D-MAVT
D-MATL

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Analysis III

Exam Solutions

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1. We Laplace transform both sides of the differential equation

$$\mathcal{L}\left[y''(t) + y(t)\right] = s^2 Y(s) - sy(0) - y'(0) + Y(s) = (s^2+1)Y(s)$$

$$\mathcal{L}[u(t-2) + u(t+2)] = \frac{1}{s}(e^{-2s} + e^{2s}).$$

Solving for $Y(s)$ leads to

$$Y(s) = \frac{1}{s(s^2+1)}(e^{-2s} + e^{2s}).$$

A partial fraction decomposition leads us to

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

We transform back

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] = 1 - \cos(t)$$

and use the t-shift property to get the solution

$$y(t) = (1 - cos(t-2))u(t-2) + (1 - cos(t+2))u(t+2)$$

2. a) D'Alembert's formula for the solution of the wave equation is given by

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy.$$

With the given initial conditions and c=2 we get

$$\begin{aligned} u(1,1) &= \frac{1}{2}\left(f(3) + f(-1) + \frac{1}{2}\int_{-1}^{3} g(y)dy\right) \\ &= \frac{1}{2}\left(0 + 3 + \frac{1}{2}\int_{-1}^{3} 1_{|y|\leq 2} dy\right) \\ &= \frac{1}{2}\left(3 + \frac{1}{2}3\right) = \frac{9}{4} \end{aligned}$$

 Please turn!

b)

$$\begin{aligned}\lim_{t\to\infty} u(1,t) &= \lim_{t\to\infty} \frac{1}{2}\left(f(1+2t) + f(1-2t) - \frac{1}{2}\int_{1-2t}^{1+2t} 1_{|y|\le 2} dy\right)\\ &= \lim_{t\to\infty} \frac{1}{4}\int_{1-2t}^{1+2t} 1_{|y|\le 2} dy\\ &= 1\end{aligned}$$

3. Using separation of variables we set $v(x,t) = F(x)G(t)$ and obtain

$$u_t = F(x)\dot{G}(t) \quad \text{and} \quad u_{xx} = F''(x)G(t)$$

which plugged into the PDE gives

$$F(x)\dot{G}(t) = c^2 F''(x)G(t) \quad \Leftrightarrow \quad \frac{\dot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = -k, \tag{1}$$

where k is a constant. Hence the differential equations to solve are

$$\begin{cases} F''(x) + kF(x) = 0, \\ \dot{G}(t) + c^2 kG(t) = 0 \end{cases}$$

Even without boundary conditions we see that if $k < 0$, then

$$\begin{cases} F(x) = Ae^{\sqrt{-k}x} + Be^{-\sqrt{-k}x}, \\ G(t) = e^{-c^2 kt}, \end{cases}$$

from which $u(x,t) = e^{-c^2kt}\left(Ae^{\sqrt{-k}x} + Be^{-\sqrt{-k}x}\right)$ will increase as t increases, which is physically impossible. Thus $k \ge 0$, and we can write $k = p^2$. Then

$$\begin{cases} F_p(x) = A(p)\cos(px) + B(p)\sin(px), \\ G_p(t) = e^{-c^2p^2t}. \end{cases}$$

A generalisation of the Superposition Principle leads to the solution

$$u(x,t) = \int_0^\infty \left(A(p)\cos(px) + B(p)\sin(px)\right) e^{-c^2p^2t}\, dp$$

where

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(pv)\, dv \quad \text{und} \quad B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(pv)\, dv.$$

and $c^2 = 4$.

As f is an even function, we have that $B(p) = 0$. For $A(p)$ we obtain:

$$\begin{aligned} A(p) &= \frac{2}{\pi}\int\limits_0^1 \cosh(v)\cos(pv)\,dv \\ &= \frac{2}{\pi}\sinh(1)\cos(p) + \frac{2p}{\pi}\cosh(1)\sin(p) - p^2 A(p) \end{aligned}$$

Here we partially integrated two times. Hence, we get

$$A(p) = \frac{1}{1+p^2}\frac{2}{\pi}\left(\sinh(1)\cos(p) + p\cosh(1)\sin(p)\right)$$

Consequently the solution is given by

$$u(x,t) = \frac{2}{\pi}\int\limits_0^\infty \frac{1}{1+p^2}\left(\sinh(1)\cos(p) + p\cosh(1)\sin(p)\right)\cos(px)e^{-4p^2t}\,dp.$$

4. We determine the Fourier series for the even extension with period $2L = 4$:

$$f_e(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}x\right)$$

The Fourier coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{L}\int_0^L f(x)dx \\ &= \frac{1}{2}\left(\int_0^1 xdx + 1\right) \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{2L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ &= \int_0^1 x\cos\left(\frac{n\pi}{2}x\right)dx + \int_1^2 \cos\left(\frac{n\pi}{2}x\right)dx \\ &= \frac{4}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) \end{aligned}$$

