi) = b$.

Solution:

The only functions with second derivative zero are the linear functions

$$w(x) = \alpha x + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

Imposing the boundary conditions we find the right coefficients

$$\begin{cases} a = w(0) = \alpha \cdot 0 + \beta \\ b = w(\pi) = \alpha \cdot \pi + \beta \end{cases} \quad \Leftrightarrow \quad \begin{cases} \alpha = \frac{b-a}{\pi} \\ \beta = a \end{cases} \quad \Leftrightarrow \quad \boxed{w(x) = \frac{b-a}{\pi}x + a.}$$

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (2).

Solution:

The PDE doesn't change because w is independent of time and has second derivative in x zero. The boundary conditions become homogeneous (that's why we chose this w)

$$v(0,t) = u(0,t) - w(0) = a - a = 0 \quad \text{and} \quad v(\pi,t) = u(\pi,t) - w(\pi) = b - b = 0.$$

The initial position of the wave changes in

$$v(x,0) = u(x,0) - w(x) = \frac{b-a}{\pi}x + a - \frac{b-a}{\pi}x - a = 0,$$

while the initial speed doesn't change (because, again, w is independent of time). Finally

$$\boxed{\begin{cases} v_{tt} = c^2 v_{xx}, & t \geq 0,\ x \in [0,\pi] \\ v(0,t) = v(\pi,t) = 0, & t \geq 0 \\ v(x,0) = 0, & x \in [0,\pi] \\ v_t(x,0) = x. & x \in [0,\pi] \end{cases}}$$

c) (i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

Solution:

This is a standard homogeneous wave equation with homogeneous boundary conditions. The formula from the script is

$$v(x,t) = \sum_{n=1}^{+\infty} \big(B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)\big) \sin\left(\frac{n\pi}{L}x\right), \quad \lambda_n = \frac{cn\pi}{L}$$
$$\overset{(L=\pi)}{=} \sum_{n=1}^{+\infty} \Big(B_n \cos(cnt) + B_n^* \sin(cnt)\Big) \sin(nx).$$

The coefficients $B_n = 0$, because the initial position is zero, while the coefficients B_n^* are the Fourier series coefficients of the odd, 2π-periodic extension of the initial speed datum $v_t(x,0) = x$, that is:

$$B_n^* = \frac{2}{\pi\lambda_n} \int_0^\pi x \sin(nx)\, dx = \frac{2}{\pi cn} \int_0^\pi x \sin(nx)\, dx$$
$$= \frac{2}{\pi cn} \left(\frac{\sin(nx) - nx\cos(nx)}{n^2} \Bigg|_0^\pi \right) = \frac{2}{\pi cn} \left(\frac{-\pi(-1)^n}{n} \right) = -\frac{2(-1)^n}{cn^2} = \frac{2(-1)^{(n+1)}}{cn^2}$$

Finally we get the following equivalent expressions

$$\boxed{v(x,t) = \sum_{n=1}^{+\infty} B_n^* \sin(cnt)\sin(nx) = \sum_{n=1}^{+\infty} \frac{2(-1)^{(n+1)}}{cn^2} \sin(cnt)\sin(nx).}$$

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (2).

Solution:

We get the following equivalent expressions

$$\boxed{u(x,t) = v(x,t) + w(x) = \left(\sum_{n=1}^{+\infty} \frac{2(-1)^{(n+1)}}{cn^2} \sin(cnt)\sin(nx) \right) + \frac{b-a}{\pi}x + a.}$$

6. Separation of variable

Consider the following time-dependent version of the heat equation on the interval $[0, L]$. We also impose boundary conditions and we look for a solution $u = u(x,t)$ such that:

$$\begin{cases} u_t = t^3 u_{xx}, & x \in [0,L], t \in (0,+\infty), \\ u(0,t) = 0, & t \in [0,+\infty), \\ u(L,t) = 0, & t \in [0,+\infty), \\ u(x,0) = \sin(\frac{3\pi x}{L}) + 2\sin(\frac{\pi x}{L}) & x \in [0,L]. \end{cases}$$

See the next page!

Find the solution $u(x,t)$ using separation of variable. Proceed as in the lecture and adapt the steps if necessary.

