# Problems and suggested solutions

**Laplace Transforms:** ( $F = \mathcal{L}(f)$ ) ($u$ = Heaviside function, $\delta$ = Dirac's delta function)

| | $f(t)$ | $F(s)$ |
|---|---|---|
| 1) | $1$ | $\frac{1}{s}$ |
| 2) | $t$ | $\frac{1}{s^2}$ |
| 3) | $t^2$ | $\frac{2}{s^3}$ |
| 4) | $t^n,\ n \in \mathbb{Z}_{\geq 0}$ | $\frac{n!}{s^{n+1}}$ |

| | $f(t)$ | $F(s)$ |
|---|---|---|
| 5) | $t^a,\ a > 0$ | $\frac{\Gamma(a+1)}{s^{a+1}}$ |
| 6) | $e^{at}$ | $\frac{1}{s-a}$ |
| 7) | $\cos(\omega t)$ | $\frac{s}{s^2+\omega^2}$ |
| 8) | $\sin(\omega t)$ | $\frac{\omega}{s^2+\omega^2}$ |

| | $f(t)$ | $F(s)$ |
|---|---|---|
| 9) | $\cosh(at)$ | $\frac{s}{s^2-a^2}$ |
| 10) | $\sinh(at)$ | $\frac{a}{s^2-a^2}$ |
| 11) | $u(t-a)g(t-a)$ | $\mathcal{L}(g)e^{-as}$ |
| 12) | $\delta(t-a)$ | $e^{-as}$ |

**Fourier transforms:**

| | $f(x)$ | $\widehat{f}(\omega)$ |
|---|---|---|
| 1) | $e^{-ax^2}$ | $\frac{1}{\sqrt{2a}}e^{\frac{-\omega^2}{4a}}$ |

| | $f(x)$ | $\widehat{f}(\omega)$ |
|---|---|---|
| 2) | $\begin{cases} e^{-ax}, & x \geq 0, \\ 0, & x < 0. \end{cases}$ | $\frac{1}{\sqrt{2\pi}(a+i\omega)}$ |

| | $f(x)$ | $\widehat{f}(\omega)$ |
|---|---|---|
| 3) | $\begin{cases} 1, & \lvert x\rvert < 1, \\ 0, & \lvert x\rvert > 1. \end{cases}$ | $\sqrt{\frac{2}{\pi}}\frac{\sin(\omega)}{\omega}$ |

**Indefinite Integrals:** $\left(n \in \mathbb{Z}_{\geq 1}\right)$

1) $$\int x\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{\cos\left(\frac{n\pi}{L}x\right) + \left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2} \quad (+\text{constant})$$

2) $$\int x^2\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{\left(\left(\frac{n\pi}{L}\right)^2x^2 - 2\right)\sin\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right)x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3} \quad (+\text{constant})$$

3) $$\int x\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{\sin\left(\frac{n\pi}{L}x\right) - \left(\frac{n\pi}{L}\right)x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2} \quad (+\text{constant})$$

4) $$\int x^2\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{\left(2 - \left(\frac{n\pi}{L}\right)^2x^2\right)\cos\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3} \quad (+\text{constant})$$

5) $$\int \frac{1}{1+x^2}\,dx = \arctan(x) \quad (+\text{constant})$$

**You can use these formulas without justification.**

# Question 1

**1.MC1 [3 Points]** Consider the following initial value problem:

$$\begin{cases} f'''(t) = \cos(2t) + e^{-3t}, & t > 0, \\ f(0) = 0 \quad f'(0) = 0, & f''(0) = \pi. \end{cases}$$

Find the Laplace transform $\mathcal{L}(f) = F$ of the function $f$.

(A) $F(s) = \frac{1}{s^2(s^2+2)} + \frac{1}{s^3(s+3)} - \frac{\pi}{s^3}$.

(B) $F(s) = \frac{1}{s(s^2+2)} + \frac{1}{s^2(s+3)} + \frac{\pi}{s^2}$.

(C) $F(s) = \frac{1}{s^2(s^2+4)} + \frac{1}{s^3(s+3)} + \frac{\pi}{s^3}$.

(D) $F(s) = \frac{1}{s^2(s^2+4)} + \frac{1}{s^3(s+3)} - \frac{\pi}{s^3}$.

**Solution:**

(C) The solution is $F(s) = \frac{1}{s^2(s^2+4)} + \frac{1}{s^3(s+3)} + \frac{\pi}{s^3}$.