5. (i) The stationary solution is given by

$$w(x) = \frac{x}{\pi} + 2$$

(ii) The boundary value problem for v with homogeneous boundary conditions reads as

$$\begin{cases} v_t(x,t) = c^2 v_{xx}(x,t), & 0 \le x \le \pi, t \ge 0, \\ v(0,t) = 0, & t \ge 0 \\ v(\pi,t) = 0, & t \ge 0 \\ v(x,0) = x(\pi - x), & 0 \le x \le \pi. \end{cases}$$

(iii) Using separation of variables we set $v(x,t) = F(x)G(t)$ and obtain

$$v_t = F(x)\dot{G}(t) \quad \text{and} \quad u_{xx} = F''(x)G(t)$$

which plugged into the PDE gives

$$F(x)\dot{G}(t) = c^2 F''(x)G(t) \quad \Leftrightarrow \quad \frac{\dot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k, \tag{2}$$

where k is a constant. The boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 0$, as otherwise u would be trivial, translate into

$$F(0) = 0 \quad \text{and} \quad F(\pi) = 0.$$

Consequently we first need to solve the following initial value problem (which is an ODE):

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(\pi) = 0. \end{cases}$$

In order to have non trivial solutions we need $k < 0$. Then the general solution is given by $F(x) = A\cos(\sqrt{-k}x) + B\sin(\sqrt{-k}x)$. Setting $p := \sqrt{-k}$, from the first boundary condition we obtain the requirement that

$$F(x) = B\sin(px),$$

and from the second

$$p_n = n, \quad n \in \mathbb{N} \quad \Rightarrow \quad F_n(x) = B_n \sin(p_n x).$$

Now we solve the ODE for $G(t)$

$$\dot{G}_n(t) = -c^2 p_n^2 G(t) =: -\lambda_n^2 G(t), \quad \lambda_n = cp_n = cn.$$

The solutions are clearly given by

$$G_n(t) = D_n e^{-\lambda_n^2 t}.$$

Consequenly for any n, we obtain the solution

$$u_n(x,t) = F_n(x) \cdot G_n(t) = B_n \sin(nx) \cdot D_n e^{-\lambda_n^2 t} =: C_n \sin(nx) e^{-\lambda_n^2 t}.$$

From the Superpositionprinciple, as the PDE is linear and homogeneous,we obtain that any (uniformly convergent) series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} C_n \sin(nx)\, e^{-\lambda_n^2 t}$$

is a solution of the boundary value problem.
Finally we need to solve for the initial conditions, i.e find the coefficients C_n such that for $x \in [0, \pi]$

$$u(x,0) = \sum_{n=0}^{\infty} C_n \sin(nx) = x(\pi - x)$$

As only terms involving sine appear on the left hand side, the C_n will be the Fourier coefficients of the Fourier series of the odd 2π-periodic extension of $x(\pi - x)$. Let us derive this Fourier series:
We determine the Fourier integral:

$$\begin{aligned}
\int_0^{\pi} x^2 \sin(nx)dx &= x^2 \frac{-1}{n} \cos(nx)\Big|_0^{\pi} - \int_0^{\pi} 2x \frac{-1}{n} \cos(nx)dx \\
&= \frac{-\pi^2}{n}(-1)^n + 2\int_0^{\pi} \frac{-1}{n^2} \sin(nx)dx \\
&= \frac{-\pi^2}{n}(-1)^n + 2\frac{1}{n^3} \cos(nx)\Big|_0^{\pi} \\
&= \frac{-\pi^2}{n}(-1)^n + 2\frac{1}{n^3}((-1)^n - 1).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\int_0^{\pi} x \sin(nx)dx &= x\frac{-1}{n} \cos(nx)|_0^{\pi} - \int_0^{\pi} \frac{-1}{n} \cos(nx)dx \\
&= \frac{-1\pi}{n}(-1)^n.
\end{aligned}$$

Therefore the Fourier series of the odd 2π-periodic extension of $x(\pi - x)$ is given by

$$\sum_{n=1}^{\infty} b_n \sin(nx),$$

with

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\
&= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)dx \\
&= 2\frac{-1\pi}{n}(-1)^n - \frac{2}{\pi}\left(\frac{-\pi^2}{n}(-1)^n + 2\frac{1}{n^3}((-1)^n - 1)\right) \\
&= \frac{4}{n^3\pi}(1 - (-1)^n).
\end{aligned}$$

Hence, we get the solution

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)e^{-\lambda_n^2 t}$$

(iv) Finally the solution for the initial boundary value problem reads as

$$u(x,t) = v(x,t) + \frac{x}{\pi} + 2 = \sum_{n=1}^{\infty} \frac{4}{n^3\pi}(1 - (-1)^n)\sin(nx)e^{-(cn)^2 t} + \frac{x}{\pi} + 2$$