Solution:

We use separation of variable $u(x,t) = F(x)G(t)$. The differential equation becomes:

$$F(x)\dot{G}(t) = t^3 F''(x)G(t),$$

which is convenient to rewrite as

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{t^3 G(t)}$$

because it becomes clear that we are comparing a function of x with a function of t, and the only way that this equality might be true is that both these functions are equal and constant:

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{t^3 G(t)} = k, \qquad k \in \mathbb{R}.$$

The boundary conditions are

$$u(0,t) = F(0)G(t) = 0 \quad \text{and} \quad u(L,t) = F(L)G(t) = 0 \quad \forall t \in [0, +\infty)$$

which in order to be true, excluding the trivial solution $G(t) \equiv 0$, become:

$$F(0) = F(L) = 0.$$

In other words the initial PDE with boundary conditions becomes the system of coupled equations

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(L) = 0, \end{cases} \qquad \text{and} \quad \dot{G}(t) = kt^3 G(t).$$

We first solve the system for $F(x)$, distinguishing the cases of k positive, zero, or negative. For $k > 0$ the general solution of the ODE is

$$F(x) = C_1 \mathrm{e}^{\sqrt{k}x} + C_2 \mathrm{e}^{-\sqrt{k}x},$$

which is, however, <u>not</u> compatible with the boundary conditions, in the sense that the only solution of this form satisfying the boundary conditions is the trivial solution: $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_1 + C_2 \quad \Leftrightarrow \quad C_2 = -C_1 \quad \Longrightarrow F(x) = C_1 \left(\mathrm{e}^{\sqrt{k}x} - \mathrm{e}^{-\sqrt{k}x}\right)$$

but then imposing the other condition:

$$0 = F(L) = C_1 \left(\mathrm{e}^{\sqrt{k}L} - \mathrm{e}^{-\sqrt{k}L}\right) \quad \Leftrightarrow \quad \begin{array}{l} \text{either } C_1 = 0 \\ \text{or } \mathrm{e}^{2\sqrt{k}L} = 1 \end{array}$$

which implies $C_1 = 0$ (and consequently $C_2 = -C_1 = 0$) because $2\sqrt{k}L \neq 0$ and therefore its exponential is not 1.

For $k=0$ the general solution is $F(x) = C_1x + C_2$ which is also not compatible with boundary conditions unless $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_2 \qquad \Longrightarrow F(x) = C_1 x$$

and then

$$0 = F(L) = C_1 L \quad \Leftrightarrow \quad C_1 = 0.$$

It remains the case $k < 0$, in which its convenient to write it in the form $k = -p^2$ for positive real number p, and general solutions of $F'' = -p^2 F$ are:

$$F(x) = A\cos(px) + B\sin(px).$$

We impose the boundary conditions:

$$0 = F(0) = A \qquad \Longrightarrow F(x) = B\sin(px)$$

and

$$0 = F(L) = B\sin(pL) \quad \overset{(\text{if } B\neq 0)}{\Leftrightarrow} \quad pL = n\pi, \quad n \in \mathbb{Z}_{\geq 1}$$

<u>Conclusion:</u> we have a nontrivial solution for each $n \geq 1$, $k = k_n = -\frac{n^2\pi^2}{L^2}$:

$$F_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right).$$

The corresponding equation for $G(t)$ is

$$\dot{G} = -t^3\frac{n^2\pi^2}{L^2}G$$

which has general solution

$$G_n(t) = C_n \mathrm{e}^{-\frac{n^2\pi^2}{4L^2}t^4}.$$

The conclusion is that for every $n \geq 1$ we have a solution

$$u_n(x,t) = F_n(x)G_n(t) = A_n \mathrm{e}^{-\frac{n^2\pi^2}{4L^2}t^4}\sin\left(\frac{n\pi}{L}x\right), \qquad \text{with} \quad A_n = B_nC_n.$$

Then by the Superposition Principle, the function

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(x,t) = \sum_{n=1}^{+\infty} A_n \mathrm{e}^{-\frac{n^2\pi^2}{4L^2}t^4}\sin\left(\frac{n\pi}{L}x\right)$$

is also a solution. By imposing the initial condition $u(x,0) = \sin(\frac{3\pi x}{L}) + 2\sin(\frac{\pi x}{L})$, we obtain

$$\sum_{n=1}^{+\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = \sin(\frac{3\pi x}{L}) + 2\sin(\frac{\pi x}{L}).$$

Therefore,

$$A_n = \begin{cases} 2 & \text{if} \quad n=1 \\ 1 & \text{if} \quad n=3 \\ 0 & \text{otherwise.} \end{cases}$$

Hence the final solution is given by,

$$\boxed{u(x,t) = 2\mathrm{e}^{-\frac{\pi^2}{4L^2}t^4}\sin\left(\frac{\pi}{L}x\right) + 1\mathrm{e}^{-\frac{3^2\pi^2}{4L^2}t^4}\sin\left(\frac{3\pi}{L}x\right)}$$

7. Fourier Series

Compute the complex Fourier series of the function $f(x) = 5e^{i\frac{4\pi}{L}x} + x$ on the interval $[-L, L]$.

