D-MAVT
D-MATL

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Analysis III
Winter Exam Solutions 2022

Exercise	1	2	3	4	5	6	Total
Value	8	10	10	12	15	15	70

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1. Classification of PDEs (*8 Points*)

Definition : The *order* of a PDE is the order of the highest derivative in the PDE.

Consider the following PDEs (in what follows, $u = u(x, y)$ is a function of two variables x and y).

State the order of the PDE and for second order PDE, classify each of them (hyperbolic, parabolic, elliptic, mixed type (you don't need to draw the region in this case)).

a) $u_{xx} + yu_{yy} = \tan(u)$.

b) $u_x + au_y + u^3 = 0$, where $a > 0$ is a positive constant.

c) $y^2 u_{xxx} + (\pi + 2)u_{yy}u_{xx} + u_x = 2u_y + u$.

d) $\pi u_{xx} + 2eu_{xy} + \pi u_{yy} = 0$, where e is the Euler's number ($e \approx 2.718$).

Solution:

A general second order, linear, PDE has the form:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)\,,$$

where A, B, C can be themselves functions of the variables (x, y). The PDE is called hyperbolic, parabolic or elliptic, if the coefficient $AC-B^2$ is, respectively, smaller, equal or greater than zero. When the sign of the coefficient is not constant the equation is of mixed type.

a) (*3 Points*) It's a second order PDE and $AC - B^2 = y$ which changes sign, so the PDE is of mixed type.
b) (*1 Point*) It's a first order PDE.
c) (*1 Point*) It's a third order PDE.
d) (*3 Points*) It's a second order PDE and $AC - B^2 = \pi^2 - e^2 > 0 \implies$ elliptic.

2. Laplace Transform (*10 Points*)

Find the solution $f(t)$ of the following initial value problem:

$$\begin{cases} f''(t) = 3 + u(t-a) - \delta(t-\pi), \quad t > 0, \\ f(0) = b, \quad f'(0) = c, \end{cases}$$

where $a, b, c > 0$ are positive constants.

Solution:

We apply the Laplace transform to the ODE in the initial value problem. We denote by $F = \mathcal{L}(f)$ the Laplace transform of the function f, and we denote the variable in the new domain by s as usual (so $F = F(s)$).

The first term to transform is the second derivative f'', for which we use the formula:

$$\mathcal{L}(f'') = s^2 F - sf(0) - f'(0) = s^2 F - sb - c.$$

The term in the right hand side becomes by linearity and using the table of the Laplace transform above,

$$\begin{aligned} \mathcal{L}(3 + u(t-a) - \delta(t-\pi)) &= \mathcal{L}(3) + \mathcal{L}(u(t-a)) - \mathcal{L}(\delta(t-\pi)) \\ &= \frac{3}{s} + \frac{e^{-as}}{s} - e^{-\pi s}. \end{aligned}$$

In conclusion the ODE becomes the following algebraic equation:

$$s^2 F - sb - c = \frac{3}{s} + \frac{e^{-as}}{s} - e^{-\pi s} \quad \Longrightarrow \quad F = \frac{3}{s^3} + \frac{e^{-as}}{s^3} - \frac{e^{-\pi s}}{s^2} + \frac{b}{s} + \frac{c}{s^2}.$$

The last step is to take the inverse Laplace transform of F. We use the t-shifting property for the second and third term,

$$\begin{aligned} f(t) = \mathcal{L}^{-1}(F) &= \mathcal{L}^{-1}\left(\frac{3}{s^3} + \frac{e^{-as}}{s^3} - \frac{e^{-\pi s}}{s^2} + \frac{b}{s} + \frac{c}{s^2}\right) \\ &= \frac{3}{2}\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(e^{-as}\frac{2}{s^3}\right) - \mathcal{L}^{-1}\left(e^{-\pi s}\frac{1}{s^2}\right) + b\mathcal{L}^{-1}\left(\frac{1}{s}\right) + c\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) \\ &= \frac{3}{2}t^2 + \frac{1}{2}u(t-a)(t-a)^2 - u(t-\pi)(t-\pi) + b + ct. \end{aligned}$$

Hence,

$$\boxed{f(t) = \frac{3}{2}t^2 + \frac{1}{2}u(t-a)(t-a)^2 - u(t-\pi)(t-\pi) + b + ct.}$$

See the next page!

