Prof. Dr. A. Iozzi ETH Zürich **Spring 2013**

D–MATL, D–MAVT

Exam for Analysis III

Exercises

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1. (*2 points*)

a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function in the Schwartz space $\mathcal{S}(\mathbb{R})$ (the space where the Fourier transform is a bijection) and let us define, for all $k \in \mathbb{R}$,

$$g(k) := \int_{\mathbb{R}} dx\ e^{-ikx} f(x).$$

Find

$$\int_{\mathbb{R}} dk\ e^{iky} g(k),$$

for all $y \in \mathbb{R}$.

b) Let $u = u(x, y)$ and let us consider the partial differential equations

$$u_{xx} + 2u_{xy} + u_{yy} + 3u_x + xu = 0, \tag{1}$$

$$u_{xx} + 2u_{xy} + 2u_{yy} + u_y = 0, \tag{2}$$

$$u_{xx} + 8u_{xy} + 2u_{yy} + e^x u = 0. \tag{3}$$

Which PDE is hyperbolic? Parabolic? Elliptic?

2. (*3 points*) Use the Laplace transform to find the solution $f : [0, \infty[\to \mathbb{R}$ of the ordinary differential equation

$$f''(t) + 4f'(t) + 4f(t) = t^3 e^{-2t}, \qquad f(0) = f'(0) = 0.$$

3. (*4 points*) Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic odd function, defined by extending

$$\forall x \in [0, \pi], \qquad f(x) := \begin{cases} x, & 0 \le x \le \pi/2; \\ \pi - x, & \pi/2 \le x \le \pi. \end{cases}$$

Show that

$$f(x) = \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)^2} (-1)^k \sin((2k+1)x). \tag{4}$$

4. (*4 points*) Find the solution of the system

$$\left\{ \begin{array}{lll} u_t = u_{xx} & 0 < x < \pi,\ t > 0; & (\textbf{PDE}) \\ u(0,t) = \pi, \quad u(\pi,t) = 0 & t \geq 0; & (\textbf{BC}) \\ u(x,0) = \left\{ \begin{array}{ll} \pi, & 0 \leq x \leq \pi/2 \\ -2x + 2\pi, & \pi/2 \leq x \leq \pi \end{array} \right. & & (\textbf{IC}) \end{array} \right. \tag{5}$$

with the following steps:

a) Find a particular solution u_p of **(PDE)+(BC)** that is independent of time.

b) Consider the function $v := u - u_p$, where u is the solution of the system (5) and u_p is the particular solution found in **(a)**. Determine the system (including initial and boundary conditions) of which $v(x,t)$ is a solution.

c) Use the equation (4) in Exercise 3, to compute $v(x,t)$. Find $u(x,t)$.

Hint: The formula for the general solution of **(PDE)** with homogeneous boundary conditions can be used without proof.

5. (*8 points*) We look for the solution $u(x,y)$ of the system

$$\left\{ \begin{array}{lll} u_{xx} + u_{yy} = 0, & 0 < x < \pi,\ 0 < y < \pi; & (\textbf{PDE}) \\ u_x(0,y) = 0 = u_x(\pi,y), & 0 < y < \pi; & (\textbf{BC1}) \\ u(x,0) = 0, & 0 \leq x \leq \pi; & (\textbf{BC2}) \\ u(x,\pi) = \cos(x) + 3\cos(20x), & 0 \leq x \leq \pi. & (\textbf{BC3}) \end{array} \right. \tag{6}$$

a) Assume that $u(x,y) = X(x)Y(y)$. Introduce a separation constant, so that the PDE $u_{xx} + u_{yy} = 0$ can be rewritten as two ordinary differential equations, one for X and one for Y.

b) Use **(BC1)** to find $X(x)$. Consider all possibilities for the separation konstant that appears in the ODE for X.

c) Use **(BC2)** and **b)** to find $Y(y)$.

d) Write all solutions of **(PDE)+(BC1)+(BC2)**. What is the principle according to which

$$u(x,y) = a_0 y + \sum_{n=1}^{\infty} a_n \cos(nx) \sinh(ny)$$

is a solution of **(PDE)+(BC1)+(BC2)**?

e) Use **(BC3)** to determine the solution of the system (6).

6. (*3 points*) Let $u(x,t)$ be the solution of the one-dimensional wave equation

$$\left\{ \begin{array}{ll} u_{tt} = u_{xx} & x \in \mathbb{R},\ t > 0; \\ u(x,0) = f(x) = \left\{ \begin{array}{ll} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{array} \right. & x \in \mathbb{R}; \\ u_t(x,0) = g(x) = \left\{ \begin{array}{ll} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{array} \right. & x \in \mathbb{R}. \end{array} \right.$$

a) Compute $u(0, \frac{1}{2})$ using a well-known formula (from the textbook or the course notes). You can just use the formula without deriving it.

b) Compute $\lim_{t \to \infty} u(x, t)$ for any $x \in \mathbb{R}$.

Hint for a) and b): You need not compute explicitly $u(x, t)$ for every x and t.