Solution:

The complex Fourier coefficients for f are, for $n \neq 0$,

$$c_n = \frac{1}{2L}\int_{-L}^{L}\left(5e^{i\frac{4\pi}{L}x}+x\right)e^{-i\frac{n\pi}{L}x}\,dx = \frac{1}{2L}\int_{-L}^{L}5e^{i\frac{4\pi}{L}x}e^{-i\frac{n\pi}{L}x}\,dx+\frac{1}{2L}\int_{-L}^{L}xe^{-i\frac{n\pi}{L}x}\,dx =: I_1+I_2.$$

Then

$$I_1 = \frac{1}{2L}\int_{-L}^{L}5e^{i4x}e^{-i\frac{n\pi}{L}x}\,dx = \begin{cases} 5 & \text{if} \quad n=4 \\ 0 & \text{otherwise.}\end{cases}$$

And

$$\begin{aligned}
I_2 = \frac{1}{2L}\int_{-L}^{L}xe^{-i\frac{n\pi}{L}x}\,dx &= \frac{L}{2\pi^2}\int_{-\pi}^{\pi}ye^{-iny}\,dy \\
&= \frac{L}{2\pi^2}\left(-\frac{y}{in}e^{-iny}\Big|_{-\pi}^{\pi} + \frac{1}{in}\int_{-\pi}^{\pi}e^{-iny}\,dy\right) \\
&= \frac{L}{2\pi^2}\left(-\frac{\pi}{in}e^{-in\pi} - \frac{\pi}{in}e^{in\pi} + \frac{1}{n^2}e^{-iny}\Big|_{-\pi}^{\pi}\right) \\
&= \frac{L}{2\pi^2}\left(-\frac{\pi}{in}e^{-in\pi} - \frac{\pi}{in}e^{in\pi} + \frac{1}{n^2}e^{-in\pi} - \frac{1}{n^2}e^{in\pi}\right) \\
&= \frac{(-1)^nL}{2\pi^2}\left(-\frac{\pi}{in} - \frac{\pi}{in} + \frac{1}{n^2} - \frac{1}{n^2}\right) \\
&= -\frac{(-1)^nL}{in\pi} = i\,\frac{(-1)^nL}{n\pi}.
\end{aligned}$$

Therefore

$$c_n = I_1 + I_2 = \begin{cases} 5 + \frac{iL}{4\pi} & \text{if} \quad n=4 \\ i\,\frac{(-1)^nL}{n\pi} & \text{otherwise}\end{cases}$$

and for $n = 0$,

$$c_0 = \frac{1}{2L}\int_{-L}^{L}\left(5e^{i\frac{4\pi}{L}x} + x\right)dx = \frac{1}{2L}\left(\frac{5L}{4\pi i}e^{i\frac{4\pi}{L}x} + \frac{x^2}{2}\right)\Big|_{-L}^{L} = 0.$$

Therefore the complex Fourier series of f is

$$f(x) = \left(5 + \frac{iL}{4\pi}\right) e^{i\frac{4\pi}{L}x} + \sum_{\substack{n=-\infty \\ n\neq 0 \\ n\neq 4}}^{\infty} i\,\frac{(-1)^n L}{n\pi} e^{i\frac{n\pi}{L}x}.$$

8. Fourier transform

Compute the Fourier transform of the function $f(x) = e^{-ax}u(x-b)$, where $a, b > 0$ are positive constants and u is the Heaviside function.

