Problems and suggested solution

Laplace Transforms: ($F = \mathcal{L}(f)$)

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geq 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)g(t-a)$	$\mathcal{L}(g)e^{-as}$
12)	$\delta(t-a)$	e^{-as}

(Γ = Gamma function, u = Heaviside function, δ = Dirac's delta function)

Fourier transforms:

	$f(x)$	$\widehat{f}(\omega)$
1)	e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{\frac{-\omega^2}{4a}}$

	$f(x)$	$\widehat{f}(\omega)$
2)	$\begin{cases} e^{-ax}, & x \geq 0, \\ 0, & x < 0. \end{cases}$	$\frac{1}{\sqrt{2\pi}(a+i\omega)}$

	$f(x)$	$\widehat{f}(\omega)$
3)	$\begin{cases} 1, & \lvert x\rvert < 1, \\ 0, & \lvert x\rvert > 1. \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(\omega)}{\omega}$

Indefinite Integrals: $\left(n \in \mathbb{Z}_{\geq 1}\right)$

1)	$\displaystyle\int x\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{\cos\left(\frac{n\pi}{L}x\right) + \left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}$ (+constant)
2)	$\displaystyle\int x^2\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{\left(\left(\frac{n\pi}{L}\right)^2x^2 - 2\right)\sin\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right)x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3}$ (+constant)
3)	$\displaystyle\int x\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{\sin\left(\frac{n\pi}{L}x\right) - \left(\frac{n\pi}{L}\right)x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}$ (+constant)
4)	$\displaystyle\int x^2\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{\left(2 - \left(\frac{n\pi}{L}\right)^2x^2\right)\cos\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3}$ (+constant)
5)	$\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x)$ (+constant)

Question 1

1.MC1 [3 Points] Let f be a solution of the following ordinary differential equation (ODE),

$$\begin{cases} f''(t) + \omega^2 f(t) = 0, & t > 0 \\ f(0) = 1, & f'(0) = 2\omega, \end{cases}$$

where $\omega > 0$ is a positive constant. Find the Laplace transform $\mathcal{L}(f) = F$ of the function f.

(A) $F(s) = \frac{s}{s^2+\omega^2} + \frac{2\omega}{s^2+\omega^2}$.
(B) $F(s) = \frac{1}{s^2+\omega^2} + \frac{2s\omega}{s^2+\omega^2}$.
(C) $F(s) = \frac{2\omega}{s+\omega^2}$.
(D) $F(s) = \frac{2\omega}{s^2+\omega}$.

Solution:

The solution is (A) $F(s) = \frac{s}{s^2+\omega^2} + \frac{2\omega}{s^2+\omega^2}$.

We apply the Laplace transform to the ODE in the initial value problem. We denote by $F = \mathcal{L}(f)$ the Laplace transform of the function f, and we denote the variable in the new domain by s as usual (so $F = F(s)$).

The first term to transform is the second derivative f'', for which we use the formula:

$$\mathcal{L}(f'') = s^2 F - s f(0) - f'(0) = s^2 F - s - 2\omega\,.$$

Then we have $\mathcal{L}(\omega^2 f) = \omega^2 F$ (by linearity). In conclusion the ODE becomes the following algebraic equation:

$$s^2 F - s - 2\omega + \omega^2 F = 0 \qquad \Longrightarrow \qquad F = \frac{s}{s^2+\omega^2} + \frac{2\omega}{s^2+\omega^2}.$$

1.MC2 [3 Points] Find the inverse Laplace transform of the following function

$$F(s) = \frac{s+2}{s^2 - 10s + 25}.$$

(A) $f(t) = e^{-5t}(1+7t)$.
(B) $f(t) = e^{5t}(1+7t)$.
(C) $f(t) = e^{5t}(1-7t)$.
(D) $f(t) = e^{-5t}(1-7t)$.

Solution:

The solution is (B) $f(t) = e^{5t}(1+7t)$.

