D-MAVT
D-MATL

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Analysis III

Exam Solutions

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1. (*10 Points*) Solve the initial value problem

$$\begin{cases} y''' + y'' = \delta(t-6), \quad t \geq 0 \\ y(0) = 0 = y'(0) \\ y''(0) = 1 \end{cases}$$

Hint: You may use the identity

$$\frac{1}{s^2(s+1)} = \frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}$$

Solution: Recall the formulas

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0)$$

And

$$\mathcal{L}(y''') = s^3\mathcal{L}(y) - s^2y(0) - sy'(0) - y''(0).$$

Using the initial conditions given in the problem, one imediately gets that the left hand side of the ODE transforms as

$$\mathcal{L}(y'''-y'') = s^3\mathcal{L}(y)-s^2y(0)-sy'(0)-y''(0)+(s^2\mathcal{L}(y)-sy(0)-y'(0)) = s^2(s+1)\mathcal{L}(y)-1$$

The Laplace tranform of the right hand side can be computed using the formula in the cover-page of the exam as

$$\mathcal{L}(\delta(t-6)) = e^{-6s}$$

The transformed ODE thus looks as follows

$$s^2(s-1)\mathcal{L}(y)(s) - 1 = e^{-6s}.$$

This can then be rewritten as

$$\mathcal{L}(y)(s) = (e^{-6s}+1) \cdot \frac{1}{s^2(s+1)}.$$

We use the hint to compute the partial fraction decomposition and we have

$$\frac{1}{s^2(s-1)} = \frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2} = \mathcal{L}(e^{-t}) - \mathcal{L}(1) + \mathcal{L}(t) = \mathcal{L}(e^{-t}+t-1).$$

Plugging this back into the transformed ODE gives

$$\mathcal{L}(y)(s) = (e^{-6s} + 1) \cdot \mathcal{L}(e^{-t} + t - 1) = e^{-6s}\mathcal{L}(e^{-t} + t - 1) + \mathcal{L}(e^{-t} + t - 1),$$

and the second shifting theorem ($t-$shifting) gives that the first term on the right hand can be rewritten as

$$e^{-6s}\mathcal{L}(e^{-t}+t-1) = \mathcal{L}\left((e^{-(t-6)} + (t-6) - 1)u(t-6)\right) = \mathcal{L}\left((e^{-t+6} + t - 7)u(t-6)\right).$$

Plugging this into the previous equation gives

$$\mathcal{L}(y)(s) = (e^{-6s} + 1) \cdot \mathcal{L}(e^{-t} + t - 1) = \mathcal{L}\left((e^{-t+6} + t - 7)u(t-6) + e^{-t} + t - 1\right).$$

Applying the inverse Laplace transform gives the solution

$$y(t) = \left((e^{(-t+6)} + t - 7)u(t-6) + e^{-t} + t - 1\right).$$

An alternative start can be as follows:

First substitute $v := y''$. Then one obtains the following initial value problem for v:

$$\begin{cases} v' + v = \delta(t-6) \\ v(0) = 1 \end{cases}$$

Using the formula

$$\mathcal{L}(v') = s\mathcal{L}(v) - v(0)$$

The ODE transforms as

$$(s+1)\mathcal{L}(v) = e^{-6s} + 1 \iff \mathcal{L}(v) = \frac{e^{-6s}+1}{s+1}$$

Plugging $y'' = v$ in the ODE and using the fomula for the second derivative, with the initial condition, gives

$$\mathcal{L}(v) = \mathcal{L}(y'') = s^2\mathcal{L}(y),$$

and from here the calculation proceeds just as in the first solution.

2. (*10 Points*) Given the 2π periodic function

$$f(x) = \begin{cases} -1 & -\pi < x < 0, \\ 1 & 0 < x < \pi, \end{cases}$$

calculate the Fourier Series of $f(x)$.

Solution: As the function the odd, the Fourier coefficients a_n of the cosine terms in the Fourier expansion of f are zero.

The coefficients of the sine terms can be computed as follows. Let $n \geq 1$, then we have with $2L = 2\pi$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx \\
&= -\frac{2}{n\pi}\cos(nx)\Big|_0^{\pi} \\
&= -\frac{2}{n\pi}\left((-1)^n - 1\right) \\
&= \begin{cases} \frac{4}{n\pi} & \text{if } n = 2j+1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Consequently, the Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} -\frac{2}{n\pi}\left((-1)^n - 1\right)\sin(nx) = \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi}\sin((2j+1)x).$$

3. (*6+4 Points*) Let $u(x,t)$ be the solution of the one dimensional wave Equation.