Solution:

$$\begin{aligned}
\mathcal{F}(f)(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\omega}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ax} u(x-b) e^{-ix\omega}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{b}^{+\infty} e^{-(i\omega+a)x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{-1}{(i\omega+a)} e^{-(i\omega+a)x} \Big|_{b}^{+\infty} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{(i\omega+a)} e^{-(i\omega+a)b}.
\end{aligned}$$

9. Laplace Equation on a rectangle

Find the solution od the following Laplace equation on the rectangle

$$R = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$$

$$\begin{cases}
u_{xx} + u_{yy} = 0\,, & (x, y) \in R \\
u(x, 0) = u(x, 1) = 0\,, & 0 \leq x \leq 1 \\
u(0, y) = 0\,, & 0 \leq y \leq 1 \\
u(1, y) = \sin\left(\pi(1-y)\right)\,. & 0 \leq y \leq 1
\end{cases}$$

You can manipulate appropriately any formula that can be useful from the lecture notes (or, alternatively, solve it via separation of variables from scratch).

Solution 1 (symmetry along the $x = y$ axis):

To solve it we use appropriately a formula learnt in the lecture notes for a similar problem: the Laplace equation on a rectangle with only nonzero boundary function the one on the side of the rectangle parallel to the x axis (while here is the one parallel to the y axis).

More precisely, let $R' = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x \le a, 0 \le y \le b\}$ be a rectangle of sides lengths (a, b) and consider the problem:

$$\begin{cases} v_{xx} + v_{yy} = 0\,, & (x,y) \in R' \\ v(0,y) = v(a,y) = 0\,, & 0 \le y \le b \\ v(x,0) = 0\,, & 0 \le x \le a \\ v(x,b) = f(x)\,. & 0 \le x \le a \end{cases} \tag{3}$$

where $f(x)$ is an arbitrary function with $f(0) = f(a) = 0$. By applying a symmetry along the $x = y$ axis (that is: exchanging x and y) we observe that $v(x, y)$ solves (3) if and only if $u(x, y) := v(y, x)$ solves the problem:

$$\begin{cases} u_{xx} + u_{yy} = 0\,, & (x,y) \in R \\ u(x,0) = u(x,a) = 0\,, & 0 \le x \le b \\ u(0,y) = 0\,, & 0 \le y \le a \\ u(b,y) = f(y)\,. & 0 \le y \le a \end{cases}$$

on the mirrored rectangle $R = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x \le b, 0 \le y \le a\}$. From the lecture notes we know that the general solution to (3) is

$$v(x,y) = \sum_{n=1}^{+\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right)$$

(with coefficient A_n determined by imposing the only nontrivial boundary condition with the function $f(x)$, but for the moment we leave it). In our problem we substitute the values $a = b = 1$, and apply the reflection along the $x = y$ axis to obtain the general solution

$$u(x,y) = v(y,x) = \sum_{n=1}^{+\infty} A_n \sin(n\pi y) \sinh(n\pi x)\,.$$

Now we find the coefficient A_n by imposing the boundary condition. It is useful to observe that the boundary function is nothing but just $f(y) = \sin(\pi(1-y)) = \sin(\pi y)$, so:

$$u(1,y) = \sum_{n=1}^{+\infty} A_n \sin(n\pi y) \sinh(n\pi) = \sin(\pi y)\,.$$

By uniqueness of the Fourier series representation of a function we obtain that the only nonzero coefficient is:

$$A_1 = \frac{1}{\sinh(\pi)},$$

so finally the solution is:

$$\boxed{u(x,y) = \frac{\sinh(\pi x)\sin(\pi y)}{\sinh(\pi)}}$$

Solution 2 (separation of variables from scratch):

We search for particular solutions of the PDE with separated variables $u(x,y) = F(x)G(y)$, for which the PDE becomes:

$$F''G + FG'' = 0 \quad \Leftrightarrow \quad \frac{F''}{F} = -\frac{G''}{G} = k,$$

for some fixed $k \in \mathbb{R}$. The first two boundary conditions $u(x,0) = u(x,1) = 0$ translate into $G(0) = G(1) = 0$, so that:

$$\begin{cases} G'' = -kG\,, \\ G(0) = G(1) = 0\,, \end{cases} \tag{4}$$

while the boundary condition $u(0,y) = 0$ becomes $F(0) = 0$, so that

$$\begin{cases} F'' = kF\,, \\ F(0) = 0\,. \end{cases} \tag{5}$$

We ignore the last boundary condition for the moment, as we are going to use it only at the end. First we want to solve the system (4), which has nontrivial solutions only for $k > 0$. For positive values of k the differential equation has general solution:

$$G(y) = A\cos(\sqrt{k}y) + B\sin(\sqrt{k}y)\,.$$

By imposing $G(0) = 0$ we obtain $A = 0$, while imposing $G(1) = 0$ we obtain that $\sqrt{k} = n\pi$ for some integer number $n \geq 1$. In conclusion we have one solution for each $n \geq 1$, with constant $k = n^2\pi^2$, of the form:

$$G_n(y) = B_n \sin(n\pi y)\,.$$

The corresponding differential equation for F is $F'' = n^2\pi^2 F$, which has general solution:

$$F_n(x) = A_n^* \mathrm{e}^{n\pi x} + B_n^* \mathrm{e}^{-n\pi x}\,.$$

By imposing $F(0) = 0$ we obtain $B_n^* = -A_n^*$, so that $F_n(x) = 2A_n^* \sinh(n\pi x)$. Putting this together with G_n and renaming the constants we obtain a solution of the Laplace equation on this rectangle, for each n:

$$u_n(x,y) = A_n \sinh(n\pi x)\sin(n\pi y)\,,$$

and by the superposition principle a general solution of the form:

$$u(x,y) = \sum_{n=1}^{+\infty} A_n \sinh(n\pi x)\sin(n\pi y)\,.$$

Finally we obtain the coefficients A_n by imposing the last boundary condition that we did not yet consider:

$$u(1,y) = \sum_{n=1}^{+\infty} A_n \sinh(n\pi)\sin(n\pi y) = \sin(\pi(1-y)) \qquad \Longrightarrow$$

$$\Longrightarrow \qquad \begin{cases} A_1 = 1/\sinh(\pi), \\ A_n = 0, \quad n \geq 2 \end{cases}$$

which yields to the solution

$$\boxed{u(x,y) = \frac{\sinh(\pi x)\sin(\pi y)}{\sinh(\pi)}}$$

10. Heat Equation with inhomogeneous boundary conditions

Consider the following problem:

$$\begin{cases} u_t = c^2 u_{xx}, & x \in [0,\pi],\ t \geq 0 \\ u(0,t) = 2, & t \geq 0 \\ u(\pi,t) = 3, & t \geq 0 \\ u(x,0) = f(x), & x \in [0,\pi] \end{cases} \tag{6}$$

where

$$f(x) = \sin(x) - 3\sin(3x) + \frac{x}{\pi} + 2.$$

The boundary conditions are not homogeneous, therefore one cannot directly apply the formulas known. You should argue as follows:

a) Construct a function $w(x)$ with $w(0) = 2$, $w(\pi) = 3$ and $w'' = 0$.

Solution:

The only linear (= second derivative zero) function passing through those points as requested is

$$w(x) = \frac{x}{\pi} + 2.$$

b) Let u be a solution of the above problem (6). State the corresponding problem solved by the function $v(x,t) := u(x,t) - w(x)$.

Solution:

The boundary value problem for v with homogeneous boundary conditions reads as

$$\begin{cases} v_t = c^2 v_{xx}, & x \in [0,\pi],\ t \geq 0 \\ v(0,t) = 0, & t \geq 0 \\ v(\pi,t) = 0, & t \geq 0 \\ v(x,0) = \widetilde{f}(x), & x \in [0,\pi] \end{cases}$$

where

$$\widetilde{f}(x) = f(x) - w(x) = \sin(x) - 3\sin(3x).$$

c) Solve the problem for v using the method of separation of variables from scratch. Show all the steps of the method of separation of variables.

Solution:

Using separation of variables we set $v(x,t) = F(x)G(t)$ and obtain

$$v_t = F\dot{G} \quad \text{and} \quad v_{xx} = F''G$$

which plugged into the PDE give

$$F\dot{G} = c^2 F''G \quad \Leftrightarrow \quad \frac{\dot{G}}{c^2 G} = \frac{F''}{F} = k, \tag{7}$$

where $k \in \mathbb{R}$ is a constant.
The boundary conditions $v(0,t) = v(\pi,t) = 0$, translate into

$$F(0) = F(\pi) = 0.$$

Consequently we first need to solve the following IVP for $F = F(x)$:

$$\begin{cases} F'' = kF, \\ F(0) = F(\pi) = 0. \end{cases}$$

In order to have non trivial solutions we need $k < 0$. In fact for $k = 0$ we get a linear function $F(x) = Ax + B$ which can be zero in two distinct points $(0,\pi)$ only if it's identically zero. While for $k > 0$ the solution is $F(x) = A\mathrm{e}^{\sqrt{k}x} + B\mathrm{e}^{-\sqrt{k}x}$ which again can be zero in the two mentioned distinct points if and only if:

$$\begin{cases} 0 = F(0) = A + B \\ 0 = F(\pi) = A\mathrm{e}^{\sqrt{k}\pi} + B\mathrm{e}^{-\sqrt{k}\pi} \end{cases} \quad \Leftrightarrow \quad A = B = 0$$