We apply the Laplace transform to the ODE in the initial value problem. We denote by $F = \mathcal{L}(f)$ the Laplace transform of the function $f$, and we denote the variable in the new domain by $s$ as usual.

The first term to transform is the third derivative $f'''$, for which we use the formula:

$$\mathcal{L}(f''') = s^3F(s) - s^2f(0) - sf'(0) - f''(0) = s^3F(s) - \pi.$$

The term in the right hand side becomes by linearity and using the table of the Laplace transform above,

$$\mathcal{L}(\cos(2t) + e^{-3t}) = \mathcal{L}(\cos(2t)) + \mathcal{L}(e^{-3t}) = \frac{s}{s^2+4} + \frac{1}{s+3}.$$

In conclusion the ODE becomes the following algebraic equation:

$$s^3F(s) - \pi = \frac{s}{s^2+4} + \frac{1}{s+3}.$$

Therefore,

$$F(s) = \frac{1}{s^2(s^2+4)} + \frac{1}{s^3(s+3)} + \frac{\pi}{s^3}$$

**1.MC2 [3 Points]** Find the inverse Laplace transform of

$$F(s) = \left(\frac{2}{s^3} + 1\right) e^{-as} + \frac{1}{s^2 - b^2},$$

where $a$ and $b$ are two positive constants.

(A) $f(t) = u(t-a)(t-a) + \delta(t-a) + \frac{1}{b}\sinh(bt)$.

(B) $f(t) = u(t-a)(t-a)^2 + \delta(t-a) + \frac{1}{b}\sinh(bt)$.

(C) $f(t) = u(t-a)(t-a)^2 + \delta(t+a) + \frac{1}{b^2}\sinh(bt)$.

(D) $f(t) = u(t-a)(t-a)^2 + \delta(t-a) + \sinh(bt)$.

**Solution:**

(B) The solution is $f(t) = u(t-a)(t-a)^2 + \delta(t-a) + \frac{1}{b}\sinh(bt)$.

For the first term we have,

$$\left(\frac{2}{s^3}+1\right)e^{-as} = \frac{2}{s^3}e^{-as} + e^{-as} \implies \mathcal{L}^{-1}\left(\left(\frac{2}{s^3}+1\right)e^{-as}\right) = \mathcal{L}^{-1}\left(\frac{2}{s^3}e^{-as}\right) + \mathcal{L}^{-1}\left(e^{-as}\right)$$
$$= u(t-a)(t-a)^2 + \delta(t-a).$$

And for the second term we have

$$\frac{1}{s^2-b^2} = \frac{1}{b}\frac{b}{s^2-b^2} \implies \mathcal{L}^{-1}\left(\frac{1}{s^2-b^2}\right) = \frac{1}{b}\mathcal{L}^{-1}\left(\frac{b}{s^2-b^2}\right) = \frac{1}{b}\sinh(bt).$$

Hence, the solution is given by

$$f(t) = u(t-a)(t-a)^2 + \delta(t-a) + \frac{1}{b}\sinh(bt).$$

**1.MC3 [3 Points]** Let $f$ be a continuous function such that $\lim_{x\to\infty} f(x) = 0$. Solve the following differential equation using the Fourier transform

$$f''''(x) + 2f''(x) + f(x) = g(x).$$

where $g$ is a given by

$$g(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

(A) $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}(\omega^2+1)^2}\sin(\omega)e^{i\omega x}\,d\omega$.

(B) $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}\omega(\omega^4-\omega^2+1)^2}\sin(\omega)e^{i\omega x}\,d\omega$.

(C) $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}\omega(\omega^4-\omega^2+1)^2}\cos(\omega)e^{i\omega x}\,d\omega$.

(D) $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}\omega(\omega^2-1)^2}\sin(\omega)e^{i\omega x}\,d\omega$.

**Solution:**

(D) The solution is $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}\omega(\omega^2-1)^2}\sin(\omega)e^{i\omega x}\,d\omega$.

