D–MAVT, D–MATL

Exam Analysis III

401-0363-10L

Last Name

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Please take note of the information on the answer-booklet.

Laplace Transforms: ($F = \mathcal{L}(f)$) (u = Heaviside function, δ = Dirac's delta function)

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geq 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)g(t-a)$	$\mathcal{L}(g)e^{-as}$
12)	$\delta(t-a)$	e^{-as}

Fourier transforms:

	$f(x)$	$\widehat{f}(\omega)$
1)	e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{\frac{-\omega^2}{4a}}$

	$f(x)$	$\widehat{f}(\omega)$
2)	$\begin{cases} e^{-ax}, & x \geq 0, \\ 0, & x < 0. \end{cases}$	$\frac{1}{\sqrt{2\pi}(a+i\omega)}$

	$f(x)$	$\widehat{f}(\omega)$
3)	$\begin{cases} 1, & \lvert x\rvert < 1, \\ 0, & \lvert x\rvert > 1. \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(\omega)}{\omega}$

Indefinite Integrals: $\left(n \in \mathbb{Z}_{\geq 1}\right)$

1)	$\displaystyle\int x\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{\cos\left(\frac{n\pi}{L}x\right) + \left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}$ (+constant)
2)	$\displaystyle\int x^2\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{\left(\left(\frac{n\pi}{L}\right)^2x^2 - 2\right)\sin\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right)x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3}$ (+constant)
3)	$\displaystyle\int x\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{\sin\left(\frac{n\pi}{L}x\right) - \left(\frac{n\pi}{L}\right)x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}$ (+constant)
4)	$\displaystyle\int x^2\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{\left(2 - \left(\frac{n\pi}{L}\right)^2x^2\right)\cos\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right)x\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3}$ (+constant)
5)	$\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x)$ (+constant)

You can use these formulas without justification.

Question 1

1.MC1 [3 Points] Consider the following initial value problem:

$$\begin{cases} f'''(t) = \cos(2t) + e^{-3t}, & t > 0, \\ f(0) = 0 \quad f'(0) = 0, & f''(0) = \pi. \end{cases}$$

Find the Laplace transform $\mathcal{L}(f) = F$ of the function f.

(A) $F(s) = \frac{1}{s^2(s^2+2)} + \frac{1}{s^3(s+3)} - \frac{\pi}{s^3}$.

(B) $F(s) = \frac{1}{s(s^2+2)} + \frac{1}{s^2(s+3)} + \frac{\pi}{s^2}$.

(C) $F(s) = \frac{1}{s^2(s^2+4)} + \frac{1}{s^3(s+3)} + \frac{\pi}{s^3}$.

(D) $F(s) = \frac{1}{s^2(s^2+4)} + \frac{1}{s^3(s+3)} - \frac{\pi}{s^3}$.

1.MC2 [3 Points] Find the inverse Laplace transform of

$$F(s) = \left(\frac{2}{s^3} + 1\right) e^{-as} + \frac{1}{s^2 - b^2},$$

where a and b are two positive constants.

(A) $f(t) = u(t-a)(t-a) + \delta(t-a) + \frac{1}{b}\sinh(bt)$.

(B) $f(t) = u(t-a)(t-a)^2 + \delta(t-a) + \frac{1}{b}\sinh(bt)$.

(C) $f(t) = u(t-a)(t-a)^2 + \delta(t+a) + \frac{1}{b^2}\sinh(bt)$.

(D) $f(t) = u(t-a)(t-a)^2 + \delta(t-a) + \sinh(bt)$.

1.MC3 [3 Points] Let f be a continuous function such that $\lim_{x\to\infty} f(x) = 0$. Solve the following differential equation using the Fourier transform

$$f''''(x) + 2f''(x) + f(x) = g(x).$$

where g is a given by

$$g(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

(A) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}(\omega^2+1)^2} \sin(\omega) e^{i\omega x}\, d\omega$.

(B) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}\omega(\omega^4-\omega^2+1)^2} \sin(\omega) e^{i\omega x}\, d\omega$.

(C) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}\omega(\omega^4-\omega^2+1)^2} \cos(\omega) e^{i\omega x}\, d\omega$.

(D) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}\omega(\omega^2-1)^2} \sin(\omega) e^{i\omega x}\, d\omega$.