For $k < 0$ we can set $k = -p^2$ and general solution is $F(x) = A\cos(px) + B\sin(px)$. The first homogeneous boundary condition $F(0) = 0$ forces $A = 0$ so that

$$F(x) = B\sin(px),$$

and from the second $F(\pi) = 0$:

$$p_n = n, \quad n \geq 1 \quad \text{integer} \quad \rightsquigarrow \quad F_n(x) = B_n \sin(nx).$$

Now we solve the ODE for $G = G(t)$:

$$\dot{G}_n = -c^2 n^2 G =: -\lambda_n^2 G, \qquad \text{where } \lambda_n = cn.$$

The solution is given by

$$G_n(t) = D_n e^{-\lambda_n^2 t}.$$

Consequenly for any $n \geq 1$, we obtain the solution

$$v_n(x,t) = F_n(x)G_n(t) = B_n \sin(nx) D_n e^{-\lambda_n^2 t} =: C_n \sin(nx) e^{-\lambda_n^2 t}.$$

and, by the superposition principle, general solution

$$v(x,t) = \sum_{n=1}^{+\infty} v_n(x,t) = \sum_{n=1}^{+\infty} C_n \sin(nx)\, e^{-\lambda_n^2 t}.$$

We now have to impose the initial condition at time $t = 0$, that is

$$v(x,0) = \sum_{n=1}^{+\infty} C_n \sin(nx) = \sin(x) - 3\sin(3x).$$

In general to solve this we need to find some Fourier series, but this case is particularly simple as the function is already in this form. We obtain

$$\begin{cases} C_1 = 1, \\ C_3 = -3, \\ C_n = 0. \quad n \neq 1, 3 \end{cases}$$

Therefore the solution is

$$v(x,t) = \sin(x)e^{-\lambda_1^2 t} - 3\sin(3x)e^{-\lambda_3^2 t} = \sin(x)e^{-c^2 t} - 3\sin(3x)e^{-9c^2 t}.$$

d) Find the solution u of the original problem (6) .

Solution:

We get

$$\boxed{u(x,t) = v(x,t) + w(x) = \sin(x)e^{-c^2 t} - 3\sin(3x)e^{-9c^2 t} + \frac{x}{\pi} + 2.}$$

11. Laplace equation in an unbounded region

Find the general solution for the following problem:

$$\begin{cases} u_{xx} + u_{yy} = 0, & -\infty \le x \le \infty, 0 \le y, \\ u(x,0) = f(x), & -\infty \le x \le \infty, \end{cases} \tag{8}$$

where $f(x)$ is any arbitrary function.

You must proceed as follows.

a) Show that you can transform the system (8) into

$$\begin{cases} -w^2 \widehat{u}(w,y) + \frac{\partial^2}{\partial y^2}\widehat{u}(w,y) = 0, \\ \widehat{u}(w,0) = \widehat{f}(w). \end{cases} \tag{9}$$

Where $\widehat{u}(w,y)$ denotes the Fourier transform of $u(x,y)$ with respect to the x variable. That is:

$$\widehat{u}(w,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y)e^{-iwx}\, dx.$$

Solution:

We take the Fourier transform with respect to the x variable of the first line, $u_{xx} + u_{yy} = 0$. The right hand side is of course zero. And the left hand side is

$$\mathcal{F}(u_{xx} + u_{yy})(w,y) = \widehat{u_{xx}}(w,y) + \widehat{u_{yy}}(w,y) = -w^2\widehat{u}(w,y) + \frac{\partial^2}{\partial y^2}\widehat{u}(w,y).$$

Where we have used the linearity and the Fourier transform of a derivative. (See properties of the Fourier transform on page 44 in the Lecture notes.)