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) = \begin{cases} \cos(x), & |x| \leq 2\pi \\ 0, & |x| > 2\pi \end{cases}, & x \in \mathbb{R} \\ u_t(x,0) = g(x) = \begin{cases} 0, & |x| \leq 2\pi \\ e^{-x}, & |x| > 2\pi \end{cases}, & x \in \mathbb{R} \end{cases}$$

a) Find $u(0,\pi)$. You may use the D'Alambert solution. *Solution:* The formula for D'Alambert solution is given by

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t) + \int_{x-t}^{x+t} g(y)\,dy\right).$$

Setting $(x,t) = (0,\pi)$ immediately gives

$$u(0,t) = \frac{1}{2}\left(f(0+\pi) + f(0-\pi) + \int_{0-\pi}^{0+\pi} g(y)\,dy\right) = \frac{1}{2}\left(\cos(\pi) + \cos(-\pi) + \int_{-\pi}^{\pi} 0\,dy\right) = -1.$$

b) Find $\lim_{a\to\infty} u(a,a)$.

Solution: Setting $(x,y) = (a,a)$ in D'Alambert's formula, we immediately obtain that

$$\begin{aligned}
\lim_{a\to\infty} u(a,a) &= \lim_{a\to\infty, |a|>2\pi} \frac{1}{2}\left(f(2a) + f(0) + \int_0^{2a} g(y)\,dy\right) \\
&= \lim_{a\to\infty, |a|>2\pi} \frac{1}{2}\left(0 + 1 + \int_0^{2\pi} 0\,dy + \int_{2\pi}^{2a} e^{-y}\,dy\right) \\
&= \frac{1}{2}\left(1 + \int_{2\pi}^{\infty} e^{-y}\,dy\right) \\
&= \frac{1}{2}\left(1 + e^{-2\pi}\right).
\end{aligned}$$

where, as we were considering $a \to \infty$, we assumed without loss of generality that $|2a| > 2\pi$.

4. (*10 Points*)
Consider the string of length $L = \pi$ and $c^2 = 1$ with zero initial velocity, initial deflection $u(x,0) = x(x-\pi)$ and fixed endpoints. The deflection $u(x,t)$ is a solution of the following PDE

$$\begin{cases} u_{tt} = u_{xx}, & \\ u(0,t) = 0 = u(\pi,t), & t \geq 0 \\ u(x,0) = x(x-\pi), & 0 \leq x \leq \pi \\ u_t(x,0) = 0, & 0 \leq x \leq \pi \end{cases}$$

Find $u(x,t)$.

Solution: The initial velocity is 0. From the lecture notes we know that the solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx) \cos(nt),$$

is a solution of the boundary value problem.
We need to solve for the initial conditions, i.e find the coefficients C_n such that

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin(nx) = x(x-\pi)$$

As only Terms in sine appear, it is clear that the C_n will be the coefficients of the odd 2π-periodic extensions of $x(x-\pi)$. These can be computed as follows: for $n \geq 1$ one has

$$\begin{aligned} C_n &= \frac{2}{\pi} \int_0^{\pi} x(x-\pi) \sin(nx)\, dx \\ &= \underbrace{\frac{-2x(x-\pi)}{n\pi} \cos(nx) \Big|_0^{\pi}}_{=0} + \frac{2}{n\pi} \int_0^{\pi} (2x-\pi) \cos(nx)\, dx \\ &= \underbrace{\frac{2(2x-\pi)}{n^2\pi} \sin(nx) \Big|_0^{\pi}}_{=0} - \frac{4}{n^2\pi} \int_0^{\pi} \sin(nx)\, dx \\ &= -\frac{4}{n^3\pi} \cos(nx) \Big|_0^{\pi} \\ &= -\frac{4}{n^3\pi} (\cos(n\pi) - 1) \\ &= \begin{cases} \frac{8}{n^3\pi}, & \text{if } n = 2j+1 \\ 0, & \text{if } n = 2j. \end{cases} \end{aligned}$$

Consequently we obtain that the solution is:

$$\begin{aligned} u(x,t) &= -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(\cos(n\pi)-1)}{n^3}\sin(nx)\cos(nt) \\ &= \frac{8}{\pi}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3}\sin((2j+1)x)\cos((2j+1)t)\,. \end{aligned}$$

5. (*10 Points*) Let $0 < x, y < \pi$. Consider the system

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x, y < \pi \\ u(0,y) = 0 = u(\pi,y), & 0 < y < \pi, \\ u(x,0) = 0, & 0 < x < \pi, \\ u(x,\pi) = 3\sin(2x), & 0 < x < \pi, \end{cases}$$

a) Find the general solution $u(x,y)$ of

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x, y < \pi \\ u(0,y) = 0 = u(\pi,y), & 0 < y < \pi, \\ u(x,0) = 0, & 0 < x < \pi, \end{cases}$$

via a separation of variables argument (please show the details).

