ETH zürich

Analysis III
Prof Dr. A. Iozzi
31 January 2023

Laplace Transforms: $(F = \mathcal{L}(f))$ (u = Heaviside function, δ = Dirac's delta function)

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geq 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)g(t-a)$	$\mathcal{L}(g)e^{-as}$
12)	$\delta(t-a)$	e^{-as}

Fourier transforms:

	$f(x)$	$\hat{f}(\omega)$
1)	e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{\frac{-\omega^2}{4a}}$

	$f(x)$	$\hat{f}(\omega)$
2)	$\begin{cases} e^{-ax}, & x \geq 0, \\ 0, & x < 0. \end{cases}$	$\frac{1}{\sqrt{2\pi}(a+i\omega)}$

	$f(x)$	$\hat{f}(\omega)$
3)	$\begin{cases} 1, & \lvert x\rvert < 1, \\ 0, & \lvert x\rvert > 1. \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(\omega)}{\omega}$

Indefinite Integrals: $\left(n \in \mathbb{Z}_{\geq 1}\right)$

1)	$\int x \cos\left(\frac{n\pi}{L}x\right) dx = \frac{\cos\left(\frac{n\pi}{L}x\right) + \left(\frac{n\pi}{L}\right) x \sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}$ (+constant)
2)	$\int x^2 \cos\left(\frac{n\pi}{L}x\right) dx = \frac{\left(\left(\frac{n\pi}{L}\right)^2 x^2 - 2\right) \sin\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right) x \cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3}$ (+constant)
3)	$\int x \sin\left(\frac{n\pi}{L}x\right) dx = \frac{\sin\left(\frac{n\pi}{L}x\right) - \left(\frac{n\pi}{L}\right) x \cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}$ (+constant)
4)	$\int x^2 \sin\left(\frac{n\pi}{L}x\right) dx = \frac{\left(2 - \left(\frac{n\pi}{L}\right)^2 x^2\right) \cos\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right) x \sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3}$ (+constant)
5)	$\int \frac{1}{1+x^2} dx = \arctan(x)$ (+constant)

You can use these formulas without justification.

Question 1

1.MC1 [3 Points] Let f be a solution of the following ordinary differential equation (ODE),

$$\begin{cases} f''(t) + \omega^2 f(t) = 0, \quad t > 0 \\ f(0) = 1, \quad f'(0) = 2\omega, \end{cases}$$

where $\omega > 0$ is a positive constant. Find the Laplace transform $\mathcal{L}(f) = F$ of the function f.

(A) $F(s) = \frac{s}{s^2+\omega^2} + \frac{2\omega}{s^2+\omega^2}$.

(B) $F(s) = \frac{1}{s^2+\omega^2} + \frac{2s\omega}{s^2+\omega^2}$.

(C) $F(s) = \frac{2\omega}{s+\omega^2}$.

(D) $F(s) = \frac{2\omega}{s^2+\omega}$.

1.MC2 [3 Points] Find the inverse Laplace transform of the following function

$$F(s) = \frac{s+2}{s^2 - 10s + 25}.$$

(A) $f(t) = e^{-5t}(1 + 7t)$.

(B) $f(t) = e^{5t}(1 + 7t)$.

(C) $f(t) = e^{5t}(1 - 7t)$.

(D) $f(t) = e^{-5t}(1 - 7t)$.

1.MC3 [3 Points] Let f be a continuous function such that $\lim_{x\to+\infty} f(x) = 0$. Solve the following differential equation using the Fourier transform

$$f(x) + f'(x) + 4f''(x) = \sqrt{2\pi}e^{-\pi x^2}.$$

(A) $f(x) = \int_{-\infty}^{\infty} \frac{1}{1+i\omega-4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{-i\omega x}\, d\omega$.

(B) $f(x) = \int_{-\infty}^{\infty} \frac{1}{1+i\omega+4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega$.

(C) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+i\omega-4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\, d\omega$.

(D) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+i\omega+4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{-i\omega x}\, d\omega$.

1.MC4 [3 Points] The complex Fourier series of the function $\cosh(ax)$ on the interval $[-\pi, \pi)$ is given by

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n a \sinh(a\pi)}{\pi(n^2+a^2)} e^{inx}.$$

Find the value of the numerical series

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2}$$

(A) $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)}$.

(B) $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)} - \frac{1}{2a^2}$.

(C) $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{2\pi}{a\sinh(a\pi)} - \frac{2}{a^2}$.