3. Fourier Series (*10 Points*)

Compute the real Fourier series of the function $f(x) = \sin(\frac{5\pi x}{L}) + \cos(\frac{4\pi x}{L}) + |x|$ on the interval $[-L, L]$. Where $|x|$ is the absolute value of x.

$$|x| = \begin{cases} x, & 0 \leq x \leq L, \\ -x, & -L \leq x \leq 0. \end{cases}$$

Solution 1:

We compute the real Fourier coefficients of $f(x)$. The first two terms have already the form of a Fourier series, so we don't need to compute their coefficients.

The function $|x|$ is an even function, therefore the coefficients $b_n = 0$. And we use the formulas given by Theorem 3.10 on page 30 in the Lecture Notes to compute the coefficient a_0 and a_n. Then,

$$a_0 = \frac{1}{L}\int_0^L |x|\ dx = \frac{1}{L}\int_0^L x\ dx = \frac{1}{L}\frac{1}{2}x^2\Big|_{x=0}^{x=L} = \frac{1}{L}\frac{1}{2}L^2 = \frac{1}{2}L.$$

And for a_n we have,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L |x| \cos\left(\frac{n\pi}{L}x\right)\ dx = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi}{L}x\right)\ dx \\ &= \frac{2}{L}\frac{\cos\left(\frac{n\pi}{L}x\right) + \left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}\Bigg|_{x=0}^{x=L} \\ &= \frac{2}{L}\frac{\cos(n\pi)}{\left(\frac{n\pi}{L}\right)^2} - \frac{2}{L}\frac{1}{\left(\frac{n\pi}{L}\right)^2} \\ &= \frac{2L}{n^2\pi^2}((-1)^n - 1). \end{aligned}$$

Hence the Fourier series is

$$\boxed{f(x) = \sin\left(\frac{5\pi x}{L}\right) + \cos\left(\frac{4\pi x}{L}\right) + \frac{1}{2}L + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}((-1)^n - 1)\cos\left(\frac{n\pi}{L}x\right).}$$

If we look at the cases when n is even or odd, we have

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even}, \\ \frac{-4L}{n^2\pi^2}, & \text{if } n \text{ is odd}. \end{cases}$$

Therefore,

$$\boxed{f(x) = \sin\left(\frac{5\pi x}{L}\right) + \cos\left(\frac{4\pi x}{L}\right) + \frac{1}{2}L - \sum_{j=0}^{\infty}\frac{4L}{(2j+1)^2\pi^2}\cos\left(\frac{(2j+1)\pi}{L}x\right).}$$

Solution 2 (Longer solution):

We can also compute the real Fourier coefficient of the function f. In this case, the function f is neither even nor odd. So we can not use the formula given by Theorem 3.10. We use instead the formulas (3.2) and (3.3) on page 26 and 27 in the Lecture Notes and we get

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\ dx = \frac{1}{2L}\int_{-L}^{L}\left(\sin\left(\frac{5\pi x}{L}\right)+\cos\left(\frac{4\pi x}{L}\right)+|x|\right)\ dx \\ &= \frac{1}{2L}\int_{-L}^{L}\sin\left(\frac{5\pi x}{L}\right)\ dx+\frac{1}{2L}\int_{-L}^{L}\cos\left(\frac{4\pi x}{L}\right)\ dx+\frac{1}{2L}\int_{-L}^{L}|x|\ dx \\ &= 0+0+\frac{1}{2L}\int_{-L}^{0}-x\ dx+\frac{1}{2L}\int_{0}^{L}x\ dx \\ &= \frac{1}{2L}\frac{(-1)}{2}x^2\bigg|_{x=-L}^{x=0}+\frac{1}{2L}\frac{1}{2}x^2\bigg|_{x=0}^{x=L} \\ &= \frac{1}{2L}\frac{1}{2}L^2+\frac{1}{2L}\frac{1}{2}L^2=\frac{1}{2}L. \end{aligned}$$