Then we take the Fourier transform of the second equation, we have

$$\widehat{u}(w,0) = \widehat{f}(w).$$

Therefore the system (8) transforms into

$$\begin{cases} -w^2\widehat{u}(w,y) + \frac{\partial^2}{\partial y^2}\widehat{u}(w,y) = 0, \\ \widehat{u}(w,0) = \widehat{f}(w). \end{cases}$$

That's exactly the system (9).

b) Show that $\widehat{u}(w,y) = \widehat{f}(w)e^{-|w|y}$ is a solution of the system (9). (Where $|w|$ is the absolute value of w).

Solution:

We check by a direct computation that $\widehat{u}(w,y) = \widehat{f}(w)e^{-|w|y}$ is a solution of (9).

$$\frac{\partial^2}{\partial y^2}\widehat{u}(w,y) = \frac{\partial^2}{\partial y^2}\left(\widehat{f}(w)e^{-|w|y}\right) = w^2\widehat{f}(w)e^{-|w|y} = w^2\widehat{u}(w,y).$$

And

$$\widehat{u}(w,0) = \widehat{f}(w)e^{-|w|\cdot 0} = \widehat{f}(w).$$

Hence, $\widehat{f}(w)e^{-|w|y}$ is a solution of (9).

c) Find the solution of the system (8). [Simplify the expression as much as possible: **no more w in your final answer**. Use the properties of the Fourier transform].

[*Hint:* *i)* $\mathcal{F}^{-1}(e^{-|w|y}) = \frac{1}{\sqrt{2\pi}}\frac{2y}{y^2+x^2}$.]

[*Hint:* *ii)* $\widehat{h}(w)\widehat{g}(w) = \frac{1}{\sqrt{2\pi}}\widehat{(h*g)}(w)$.]

Solution:

To find the solution of (8) we have to take the inverse Fourier transform of the solution of (9), i.e. the inverse Fourier transform of $\widehat{f}(w)e^{-|w|y}$. So we compute this inverse Fourier transform using the two hints.

First, using the first hint we have,

$$e^{-|w|y} = \mathcal{F}\Big(\mathcal{F}^{-1}(e^{-|w|y})\Big) \overset{i)}{=} \mathcal{F}\Big(\frac{1}{\sqrt{2\pi}}\frac{2y}{y^2+x^2}\Big)(w). \tag{10}$$

Then using the second hint,

$$\begin{aligned}
u(x,y) = \mathcal{F}^{-1}(\widehat{f}(w)e^{-|w|y}) &\overset{(10)}{=} \mathcal{F}^{-1}\left(\widehat{f}(w)\mathcal{F}\left(\frac{1}{\sqrt{2\pi}}\frac{2y}{y^2+x^2}\right)(w)\right) \\
&\overset{ii)}{=} \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}\left(f * \left(\frac{1}{\sqrt{2\pi}}\frac{2y}{y^2+x^2}\right)\right)(w)\right) \\
&= \mathcal{F}^{-1}\left(\mathcal{F}\left(f * \left(\frac{1}{2\pi}\frac{2y}{y^2+x^2}\right)\right)(w)\right) \\
&= f * \left(\frac{1}{2\pi}\frac{2y}{y^2+x^2}\right) \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x-z)\frac{2y}{y^2+z^2}\,dz.
\end{aligned}$$

Hence the solution of (8) is

$$\boxed{u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x-z)\frac{2y}{y^2+z^2}\,dz.}$$

Remark $\widehat{u}(w,y) = \widehat{f}(w)e^{wy}$ would be also a solution of the system (9). But it's not a bounded solution, that's why we excluded this solution. Indeed, $\widehat{f}(w)e^{wy}$ goes to $+\infty$ when y goes to $+\infty$.

12. Wave Equation

Consider the following 1-dimensional wave equation on the interval $[0, L]$:

$$\begin{cases}
u_{tt} = c^2 u_{xx}\,, & x \in [0,L],\ t \geq 0 \\
u(0,t) = u(L,t) = 0\,, & t \geq 0 \\
u(x,0) = 0\,, & 0 \leq x \leq L \\
u_t(x,0) = x\,, & 0 \leq x \leq L
\end{cases}$$

a) Find the solution in Fourier series. You can use the formula from the lecture notes.