We take the Fourier transform on both sides of the equation.

$$\omega^4\widehat{f}(\omega) - 2\omega^2\widehat{f}(\omega) + \widehat{f}(\omega) = \sqrt{\frac{2}{\pi}}\frac{\sin(\omega)}{\omega}.$$

We can rewrite this equality as follow

$$\widehat{f}(\omega) = \frac{1}{\omega^4 - 2\omega^2 + 1}\sqrt{\frac{2}{\pi}}\frac{\sin(\omega)}{\omega} = \frac{\sqrt{2}}{\sqrt{\pi}\omega(\omega^2-1)^2}\sin(\omega).$$

Finally, we take the inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\sqrt{2}}{\sqrt{\pi}\omega(\omega^2-1)^2}\sin(\omega)e^{i\omega x}\ d\omega.$$

**1.MC4 [1 Point]** Determine if the following function is even, odd, or neither and if it is periodic or not.

$$\cos(2\pi x + 5) + \frac{e^{ix} + e^{-ix}}{2}$$

(A) The function is periodic but not even or odd.
(B) The function is even but not periodic.
(C) The function is odd but not periodic.
(D) The function is periodic and even.

**Solution:**

(A) The function is periodic but not even or odd.

We can rewrite the function as follow

$$\cos(2\pi x + 5) + \frac{e^{ix} + e^{-ix}}{2} = \cos(2\pi x + 5) + \cos(x).$$

The sum of two cosine functions is periodic. And the function is not even or odd because

$$\cos(2\pi(-x) + 5) \neq \cos(2\pi x + 5),$$

and

$$\cos(2\pi(-x) + 5) \neq -\cos(2\pi x + 5).$$

**1.MC5 [3 Points]** Let $f$ be a $2\pi$ periodic continuous and **differentiable** function such that $f'(0) = \frac{\pi^6}{945}$. The Fourier series of $f$ on the interval $[-\pi, \pi]$ is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\pi n^7} \sin(\pi n x)$$

Find the value of the numerical series

$$\sum_{n=1}^{\infty} \frac{1}{n^6}.$$

(A) $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.
(B) $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{2\pi^6}{945}$.
(C) $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{n\pi^6}{945}$.
(D) $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} - 1$.

**Solution:**

(A) The solution is $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.

We have

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\pi n^7} \sin(\pi n x).$$

Since $f$ is differentiable and the Series is converging we can differentiate both sides.

$$f'(x) = \sum_{n=1}^{\infty} \frac{\pi n}{\pi n^7} \cos(\pi n x) = \sum_{n=1}^{\infty} \frac{1}{n^6} \cos(\pi n x).$$

We can evaluate both sides at $x = 0$,

$$\frac{\pi^6}{945} = f'(0) = \sum_{n=1}^{\infty} \frac{1}{n^6}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

**1.MC6 [3 Points]** Consider the following PDE (partial differential equation) for the function $u = u(x, y)$:

$$u_{xx} + 2\cos(x)u_{xy} + yu_{yy} - u_x + u_y = \sin(x).$$

Is the PDE hyperbolic, parabolic, elliptic or of mixed type ?

(A) hyperbolic.
(B) parabolic.
(C) elliptic.
(D) mixed type.

**Solution:**

(D) The PDE is of mixed typed because, $A = 1$, $B = \cos(x)$, and $C = y$. Therefore, $AC - B^2 = y - \cos^2(x)$ which changes sign, so the PDE is of mixed type.

**1.MC7 [3 Points]** Wave equation with D'Alembert solution.

Consider the following wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\, t \geq 0, \\ u(x,0) = e^{2x}, & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Find the value of the solution $u$ at position $x = 0$, i.e. $u(0,t)$

(A) $u(0,t) = \cos(2ct)$.
(B) $u(0,t) = \sin(2ct)$.
(C) $u(0,t) = \sinh(2ct)$.
(D) $u(0,t) = \cosh(2ct)$.

**Solution:**

(D) The solution is $u(0,t) = \cosh(2ct)$.

We can use D'Alembert's formula:

$$u(0,t) = \frac{1}{2}\left(f(ct) + f(-ct)\right) + \frac{1}{2c}\int_{-ct}^{ct} g(s)\,ds = \frac{1}{2}\left(e^{2ct} + e^{-2ct}\right) = \cosh(2ct).$$

**1.MC8 [3 Points]** Let $u(x,y) = e^{-((x-1)^2+(y-1)^2)}$. The maximum value of $u$ in the disk of radius 4 centred at 0, denoted by $D_4$, is at the point $(x,y) = (1,1)$. That is

$$\max_{D_4} u(x,y) = u(1,1).$$

Which of the following statements is true?