We have

$$F(s) = \frac{s+2}{s^2-10s+25} = \frac{s+2}{(s-5)^2} = \frac{(s-5)+7}{(s-5)^2} = \frac{1}{s-5} + \frac{7}{(s-5)^2}.$$

Therefore,

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s-5} + \frac{7}{(s-5)^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-5}\right) + \mathcal{L}^{-1}\left(\frac{7}{(s-5)^2}\right) = e^{5t} + e^{5t}7t = e^{5t}(1+7t).$$

1.MC3 [3 Points] Let f be a continuous function such that $\lim_{x\to\infty} f(x) = 0$. Solve the following differential equation using the Fourier transform

$$f(x) + f'(x) + 4f''(x) = \sqrt{2\pi}e^{-\pi x^2}.$$

(A) $f(x) = \int_{-\infty}^{\infty} \frac{1}{1+i\omega-4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{-i\omega x}\ d\omega.$

(B) $f(x) = \int_{-\infty}^{\infty} \frac{1}{1+i\omega+4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\ d\omega.$

(C) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+i\omega-4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\ d\omega.$

(D) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+i\omega+4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{-i\omega x}\ d\omega.$

Solution:

The solution is (C) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+i\omega-4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\ d\omega.$

We take the Fourier transform on both sides of the equation. (We use the table above and the property of the Fourier transform for a derivative.)

$$\widehat{f}(\omega) + i\omega\widehat{f}(\omega) - 4\omega^2\widehat{f}(\omega) = \sqrt{2\pi}\frac{1}{\sqrt{2\pi}}e^{-\frac{\omega^2}{4\pi}}.$$

Then we solve this algebraic equation

$$\widehat{f}(\omega) = \frac{1}{1+i\omega-4\omega^2}e^{-\frac{\omega^2}{4\pi}}.$$

Finally, we take the inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{1+i\omega-4\omega^2} e^{-\frac{\omega^2}{4\pi}} e^{i\omega x}\ d\omega.$$

1.MC4 [3 Points] The complex Fourier series of the function $\cosh(ax)$ on the interval $[-\pi, \pi)$ is given by

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n a \sinh(a\pi)}{\pi(n^2+a^2)} e^{inx}.$$

Find the value of the numerical series

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2}$$

(A) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)}$.

(B) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)} - \frac{1}{2a^2}$.

(C) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{2\pi}{a\sinh(a\pi)} - \frac{2}{a^2}$.

(D) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{2\pi}{a\sinh(a\pi)}$.

Solution:

The solution is (B) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)} - \frac{1}{2a^2}$.

The complex Fourier series will be equal to $\cosh(ax)$ for each $x \in [-\pi, \pi]$, or equivalently:

$$\frac{\pi\cosh(ax)}{a\sinh(a\pi)} = \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n^2+a^2} e^{inx}, \qquad \forall x \in [-\pi, \pi]. \tag{1}$$

In particular for $x = 0$ we obtain something very similar to what we need. Observe that the right hand side becomes

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n^2+a^2} \underbrace{e^{in0}}_{=1} = \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \underbrace{\frac{1}{a^2}}_{\substack{n=0 \\ \text{term}}} + 2\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} =: \frac{1}{a^2} + 2S,$$

where S is the sum we wanted to compute. Therefore equation (1) in $x = 0$ becomes:

$$\frac{\pi}{a\sinh(a\pi)} = \frac{1}{a^2} + 2S,$$

from which

$$S = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)} - \frac{1}{2a^2}.$$

1.MC5 [3 Points] Determine if the following function is even, odd, or neither and if it is periodic or not. If the function is periodic, determine the fundamental period.

$$15\cos(3x) + 3\cos(4x).$$

(A) The function is even and periodic of fundamental period 2π.

(B) The function is odd and periodic of fundamental period 2π.

(C) The function is even and periodic of fundamental period π.

(D) The function is odd and periodic of fundamental period π.

Solution:

(A) The function is even and periodic of fundamental period 2π.

First,

$$15\cos(3(-x)) + 3\cos(4(-x)) = 15\cos(-3x) + 3\cos(-4x) = 15\cos(3x) + 3\cos(4x).$$

Therefore, the function is even.

If $f(x)$ is periodic of period P_1 and $g(x)$ is periodic of period P_2, then their sum $f(x) + g(x)$ is periodic of period the least common multiple $P = \mathrm{LCM}(P_1, P_2)$ of the two periods.