(D) $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{2\pi}{a\sinh(a\pi)}$.

1.MC5 [3 Points] Determine if the following function is even, odd, or neither and if it is periodic or not. If the function is periodic, determine the fundamental period.

$$15\cos(3x) + 3\cos(4x).$$

(A) The function is even and periodic of fundamental period 2π.

(B) The function is odd and periodic of fundamental period 2π.

(C) The function is even and periodic of fundamental period π.

(D) The function is odd and periodic of fundamental period π.

1.MC6 [3 Points] Consider the following PDE (partial differential equation) for the function $u = u(x, y)$:

$$4u_{xx} + xu_x + 6u_{xy} - yu_y = -7u_{yy} + u_x.$$

Is the PDE hyperbolic, parabolic, elliptic or of mixed type ?

(A) hyperbolic.

(B) elliptic.

(C) parabolic.

(D) mixed type.

1.MC7 [3 Points] Wave equation with D'Alembert solution.

Let $u(x,t)$ be the solution of the following problem

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t>0, \\ u(x,0) = f(x) = \begin{cases} 2, & |x| \leq 14 \\ 0, & |x| > 14 \end{cases} & x \in \mathbb{R}, \\ u_t(x,0) = g(x) = \begin{cases} 1, & 3 \leq x \leq 4 \\ 0, & x \notin [3,4] \end{cases} & x \in \mathbb{R}. \end{cases}$$

Find the values of u at the point $(x,t) = (10,7)$, i.e. find $u(10,7)$

(A) $u(10,7) = 8$.

(B) $u(10,7) = 10$.

(C) $u(10,7) = \frac{3}{2}$.

(D) $u(10,7) = \frac{5}{2}$.

1.MC8 [3 Points] Let $u = u(x,y)$ be a harmonic function in D_2 the disk of radius 2 centred at 0. The maximum value of u is at $(x,y) = (\sqrt{2}, -\sqrt{2})$, i.e. $\max_{D_2} u(x,y) = u(\sqrt{2}, -\sqrt{2})$.

Which of the following statements is true?

(A) u is not constant in D_2.

(B) u is constant in D_2.

(C) There exists another point (x', y') in D_2 such that $u(x,y) = u(x', y')$.

(D) We cannot conclude that (A), (B) and (C) are true for every u.

1.MC9 [3 Points] Consider the Dirichlet problem for the Laplace equation,

$$R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2\},$$

$$\begin{cases} \Delta u = 0, & (x,y) \in R, \\ u(0,y) = u(1,y) = 0, & 0 \leq y \leq 2, \\ u(x,0) = 0, & 0 \leq x \leq 1, \\ u(x,2) = f(x) & 0 \leq x \leq 1, \end{cases}$$

where f is a continuous function.

(A) There are infinitely many solutions.

(B) We cannot say anything.

(C) There is a unique solution.

(D) There is no solution.

Question 2

2.Q1 [15 Points] Heat Equation with inhomogeneous boundary conditions

Find the general solution of the Heat equation (**with inhomogeneous boundary conditions**) for the following problem:

$$(1) \qquad \begin{cases} u_t = c^2 u_{xx}, & 0 \le x \le \pi, t \ge 0, \\ u(0,t) = 5, & t \ge 0, \\ u(\pi, t) = 8, & t \ge 0, \\ u(x,0) = f(x) + w(x), & 0 \le x \le \pi, \end{cases}$$

where w is the function that you have to find in point $a)$ and f is given by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \in [0,\pi), \\ 0 & \text{if} \quad x = \pi. \end{cases}$$

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w'' = 0$, $w(0) = 5$, and $w(\pi) = 8$.

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

c) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

d) Write down explicitly the solution $u(x,t)$ of the original problem (1).

Question 3

3.Q1 [10 Points] Dirichlet problem on a region with symmetries

Find the solution $u(r,\theta)$ of the following Dirichlet problem on the disk of radius R in polar coordinates:

$$\begin{cases} \Delta u = 0, & 0 \le r \le R, 0 \le \theta \le 2\pi, \\ u(R,\theta) = \sin^2(\theta) + 8\cos^3(\theta), & 0 \le \theta \le 2\pi. \end{cases}$$

You should give the answer without unsolved integral and you can use the formulas developed in the lecture.

[*Hint:* Don't try to find the solution in the Poisson integral form.]

[*Hint:* Remember the trigonometric formulas

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta) \qquad \text{and} \qquad \cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta).]$$