(A) $u$ is not constant in $D_4$.
(B) $u$ is constant in $D_4$.
(C) The minimum of $u$ is at the point $(x,y) = (-1,-1)$
(D) We cannot conclude that (A), (B) and (C) are true.

**Solution:**

(A) The solution is: $u$ is not constant in $D_4$.

$u$ is not an harmonic function. Indeed,

$$\partial_{xx} u(x,y) = \partial_{xx} e^{-((x-1)^2+(y-1)^2)} = \text{computations} = (4x^2 - 8x + 2)e^{-((x-1)^2+(y-1)^2)},$$

and

$$\partial_{yy} u(x,y) = \partial_{yy} e^{-((x-1)^2+(y-1)^2)} = \text{computations} = (4y^2 - 8y + 2)e^{-((x-1)^2+(y-1)^2)}.$$

Therefore,

$$\partial_{xx} u(x,y) + \partial_{yy} u(x,y) \neq 0.$$

The function $u$ is a Gaussian and therefore it's not constant in $D_4$.

**1.MC9 [3 Points]** Consider the Dirichlet problem for the Heat equation on an infinite bar,

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & x \in \mathbb{R},\ t \geq 0, \\ u(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

(A) Every constant function is a solution.

(B) There is no solution.

(C) The function $u(x,t) = 0$ is a solution.

(D) We cannot conclude that (A), (B) and (C) are true.

**Solution:**

(C) The function $u(x,t) = 0$ is a solution.

The function $u(x,t) = 0$ satisfies the PDE because $u_t = 0$ and $u_{xx} = 0$ and it satisfies the initial condition. Therefore, $u(x,t) = 0$ is a solution.

# Question 2

**2.Q1 [15 Points] Wave Equation with inhomogeneous boundary conditions**

Find the solution of the following wave equation (**with inhomogeneous boundary conditions**) on the interval $[0, \pi]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t \geq 0,\ x \in [0, \pi] \\ u(0,t) = 10\pi^2, & t \geq 0 \\ u(\pi, t) = 10\pi^2, & t \geq 0 \\ u(x,0) = 2\sin(5x) + \sin(4x) + 10\pi^2, & x \in [0, \pi] \\ u_t(x,0) = -4\sin(2x). & x \in [0, \pi] \end{cases} \tag{1}$$

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w''(x) = 0$, $w(0) = 10\pi^2$, and $w(\pi) = 10\pi^2$.

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for $v$, equivalent to (17).

c) (i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (17).

**Solution:**

a) The only functions with second derivative zero are the linear functions

$$w(x) = \alpha x + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

Imposing the boundary conditions we find the right coefficients

$$\begin{cases} 10\pi^2 = w(0) = \alpha \cdot 0 + \beta \\ 10\pi^2 = w(\pi) = \alpha \cdot \pi + \beta \end{cases} \quad \Leftrightarrow \quad \begin{cases} \alpha = 0 \\ \beta = 10\pi^2 \end{cases} \quad \Leftrightarrow \quad w(x) = 10\pi^2.$$

b) The PDE doesn't change because $w$ is independent of time and has second derivative in $x$ zero. The boundary conditions become homogeneous (that's why we chose this $w$)

$$v(0,t) = u(0,t) - w(0) = 10\pi^2 - 10\pi^2 = 0$$
$$v(\pi,t) = u(\pi,t) - w(\pi) = 10\pi^2 - 10\pi^2 = 0.$$

The initial position of the wave changes in

$$\begin{aligned} v(x,0) = u(x,0) - w(x) &= 2\sin(5x) + \sin(4x) + 10\pi^2 - 10\pi^2 \\ &= 2\sin(5x) + \sin(4x), \end{aligned}$$

while the initial speed doesn't change (because, again, $w$ is independent of time). Finally

$$\begin{cases} v_{tt} = c^2 v_{xx}, & t \geq 0,\ x \in [0,\pi] \\ v(0,t) = v(\pi,t) = 0, & t \geq 0 \\ v(x,0) = 2\sin(5x) + \sin(4x), & x \in [0,\pi] \\ v_t(x,0) = -4\sin(2x). & x \in [0,\pi] \end{cases}$$

c) (i) This is a standard homogeneous wave equation with homogeneous boundary conditions. The formula from the script is

$$v(x,t) = \sum_{n=1}^{+\infty} \Big( B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t) \Big) \sin\left(\frac{n\pi}{L} x\right), \quad \lambda_n = \frac{cn\pi}{L}$$

$$\overset{(L=\pi)}{=} \sum_{n=1}^{+\infty} \Big( B_n \cos(cnt) + B_n^* \sin(cnt) \Big) \sin(nx).$$