In this case, $15\cos(3x)$ is periodic of fundamental period $\frac{2\pi}{3}$ while $3\cos(4x)$ is periodic of fundamental period $\frac{2\pi}{4}$, therefore their sum is periodic of period

$P = \mathrm{LCM}\left(\frac{2\pi}{3}, \frac{\pi}{2}\right) = \mathrm{LCM}(4, 3) \cdot \frac{\pi}{6} = \frac{12}{6}\pi = 2\pi$

It is easy to see that no smaller number is a period.

1.MC6 [3 Points] Consider the following PDE (partial differential equation) for the function $u = u(x, y)$:

$$4u_{xx} + xu_x + 6u_{xy} - yu_y = -7u_{yy} + u_x.$$

Is the PDE hyperbolic, parabolic, elliptic or of mixed type ?

(A) hyperbolic.

(B) elliptic.

(C) parabolic.

(D) mixed type.

Solution:

(B) The PDE is elliptic because, $A = 4$, $B = 3$, and $C = 7$ therefore,

$$AC - B^2 = 4 \cdot 7 - 9 = 19 > 0.$$

The sign of the coefficient is positive, therefore the PDE is elliptic.

1.MC7 [3 Points] Wave equation with D'Alembert solution.

Let $u(x,t)$ be the solution of the following problem

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x) = \begin{cases} 2, & |x| \leq 14 \\ 0, & |x| > 14 \end{cases} & x \in \mathbb{R}, \\ u_t(x,0) = g(x) = \begin{cases} 1, & 3 \leq x \leq 4 \\ 0, & x \notin [3,4] \end{cases} & x \in \mathbb{R}. \end{cases}$$

Find the values of u at the point $(x,t) = (10,7)$, i.e. find $u(10,7)$

(A) $u(10,7) = 8$.

(B) $u(10,7) = 10$.

(C) $u(10,7) = \frac{3}{2}$.

(D) $u(10,7) = \frac{5}{2}$.

Solution:

The solution is (C) $u(10,7) = \frac{3}{2}$.

D'Alembert's formula for the solution of the wave equation is:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds.$$

With our given initial conditions, $c = 1$, $x = 10$, and $t = 7$, we get

$$\begin{aligned} u(10,7) &= \frac{1}{2}\left(f(17) + f(3)\right) + \frac{1}{2}\int_3^{17} g(s) ds \\ &= \frac{1}{2}(0+2) + \frac{1}{2}\int_3^4 1 ds \\ &= 1 + \frac{1}{2} \\ &= \frac{3}{2}. \end{aligned}$$

1.MC8 [3 Points] Let $u = u(x,y)$ be a harmonic function in D_2 the disk of radius 2 centred at 0. The maximum value of u is at $(x,y) = (\sqrt{2}, -\sqrt{2})$, i.e. $\max_{D_2} u(x,y) = u(\sqrt{2}, -\sqrt{2})$.

Which of the following statements is true?

(A) u is not constant in D_2.

(B) u is constant in D_2.

(C) There exists another point (x', y') in D_2 such that $u(x,y) = u(x', y')$.

(D) We cannot conclude that (A), (B) and (C) are true for every u.

Solution:

The solution is (D): We cannot conclude that (A), (B) and (C) are true.

The point $(x, y) = (\sqrt{2}, -\sqrt{2})$ is on the boundary of D_2.

Therefore, the maximum of u is taken on the boundary. Hence, we cannot conclude anything about u.

1.MC9 [3 Points] Consider the Neumann problem for the following PDE,

$$\begin{cases} \nabla^2 u = f, & \text{in } D_2, \\ \frac{\partial u}{\partial n} = g, & \text{on } \partial D_2, \end{cases}$$

with D_2 the disk of radius 2 centred at 0 and f and g are two given functions such that

$$\int_{D_2} f(x)\ dx = 2, \qquad \text{and} \qquad \int_{\partial D_2} g(x)\ dx = 2.$$

Which of the following is true:

(A) There are infinitely many solutions.

(B) There is no solution.

(C) There are two solutions.

(D) We cannot conclude that (A), (B), or (C) are true.

Solution:

We cannot conclude that (A), (B), or (C) are true.

