ETH Zürich | Summer 2017
Prof. Dr. H. M. Soner

Exam Analysis III
D–MAVT, D–MATL

Please fill!

Surname:	
First Name:	
Student Card Nr.:	

Please do not fill!

Exercise	Value	Points	Control
1	8		
2	7		
3	7		
4	17		
5	13		
Total			

Please do not fill!

Completeness	

Important: Before the exam starts, please

- Turn off your mobile phone and place it inside your Briefcase/Backpack, underneath the table.
- Place your Student Card (Legi) on the desk.
- Fill in the front page of the exam with your generalities.

During the exam, please

- Start every exercise on a new piece of paper.
- Put your name on the top right corner of every page.
- You are expected to motivate your answers. Please write down calculations and intermediate results.
- Provide at most **one** solution to each exercise.
- **Do not** write with **pencils**. Please avoid using **red** or **green** ink pens.

Allowed :

- 20 pages (=10 sheets) DIN A4 handwritten or typed personal summary.
- An English dictionary

Not allowed :

No further aids are allowed, such as formula collections (Papula or similar), mobile phones, communication devices or pocket calculators.

Good Luck!

Laplace Transforms:

	$f(t)$	$\mathcal{L}(f)$
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	t^2	$\frac{2!}{s^3}$
4	$t^n,\ n \in \mathbb{N}_0$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$\mathcal{L}(f)$
5	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\frac{1}{s-a}$
7	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$\mathcal{L}(f)$
9	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11	$u(t-a)$	$\frac{e^{-as}}{s}$
12	$\delta(t-a)$	e^{-as}

(Γ=Gamma function, u=Heaviside function, δ=Delta function)

1. Laplace Trasform (*8 Points*)

Let $a > 0$. Solve the following initial value problem using the Laplace transform.

$$\begin{cases} y'(t) + \int_0^t y(\tau)\cos(t-\tau)d\tau = \delta(t-a) & \text{for } t \geq 0 \\ y(0) = 0. \end{cases}$$

Please justify your answer.

2. Fourier Series (*6+1=7 Points*)

a) Given the odd, 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ -(x-\pi) & \frac{\pi}{2} \leq x \leq \pi, \end{cases}$$

determine the real Fourier series of f, i.e. state it's Fourier series and determine the coefficients.

b) Does the Fourier series converge pointwise to the function f? Please justify your answer.

3. D'Alembert (*3+4=7 Points*)

Let $u(x,t)$ be the solution of the one dimensional wave equation with speed of propagation $c > 0$ and the following initial conditions:

$$\begin{cases} u_{tt} = c^2 u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) = \begin{cases} 2\pi - |x| & |x| \leq 2\pi \\ 0 & |x| > 2\pi \end{cases} & x \in \mathbb{R} \\ u_t(x,0) = g(x) = \begin{cases} \cos^2(x) & |x| \leq 2\pi \\ \frac{1}{x^2} & |x| > 2\pi \end{cases} & x \in \mathbb{R} \end{cases}$$

Use D'Alembert's solution to determine

a) $u\left(0, \frac{\pi}{c}\right)$,

b) $\lim\limits_{a\to\infty} u\left(a, \frac{a}{c}\right)$.

4. Heat Equation (*3+12+2=17 Points*)

We consider a homogeneous conducting rod with length $L > 0$ and thermal diffusivity $a > 0$. Its ends are kept on constant temperature 0 and the initial temperature distribution along the rod is given by the function $\sin(\frac{\pi x}{L})$. At time $t = 0$ an electric current is switched on. It flows through the rod and thereby heats up the rod with a constant heating power $b > 0$ per unit length. We study the time evolution of the temperature along the rod.

This problem can be modelled mathematically in the following way. Let $u(x,t)$ denote the temperature of the rod at time t and position x. It is the solution of the following heat equation with boundary (BC) and initial (IC) conditions:

$$\begin{aligned} u_t &= a^2 u_{xx} + b && \text{for } 0 < x < L,\ t > 0 && \text{(PDE)} \\ u(0,t) &= u(L,t) = 0 && \text{for } t \geq 0 && \text{(BC)} \\ u(x,0) &= \sin\left(\frac{\pi x}{L}\right) && \text{for } 0 \leq x \leq L. && \text{(IC)} \end{aligned}$$

This PDE is inhomogeneous. We can reduce it to a homogeneous PDE by determining its stationary solution first.

a) Find the stationary (time independent) solution $v : x \mapsto v(x)$ of the the heat equation equation (PDE) which fulfils the boundary condition (BC).

b) Let $u(x,t) = v(x) + w(x,t)$. Determine the function $w(x,t)$. It is the solution of a homogeneous PDE with corresponding boundary and initial conditions. Formulate the boundary value problem for w (PDE, BC, IC) and solve it. Please justify your answer.

c) Combine the two steps above and state the solution for the temperature along the rod $u(x,t)$.

5. Laplace Equation (*2+9+2=13 Points*)
Let $u : [0,1]\times[0,1] \to \mathbb{R}$ be the solution of the following boundary value problem ($k \in \mathbb{N}$):

$$\begin{cases} \quad \Delta u = 0 & x \in (0,1), y \in (0,1) \\ u(x,0) = 1 - x & x \in [0,1] \\ u(0,y) = 1 - y & y \in [0,1] \\ u(1,y) = y & y \in [0,1] \\ u(x,1) = \sin^3(k\pi x) + x & x \in [0,1]. \end{cases} \tag{1}$$

This problem has four inhomogeneous boundary conditions. In order to reduce it to a problem with only one inhomogeneous boundary condition take the following steps:

a) Find a function $v(x,y)$ of the form $v(x,y) = ax + bxy + cy + d$, which fulfils the first three boundary conditions, i.e. determine the constants a, b, c, d such that v is a solution of

$$\begin{cases} \quad \Delta v = 0 & x \in (0,1), y \in (0,1) \\ v(x,0) = 1 - x & x \in [0,1] \\ v(0,y) = 1 - y & y \in [0,1] \\ v(1,y) = y & y \in [0,1]. \end{cases}$$

b) Use the result from above to reduce the problem to a PDE with only one inhomogeneous boundary condition. Let $u(x,y) = v(x,y) + w(x,y)$. Formulate the boundary value problem (PDE and boundary conditions) for w and solve it. Please justify your answer.
Hint:

$$\sin^3(k\pi x) = \frac{1}{4}(3\sin(k\pi x) - \sin(3k\pi x))$$

c) Combine the two steps above and state the solution u of the initial boundary value problem (1).