Solution: Using separation of variables we set $u(x,y) = F(x)G(y)$ and obtain

$$u_{xx} = F''(x)G(y) \quad \text{and} \quad u_{yy} = F(x)G''(y)$$

which plugged into the PDE gives

$$F''(x)G(y) + F(x)G''(y) = 0 \quad \Leftrightarrow \quad (-1)\cdot\frac{G''(y)}{G(y)} = \frac{F(x)}{F(x)} = k, \tag{1}$$

where k is a constant. The boundary conditions $u(0,y) = 0$ and $u(\pi,y) = 0$, as we are considering $G(y) \not\equiv 0$ as otherwise u would be trivial, translate into

$$F(0) = 0 \quad \text{and} \quad F(\pi) = 0.$$

Consequently we first need to solve the following initial value problem (which is an ODE):

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(\pi) = 0. \end{cases}$$

Simialar ODE's have already been treated various times. One has to run the following case division, depending on sign of k:

$k = 0$: In this case the general solution is given by $F_0(x) = Ax + B$. Plugging in the boundary conditions, gives as only solution $A = 0 = B$, i.e. $F_0(x) = Ax + B = 0$.

$k>0$: In this case the general solution is given by $F(x)=Ae^{\sqrt{k}x}+Be^{-\sqrt{k}x}$. Plugging in the boundary conditions, gives as only solution $A=0=B$, i.e. the trivial solution, which is clearly uninteresting.

$k<0$: In this case the general solution is given by $F(x)=A\cos(\sqrt{-k}x)+B\sin(\sqrt{-k}x)$. From the first boundary condition we obtain the requirement that

$$F(x)=B\sin(px),$$

and from the second $\sqrt{-k}=n, \quad n\in\mathbb{N}$, i.e. all the solutions are of form

$$F_n(x)=A_n\sin(nx).$$

Now we solve the second ODE coming from Equation 1, i.e we solve

$$\begin{cases} G_n(y)=-n^2G(y).,\\ G_n(0)=0. \end{cases}$$

For $n=0$, the general solution is given by

$$G_0(y)=Cy+D,$$

and the boundary condition gives $B=0$, thus

$$G_0(y)=Cy$$

For $n\geq 1$, the solutions are of forms

$$G_n(y)=B_n\sinh(ny)$$

Consequently for $n=0$ we obtain the solution

$$u_0(x,y)=F_0(x)G_0(y)=0$$

and for any $n\geq 1$, we obtain the solution

$$u_n(x,y)=F_n(x)\cdot G_n(y)=A_n\sin(nx)\cdot B_n\sinh(ny)=:C_n\sin(nx)\sinh(ny)\,.$$

From the Superpositionprinciple, as the PDE is linear and homogeneous, we obtain that any (uniformly convergent) series of the form

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y)=\sum_{n=1}^{\infty}C_n\sin(nx)\sinh(ny)\,,$$

is the general solution of the boundary value problem.

b) Find the solution of

$$\begin{cases} u_{xx}+u_{yy}=0, & 0<x,y<\pi\\ u(0,y)=0=u(\pi,y), & 0<y<\pi,\\ u(x,0)=0, & 0<x<\pi,\\ u(x,\pi)=3\sin(2x), & 0<x<\pi, \end{cases}$$

Solution:

The coefficients C_n satisfy

$$u(x, \pi) = \sum_{n=1}^{\infty} C_n \sin(nx) \sinh(n\pi) = 3 \sin(2x).$$

Comparing the coefficients on both sides, immediately gives

$$C_2 = \frac{3}{\sinh(2\pi)}, \quad C_n = 0 \text{ otherwise.}$$

Consequently the solution is given by

$$u(x, y) = \frac{3}{\sinh(2\pi)} \sin(2x) \sinh(2y).$$