Indeed, let's assume that $u = u(x)$ is a solution of the PDE. (With $x \in D_1$.) Then we integrate the PDE on D_2 and use the divergence Theorem.

$$\begin{aligned} &\int_{D_2} \nabla^2 u(x)\ dx = \int_{D_2} f(x)\ dx \iff \int_{D_2} \operatorname{div}(\nabla u(x))\ dx = \int_{D_2} f(x)\ dx \\ \iff &\int_{\partial D_2} \nabla u(x) \cdot n\ dx = \int_{D_2} f(x)\ dx \iff \int_{\partial D_2} \frac{\partial u}{\partial n}\ dx = \int_{D_2} f(x)\ dx \\ &\iff \int_{\partial D_2} g(x)\ dx = \int_{D_2} f(x)\ dx \iff 2 = 2. \end{aligned}$$

Therefore, we can not conclude anything.

Question 2

2.Q1 [15 Points] Separation of variables for the Heat equation

Consider the following time-dependent version of the Heat equation on the interval $[0,1]$. We also impose boundary conditions and we look for a solution $u = u(x,t)$ such that:

$$\begin{cases} u_t(x,t) = a(t)u_{xx}(x,t), & x \in [0,1], t \in [0,+\infty), \\ u(0,t) = 0, & t \in [0,+\infty), \\ u(1,t) = 0, & t \in [0,+\infty), \\ u(x,0) = 2\sin(3\pi x) + \sin(7\pi x), & x \in [0,1], \end{cases}$$

where $a(t)$ is a given continuous function. Find the solution $u(x,t)$ using separation of variable. Proceed as in the lecture and adapt the steps if necessary.

Solution:

We use separation of variable $u(x,t) = F(x)G(t)$. The differential equation becomes:

$$F(x)\dot{G}(t) = a(t)F''(x)G(t),$$

which is convenient to rewrite as

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{a(t)G(t)}$$

because it becomes clear that we are comparing a function of x with a function of t, and the only way that this equality might be true is that both these functions are equal and constant:

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{a(t)G(t)} = k, \qquad k \in \mathbb{R}.$$

The boundary conditions are

$$u(0,t) = F(0)G(t) = 0 \quad \text{and} \quad u(1,t) = F(1)G(t) = 0 \quad \forall t \in [0,+\infty)$$

which in order to be true, excluding the trivial solution $G(t) \equiv 0$, become:

$$F(0) = F(1) = 0.$$

In other words the initial PDE with boundary conditions becomes the system of coupled equations

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(1) = 0, \end{cases} \qquad \text{and} \quad \dot{G}(t) = ka(t)G(t).$$

We first solve the system for $F(x)$, distinguishing the cases of k positive, zero, or negative. For $k > 0$ the general solution of the ODE is

$$F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x},$$

which is, however, not compatible with the boundary conditions, in the sense that the only solution of this form satisfying the boundary conditions is the trivial solution: $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_1 + C_2 \quad \Leftrightarrow \quad C_2 = -C_1 \quad \Longrightarrow F(x) = C_1 \left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right)$$

but then imposing the other condition:

$$0 = F(1) = C_1 \left(e^{\sqrt{k}} - e^{-\sqrt{k}}\right) \quad \Leftrightarrow \quad \begin{array}{l} \text{either } C_1 = 0 \\ \text{or } e^{2\sqrt{k}} = 1 \end{array}$$

which implies $C_1 = 0$ (and consequently $C_2 = -C_1 = 0$) because $2\sqrt{k} \neq 0$ and therefore its exponential is not 1.

For $k = 0$ the general solution is $F(x) = C_1 x + C_2$ which is also not compatible with boundary conditions unless $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_2 \quad \Longrightarrow F(x) = C_1 x$$

and then

$$0 = F(1) = C_1 \quad \Leftrightarrow \quad C_1 = 0.$$

It remains the case $k < 0$, in which its convenient to write it in the form $k = -p^2$ for positive real number p, and general solutions of $F'' = -p^2 F$ are:

$$F(x) = A\cos(px) + B\sin(px).$$

We impose the boundary conditions:

$$0 = F(0) = A \quad \Longrightarrow F(x) = B\sin(px)$$

and

$$0 = F(1) = B\sin(p) \quad \overset{(\text{if } B \neq 0)}{\Leftrightarrow} \quad p = n\pi, \quad n \in \mathbb{Z}_{\geq 1}$$