For a_n we have,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\ dx \\ &= \frac{1}{L}\int_{-L}^{L}\left(\sin\left(\frac{5\pi x}{L}\right)+\cos\left(\frac{4\pi x}{L}\right)+|x|\right)\cos\left(\frac{n\pi}{L}x\right)\ dx \\ &= \underbrace{\frac{1}{L}\int_{-L}^{L}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{n\pi}{L}x\right)\ dx}_{I_1}+\underbrace{\frac{1}{L}\int_{-L}^{L}\cos\left(\frac{4\pi x}{L}\right)\cos\left(\frac{n\pi}{L}x\right)\ dx}_{I_2}+\underbrace{\frac{1}{L}\int_{-L}^{L}|x|\cos\left(\frac{n\pi}{L}x\right)\ dx}_{I_3}. \end{aligned}$$

To compute I_1 and I_2 we use the *Properties of Orthogonality of the trigonometric system* on page 24 in the Lecture Notes.

$$I_1 = 0 \qquad \text{and} \qquad I_2 = \begin{cases} 0, & \text{if } n \neq 4, \\ \frac{1}{L}L = 1, & \text{if } n = 4. \end{cases}$$

For I_3 we use that $|x|\cos\left(\frac{n\pi}{L}x\right)$ is an even function.

$$\begin{aligned} I_3 &= \frac{1}{L}\int_{-L}^{L}|x|\cos\left(\frac{n\pi}{L}x\right)\ dx = \frac{2}{L}\int_{0}^{L}x\cos\left(\frac{n\pi}{L}x\right)\ dx \\ &= \frac{2}{L}\frac{\cos\left(\frac{n\pi}{L}x\right)+\left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}\Bigg|_{x=0}^{x=L} \\ &= \frac{2}{L}\frac{\cos(n\pi)}{\left(\frac{n\pi}{L}\right)^2}-\frac{2}{L}\frac{1}{\left(\frac{n\pi}{L}\right)^2} \\ &= \frac{2L}{n^2\pi^2}((-1)^n-1). \end{aligned}$$

 See the next page!

At the end we have for a_n,

$$a_n = \begin{cases} \frac{2L}{n^2\pi^2}((-1)^n - 1), & \text{if } n \neq 4, \\ 1, & \text{if } n = 4, \end{cases}$$

or

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even and } n \neq 4, \\ 1, & \text{if } n = 4, \\ \frac{-4L}{n^2\pi^2}, & \text{if } n \text{ is odd.} \end{cases}$$

Finally we compute b_n,

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx \\ &= \frac{1}{L}\int_{-L}^{L}\left(\sin\left(\frac{5\pi x}{L}\right) + \cos\left(\frac{4\pi x}{L}\right) + |x|\right)\sin\left(\frac{n\pi}{L}x\right)\,dx \\ &= \underbrace{\frac{1}{L}\int_{-L}^{L}\sin\left(\frac{5\pi x}{L}\right)\sin\left(\frac{n\pi}{L}x\right)\,dx}_{J_1} + \underbrace{\frac{1}{L}\int_{-L}^{L}\cos\left(\frac{4\pi x}{L}\right)\sin\left(\frac{n\pi}{L}x\right)\,dx}_{J_2} + \underbrace{\frac{1}{L}\int_{-L}^{L}|x|\sin\left(\frac{n\pi}{L}x\right)\,dx}_{J_3}. \end{aligned}$$

To compute J_1 and J_2 we use the *Properties of Orthogonality of the trigonometric system* on page 24 in the Lecture Notes.

$$J_1 = \begin{cases} 0, & \text{if } n \neq 5, \\ \frac{1}{L}L = 1, & \text{if } n = 5, \end{cases} \qquad \text{and} \qquad J_2 = 0.$$

For I_3 we use that $|x|$ is an even function.

$$I_3 = \frac{1}{L}\int_{-L}^{L}|x|\sin\left(\frac{n\pi}{L}x\right)\,dx = 0.$$

Therefore

$$b_n = \begin{cases} 0, & \text{if } n \neq 5, \\ 1, & \text{if } n = 5. \end{cases}$$

Hence the Fourier series is

$$\boxed{f(x) = \frac{1}{2}L + \sum_{\substack{n=1 \\ n\neq 4}}^{\infty}\frac{2L}{n^2\pi^2}((-1)^n - 1)\cos\left(\frac{n\pi}{L}x\right) + \cos\left(\frac{4\pi x}{L}\right) + \sin\left(\frac{5\pi x}{L}\right).}$$

Please turn!