Solution:

The general solution (via Fourier series) of the wave equation on the interval $[0, L]$ with initial data $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ is:

$$u(x,t) = \sum_{n=1}^{+\infty}[B_n\cos(\lambda_n t) + B_n^*\sin(\lambda_n t)]\sin\left(\frac{n\pi}{L}x\right),$$

where:

$$\lambda_n = \frac{cn\pi}{L}\,,$$
$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\,,$$
$$B_n^* = \frac{1}{\lambda_n}\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx\,,$$

Here $f(x) = 0$, so $B_n \equiv 0$, while $g(x) = x$, so :

$$B_n^* = \frac{1}{\lambda_n}\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{1}{\lambda_n}\frac{2}{L}\int_0^L x\sin\left(\frac{n\pi}{L}x\right)dx =$$
$$= \frac{1}{\lambda_n}\frac{2}{L}\cdot\left[\frac{\sin\left(\frac{n\pi}{L}x\right) - \frac{n\pi}{L}x\cos\left(\frac{n\pi}{L}x\right)}{\frac{n^2\pi^2}{L^2}}\right]\Bigg|_{x=0}^{x=L} = \frac{1}{\lambda_n}\frac{2}{L}\cdot\left[-\frac{L^2}{n\pi}\cos(n\pi)\right] =$$
$$= \frac{\cancel{L}}{cn\pi}\cdot\frac{2}{\cancel{L}}\cdot\left[\frac{L^2}{n\pi}(-1)^{n+1}\right] = \frac{(-1)^{n+1}2L^2}{c\pi^2 n^2}\,.$$

So the solution is

$$\boxed{u(x,t) = \frac{2L^2}{c\pi^2}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^2}\sin\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)}$$

b) Remember that the solution can also be written as

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds\,,$$

where g^* is the odd, $2L$-periodic extension of the velocity initial datum g. Use this formula to compute

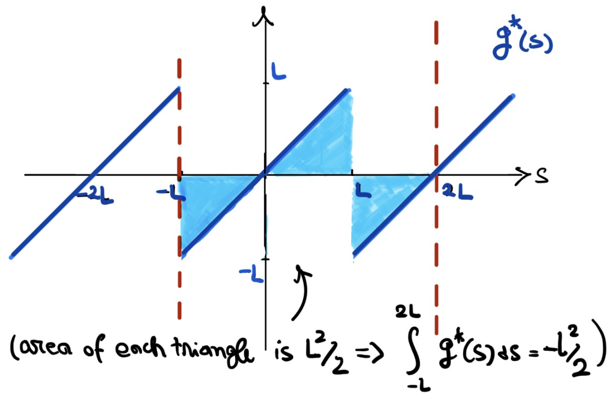
$$u\left(\frac{L}{2},\frac{3L}{2c}\right) = \ ?$$

Solution:

We observe that for $x = L/2$ and $t = 3L/2c$ the interval over which we need to integrate is:

$$[x-ct, x+ct] = [-L, 2L]$$

The odd, $2L$-periodic extension g^* of the initial datum is:



so the desired integral is :

$$\boxed{u\left(\frac{L}{2}, \frac{3L}{2c}\right) = \frac{1}{2c}\int_{-L}^{2L} g^*(s)\,ds = -L^2/4c\,.}$$

c) Compare the result from **b)** with the formula from **a)** evaluated in the point $(x,t) = (L/2, 3L/2c)$ to find the value of the following numerical series:

$$\sum_{m=0}^{+\infty} \frac{1}{(2m+1)^2} = \ ?$$

Solution:

Computing the same value $u(L/2, 3L/2c)$ from the formula obtained in **a)** we obtain:

$$u\left(\frac{L}{2}, \frac{3L}{2c}\right) = \frac{2L^2}{c\pi^2}\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} \sin\left(\frac{3n\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right),$$

and we observe that

$$\sin\left(\frac{3n\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right) = \begin{cases} -1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

so we sum only over indices of the form $n = 2m+1$, with $m = 0, 1, 2, \dots$ (starting at zero because we start from $n = 1$), for which $(-1)^{n+1} = 1$, and we obtain:

$$-\frac{L^2}{4c} \overset{\textbf{b)}}{=} u\left(\frac{L}{2}, \frac{3L}{2c}\right) = -\frac{2L^2}{c\pi^2}\sum_{m=0}^{+\infty} \frac{1}{(2m+1)^2},$$

from which the desired value of the numerical series follows:

$$\boxed{\sum_{m=0}^{+\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}\,.}$$