Conclusion: we have a non-trivial solution for each $n \geq 1$, $k = k_n = -n^2\pi^2$:

$$F_n(x) = B_n \sin(n\pi x).$$

The corresponding equation for $G(t)$ is

$$\dot{G}(t) = -a(t) n^2 \pi^2 G(t)$$

which has general solution[a]

$$G_n(t) = C_n e^{-n^2\pi^2 \int a(s)\, ds} = C_n e^{-n^2\pi^2 A(t)},$$

where

$$A(t) = \int a(s)\, ds.$$

The conclusion is that for every $n \geq 1$ we have a solution

$$u_n(x,t) = F_n(x)G_n(t) = A_n e^{-n^2\pi^2 A(t)} \sin(n\pi x), \qquad \text{with} \quad A_n = B_n C_n.$$

Then by the Superposition Principle, the function

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 A(t)} \sin(n\pi x)$$

is also a solution. By imposing the initial condition $u(x,0) = f(x)$, we have

$$u(x,0) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 A(0)} \sin(n\pi x) = f(x).$$

Therefore,

$$\sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 A(0)} \sin(n\pi x) = 2\sin(3\pi x) + \sin(7\pi x).$$

By comparing the coefficient we get $A_3 e^{-3^2\pi^2 A(0)} = 2$ and $A_7 e^{-7^2\pi^2 A(0)} = 1$ and $A_n = 0$ otherwise. Therefore, we have $A_3 = \frac{2}{e^{-3^2\pi^2 A(0)}}$ and $A_7 = \frac{1}{e^{-7^2\pi^2 A(0)}}$. Hence the final solution is given by,

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 A(t)} \sin(n\pi x) \\ &= \frac{2}{e^{-3^2\pi^2 A(0)}} e^{-3^2\pi^2 A(t)} \sin(3\pi x) + \frac{1}{e^{-7^2\pi^2 A(0)}} e^{-7^2\pi^2 A(t)} \sin(7\pi x). \end{aligned}$$

[a] A homogeneous linear differential equation of first order with variable coefficients is an equation of the form

$$y'(t) = a(t)y(t),$$

where $a = a(t)$ possibly varies with t. As it is easy to check, the general solution is given by

$$y(t) = ce^{A(t)}, \quad c \in \mathbb{R},$$

where $A(t)$ is any primitive of $a(t)$:

$$A(t) = \int a(s)\, ds.$$

Question 3

3.Q1 [10 Points] Dirichlet problem on a region with symmetries

Find the solution $u(r,\theta)$ of the following Dirichlet problem on the disk of radius R in polar coordinates:

$$\begin{cases} \nabla^2 u = 0, & 0 \leq r \leq R, 0 \leq \theta \leq 2\pi, \\ u(R,\theta) = \sin^2(\theta) + 8\cos^3(\theta), & 0 \leq \theta \leq 2\pi. \end{cases}$$

You should give the answer without unsolved integral and you can use the formulas developed in the lecture.

[*Hint:* Don't try to find the solution in the Poisson integral form.]

[*Hint:* Remember the trigonometric formulas

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta) \qquad \text{and} \qquad \cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta).\,]$$

Solution:

The solution will be

$$u(r,\theta) = \sum_{n=0}^{+\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right),$$

with coefficients found imposing

$$u(R,\theta) = \sum_{n=0}^{+\infty} R^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) = \sin^2(\theta) + 8\cos^3(\theta).$$

Using the trigonometric formulas

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta) \qquad \text{and} \qquad 8\cos^3(\theta) = 6\cos(\theta) + 2\cos(3\theta),$$

we have,

$$u(R,\theta) = \sum_{n=0}^{+\infty} R^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) = \frac{1}{2} + 6\cos(\theta) - \frac{1}{2}\cos(2\theta) + 2\cos(3\theta).$$

We obtain coefficients

$$\begin{cases} B_n = 0, & \forall n \geq 0, \\ A_n = 0, & \forall n \geq 0,\ n \neq 0, 1, 2, \text{ and } 3, \\ A_0 = \frac{1}{2}, \\ A_1 = \frac{6}{R}, \\ A_2 = -\frac{1}{2R^2}, \\ A_3 = \frac{2}{R^3}. \end{cases}$$

Finally

$$u(r,\theta) = \frac{1}{2} + \frac{6r}{R}\cos(\theta) - \frac{r^2}{2R^2}\cos(2\theta) + \frac{2r^3}{R^3}\cos(3\theta).$$