4. Wave Equation with D'Alembert solution (*12 Points*)

Let $c > 0$. Consider the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\, t \geq 0, \\ u(x,0) = \frac{1}{c}\left((x^2-2)\sin(x) + 2x\cos(x)\right), & x \in \mathbb{R}, \\ u_t(x,0) = x^2\cos(x), & x \in \mathbb{R}. \end{cases}$$

Find the solution $u(x,t)$. You may use D'Alembert formula.
[Simplify the expression up to the point of solving the integral. No further simplification is necessary].

Solution:

D'Alembert's formula for the solution of the wave equation is:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(s)\, ds.$$

With our given initial conditions we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\frac{1}{c}\left(((x+ct)^2-2)\sin(x+ct) + 2(x+ct)\cos(x+ct)\right)\right) \\ &\quad + \frac{1}{2}\left(\frac{1}{c}\left(((x-ct)^2-2)\sin(x-ct) + 2(x-ct)\cos(x-ct)\right)\right) \\ &\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} s^2\cos(s) ds \\ &= \frac{1}{2c}\Big(((x+ct)^2-2)\sin(x+ct) + 2(x+ct)\cos(x+ct)\Big) \\ &\quad + \frac{1}{2c}\Big(((x-ct)^2-2)\sin(x-ct) + 2(x-ct)\cos(x-ct)\Big) \\ &\quad + \frac{1}{2c}\left((s^2-2)\sin(s) + 2s\cos(s)\right)\Bigg|_{x-ct}^{x+ct} \\ &= \frac{1}{2c}\Big(((x+ct)^2-2)\sin(x+ct) + 2(x+ct)\cos(x+ct)\Big) \\ &\quad + \frac{1}{2c}\Big(((x-ct)^2-2)\sin(x-ct) + 2(x-ct)\cos(x-ct)\Big) \\ &\quad + \frac{1}{2c}\Big(((x+ct)^2-2)\sin(x+ct) + 2(x+ct))\cos(x+ct)\Big) \\ &\quad - \frac{1}{2c}\Big(((x-ct)^2-2)\sin(x-ct) + 2(x-ct)\cos(x-ct)\Big) \\ &= \frac{1}{c}\Big(((x+ct)^2-2)\sin(x+ct) + 2(x+ct)\cos(x+ct)\Big). \end{aligned}$$

Hence, the solution is

$$\boxed{u(x,t) = \frac{1}{c}\Big(((x+ct)^2-2)\sin(x+ct) + 2(x+ct)\cos(x+ct)\Big).}$$

5. Heat Equation with inhomogeneous boundary conditions (*15 Points*)

Find the general solution of the Heat equation (**with inhomogeneous boundary conditions**) for the following problem:

$$\begin{cases} u_t = c^2 u_{xx}, & 0 \leq x \leq L, t \geq 0, \\ u(0,t) = 0, & t \geq 0, \\ u(L,t) = L, & t \geq 0, \\ u(x,0) = f(x) + x, & 0 \leq x \leq L, \end{cases} \tag{1}$$

where $L > 0$ is a constant, and $f(x)$ is any (twice differentiable) function such that $f(0) = 0, f(L) = 0$.

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w'' = 0$, $w(0) = 0$, and $w(L) = L$.

Solution:

The only functions with second derivative zero are the linear functions

$$w(x) = \alpha x + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

Imposing the boundary conditions we find the right coefficients

$$\begin{cases} 0 = w(0) = \alpha \cdot 0 + \beta \\ L = w(L) = \alpha \cdot L + \beta \end{cases} \quad \Leftrightarrow \quad \begin{cases} \alpha = \frac{L}{L} = 1 \\ \beta = 0 \end{cases} \quad \Leftrightarrow \quad \boxed{w(x) = x.}$$

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

Solution:

The PDE doesn't change because w is independent of time and has second derivative in x zero. The boundary conditions become homogeneous (that's why we chose this w)

$$v(0,t) = u(0,t) - w(0) = 0 - 0 = 0 \quad \text{and} \quad v(L,t) = u(L,t) - w(L) = L - L = 0.$$