1.MC4 [1 Point] Determine if the following function is even, odd, or neither and if it is periodic or not.

$$\cos(2\pi x + 5) + \frac{e^{ix} + e^{-ix}}{2}$$

(A) The function is periodic but not even or odd.

(B) The function is even but not periodic.

(C) The function is odd but not periodic.

(D) The function is periodic and even.

1.MC5 [3 Points] Let f be a 2π periodic continuous and **differentiable** function such that $f'(0) = \frac{\pi^6}{945}$. The Fourier series of f on the interval $[-\pi, \pi]$ is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\pi n^7} \sin(\pi n x)$$

Find the value of the numerical series

$$\sum_{n=1}^{\infty} \frac{1}{n^6}.$$

(A) $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.

(B) $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{2\pi^6}{945}$.

(C) $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{n\pi^6}{945}$.

(D) $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} - 1$.

1.MC6 [3 Points] Consider the following PDE (partial differential equation) for the function $u = u(x, y)$:

$$u_{xx} + 2\cos(x)u_{xy} + yu_{yy} - u_x + u_y = \sin(x).$$

Is the PDE hyperbolic, parabolic, elliptic or of mixed type ?

(A) hyperbolic.

(B) parabolic.

(C) elliptic.

(D) mixed type.

1.MC7 [3 Points] Wave equation with D'Alembert solution.

Consider the following wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\, t \geq 0, \\ u(x,0) = e^{2x}, & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Find the value of the solution u at position $x = 0$, i.e. $u(0,t)$

(A) $u(0,t) = \cos(2ct)$.

(B) $u(0,t) = \sin(2ct)$.

(C) $u(0,t) = \sinh(2ct)$.

(D) $u(0,t) = \cosh(2ct)$.

1.MC8 [3 Points] Let $u(x,y) = e^{-((x-1)^2+(y-1)^2)}$. The maximum value of u in the disk of radius 4 centred at 0, denoted by D_4, is at the point $(x,y) = (1,1)$. That is

$$\max_{D_4} u(x,y) = u(1,1).$$

Which of the following statements is true?

(A) u is not constant in D_4.

(B) u is constant in D_4.

(C) The minimum of u is at the point $(x,y) = (-1,-1)$

(D) We cannot conclude that (A), (B) and (C) are true.

1.MC9 [3 Points] Consider the Dirichlet problem for the Heat equation on an infinite bar,

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & x \in \mathbb{R},\ t \geq 0, \\ u(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

(A) Every constant function is a solution.

(B) There is no solution.

(C) The function $u(x,t) = 0$ is a solution.

(D) We cannot conclude that (A), (B) and (C) are true.

Question 2

2.Q1 [15 Points] Wave Equation with inhomogeneous boundary conditions

Find the solution of the following wave equation (**with inhomogeneous boundary conditions**) on the interval $[0, \pi]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t \geq 0,\ x \in [0, \pi] \\ u(0, t) = 10\pi^2, & t \geq 0 \\ u(\pi, t) = 10\pi^2, & t \geq 0 \\ u(x, 0) = 2\sin(5x) + \sin(4x) + 10\pi^2, & x \in [0, \pi] \\ u_t(x, 0) = -4\sin(2x). & x \in [0, \pi] \end{cases} \tag{1}$$

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w''(x) = 0$, $w(0) = 10\pi^2$, and $w(\pi) = 10\pi^2$.

b) Define $v(x, t) := u(x, t) - w(x)$. Formulate the corresponding problem for v, equivalent to (17).

c) (i) Find, using the formula from the script, the solution $v(x, t)$ of the problem you have just formulated.

(ii) Write down explicitly the solution $u(x, t)$ of the original problem (17).

Question 3

3.Q1 [10 Points] Dirichlet problem for Laplace equation on a rectangle

Find the general solution $u = u(x, y)$ of the following Laplace equation on a rectangle, with nonzero boundary conditions on one edge of the rectangle:

$$\begin{cases} \nabla^2 u = 0, & 0 \leq x \leq a, 0 \leq y \leq b \\ u(x,0) = u(0,y) = 0, & 0 \leq x \leq a, 0 \leq y \leq b \\ u(x,b) = f(x), & 0 \leq x \leq a \\ u(a,y) = 0, & 0 \leq y \leq b \end{cases}$$

where f is given by

$$f(x) = x^2.$$

You can use the general formula directly to obtain the solution. For this exercise, no points will be given for detailing all the steps of the separation of variable.