The coefficients $B_n$ are the Fourier coefficients of the odd, $2\pi$-periodic extension of the initial position datum $v(x,0) = 2\sin(5x) + \sin(4x)$, that is:

$$\sum_{n=1}^{+\infty} B_n \sin(nx) = 2\sin(5x) + \sin(4x).$$

By identifying the term we have, $B_4 = 1$, $B_5 = 2$ and $B_n = 0$ otherwise.
The coefficients $B_n^*$ are the Fourier coefficients of the odd, $2\pi$-periodic extension of the initial speed datum $v_t(x,0) = -4\sin(2x)$, that is:

$$\sum_{n=1}^{+\infty} cnB_n^* \sin(nx) = -4\sin(2x).$$

By identifying the term we have $B_n^* = 0$ for $n \neq 2$ and $c2B_2^* = -4$, therefore $B_2^* = \frac{-2}{c}$.
Finally we get the following solution

$$v(x,t) = -\frac{2}{c}\sin(2ct)\sin(2x) + \cos(4ct)\sin(4x) + 2\cos(5ct)\sin(5x).$$

(ii) The solution $u(x,t)$ of the inhomogeneous problem is

$$u(x,t) = -\frac{2}{c}\sin(2ct)\sin(2x) + \cos(4ct)\sin(4x) + 2\cos(5ct)\sin(5x) + 10\pi^2.$$

# Question 3

**3.Q1 [10 Points] Dirichlet problem for Laplace equation on a rectangle**

Find the general solution $u = u(x, y)$ of the following Laplace equation on a rectangle, with nonzero boundary conditions on one edge of the rectangle:

$$\begin{cases} \nabla^2 u = 0, & 0 \leq x \leq a, 0 \leq y \leq b \\ u(x,0) = u(0,y) = 0, & 0 \leq x \leq a, 0 \leq y \leq b \\ u(x,b) = f(x), & 0 \leq x \leq a \\ u(a,y) = 0, & 0 \leq y \leq b \end{cases}$$

where $f$ is given by

$$f(x) = x^2.$$

You can use the general formula directly to obtain the solution. For this exercise, no points will be given for detailing all the steps of the separation of variable.

**Solution:**

The solution of the above Dirichlet problem is

$$u(x,y) = \sum_{n=1}^{+\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right),$$

where

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi}{a}b\right)} \int_0^a f(s) \sin\left(\frac{n\pi}{a}s\right) ds.$$

We can compute this integral explicitly using $f$,

$$\begin{aligned} A_n &= \frac{2}{a \sinh\left(\frac{n\pi}{a}b\right)} \int_0^a f(s) \sin\left(\frac{n\pi}{a}s\right) ds \\ &= \frac{2}{a \sinh\left(\frac{n\pi}{a}b\right)} \int_0^a s^2 \sin\left(\frac{n\pi}{a}s\right) ds \\ &= \frac{2}{a \sinh\left(\frac{n\pi}{a}b\right)} \frac{\left(2 - \left(\frac{n\pi}{a}\right)^2 s^2\right) \cos\left(\frac{n\pi}{a}s\right) + 2\left(\frac{n\pi}{a}\right) s \sin\left(\frac{n\pi}{a}s\right)}{\left(\frac{n\pi}{a}\right)^3}\Bigg|_0^a \\ &= \frac{2}{a \sinh\left(\frac{n\pi}{a}b\right)} \frac{\left(2 - (n\pi)^2\right)(-1)^n - 2}{\left(\frac{n\pi}{a}\right)^3} \\ &= \frac{2a^2}{\sinh\left(\frac{n\pi}{a}b\right)} \frac{\left(2 - (n\pi)^2\right)(-1)^n - 2}{(n\pi)^3}. \end{aligned}$$

Hence, the solution is

$$u(x,y) = \sum_{n=1}^{+\infty} \left( \frac{2a^2}{\sinh\left(\frac{n\pi}{a}b\right)} \frac{\left(2-(n\pi)^2\right)(-1)^n - 2}{(n\pi)^3} \right) \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right).$$