The initial condition becomes

$$v(x,0) = u(x,0) - w(x) = f(x) + x - w(x) = f(x) + x - x = f(x).$$

Finally, the boundary value problem for v with homogeneous boundary conditions reads as

$$\boxed{\begin{cases} v_t = c^2 v_{xx}, & 0 \leq x \leq L,\ t \geq 0, \\ v(0,t) = 0, & t \geq 0, \\ v(\pi,t) = 0, & t \geq 0, \\ v(x,0) = f(x), & 0 \leq x \leq L. \end{cases}}$$

c) The Fourier series of the $2L$ periodic odd extension of f is given by

$$f(x) := \sum_{n=1}^{+\infty} \frac{(4n+2)}{(n^2+\pi n-1)^3} \sin\left(\frac{n\pi}{L}x\right).$$

(i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

Solution:

This is a standard homogeneous heat equation with homogeneous boundary conditions. The formula from the script is

$$v(x,t) = \sum_{n=1}^{+\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t},$$

where $\lambda_n^2 = \left(\frac{cn\pi}{L}\right)^2$. We still need to find the coefficients B_n. Using the initial condition $v(x,0) = f(x)$ we find

$$\sum_{n=1}^{+\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{+\infty} \frac{(4n+2)}{(n^2+\pi n-1)^3} \sin\left(\frac{n\pi}{L}x\right).$$

Therefore, by identifying the terms we have,

$$B_n = \frac{(4n+2)}{(n^2+\pi n-1)^3}.$$

Finally, the solution is

$$\boxed{v(x,t) = \sum_{n=1}^{+\infty} \frac{(4n+2)}{(n^2+\pi n-1)^3} \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}.}$$

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (1).

Solution:

For u we get the following expression

$$\boxed{v(x,t) = \sum_{n=1}^{+\infty} \frac{(4n+2)}{(n^2+\pi n-1)^3} \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t} + x.}$$

6. PDE with Fourier transform (*15 Points*)

Find the solution $u = u(x,t)$ of the following equation using the Fourier transform:

$$\begin{cases} u_x + u_t + u = 0\,, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x)\,, & x \in \mathbb{R}. \end{cases}$$

[*Hint:* *You can proceed as follow:*

a) Take the Fourier transform with respect to the x variable of the PDE and the initial condition and transform them into an ODE.

b) Solve the ODE.

c) Take the inverse Fourier transform of the solution of the ODE to find the solution of the PDE.]

Solution:

We take the Fourier transform with respect to the x variable of the two equations. We use property 1) (linearity) and property 2) (x-derivative) on page 44 of the Lecture Notes and for the the t-derivative, we have the formula on top of page 67. Therefore the PDE gives

$$iw\widehat{u}(w,t) + \frac{\partial \widehat{u}}{\partial t}(w,t) + \widehat{u}(w,t) = 0$$

and the initial condition gives

$$\widehat{u}(w,0) = \widehat{f}(w).$$

That's an ODE in t, we can write it as

$$\boxed{\frac{\partial \widehat{u}}{\partial t}(w,t) = (-iw - 1)\widehat{u}(w,t)}$$

and the solution is

$$\widehat{u}(w,t) = A(w)e^{(-iw-1)t}.$$

The initial condition above implies $A(w) = \widehat{f}(w)$. Therefore

$$\boxed{\widehat{u}(w,t) = e^{-t}\widehat{f}(w)e^{-iwt}.}$$

Finally we take the inverse Fourier transform of this last equation,

$$\begin{aligned}
\mathcal{F}^{-1}(\widehat{u})(x) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \widehat{u}(w,t)e^{ixw}\,dw \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-t}\widehat{f}(w)e^{-iwt}e^{ixw}\,dw \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-t}\widehat{f}(w)e^{i(x-t)w}\,dw \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \Big(\mathcal{F}(e^{-t}f)(w)\Big)e^{i(x-t)w}\,dw \\
&= \mathcal{F}^{-1}\Big(\mathcal{F}(e^{-t}f)\Big)(x-t) \\
&= e^{-t}f(x-t).
\end{aligned}$$

Hence the solution is given by

$$\boxed{u(x,t) = e^{-t}f(x-t).}$$