ETH Zürich
Prof. Dr. A. Iozzi

Summer 2022
2 September

Exam Analysis III
D–MAVT, D–MATL

Surname:	
First Name:	
Student Card Nr.:	
Exam Nr.:	

This page contains the generalities of the student: surname, first name, student card (Legi) number and exam number. The exam number is a number that identifies uniquely the student.

By signing this page the student confirms that the above personal data are correct.

Signature

ETH Zürich
Prof. Dr. A. Iozzi

Summer 2022
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Exam Analysis III
D–MAVT, D–MATL

Student Card Nr.:	
Exam Nr.:	

Please do not fill!

Exercise	Value	Points	Control
1	8		
2	10		
3	8		
4	12		
5	15		
6	15		
Total	68		

Please do not fill!

Completeness	

Please turn!

Before the exam:

- Turn off your mobile phone and place it inside your briefcase/backpack.
- Put your bags on the floor. No bags on the desk!
- Place your Student Card (Legi) on the desk.

During the exam, please:

- Start every exercise on a new piece of paper.
- Put your **exam number** on the top right corner of every page.
- For **exercises 1, 2, and 3**: Write the answer in the boxes. You don't need to justify your answer.
- For **exercises 4, 5, and 6**: Motivate your answers. Write down calculations and intermediate results.
- Provide at most **one** solution to each exercise.
- **Do not** write with pencils. Please avoid using **red** or **green** ink pens.

After the exam:

- Make sure that every solutions sheet has your exam number on it.
- Place back the exam sheets, together with your solutions in the envelope.

Allowed aids:

- 20 pages (= 10 sheets) DIN A4 handwritten or typed personal summary.
- An English (or English-German) dictionary.
- **No** further aids are allowed. In particular neither communication devices, nor pocket calculators.

Good Luck!

See the next page!

Laplace Transforms: $\quad (F = \mathcal{L}(f))$

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geq 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)$	$\frac{1}{s}e^{-as}$
12)	$\delta(t-a)$	e^{-as}

(Γ = Gamma function, u = Heaviside function, δ = Delta function)

Indefinite Integrals (you may use): $\quad \left(n \in \mathbb{Z}_{\geq 1}\right)$

1)	$\displaystyle\int x \cos\left(\frac{n\pi}{L}x\right)\, dx = \frac{\cos\left(\frac{n\pi}{L}x\right) + \left(\frac{n\pi}{L}\right) x \sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}$ (+ constant)
2)	$\displaystyle\int x^2 \cos\left(\frac{n\pi}{L}x\right)\, dx = \frac{\left(\left(\frac{n\pi}{L}\right)^2 x^2 - 2\right) \sin\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right) x \cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3}$ (+ constant)
3)	$\displaystyle\int x \sin\left(\frac{n\pi}{L}x\right)\, dx = \frac{\sin\left(\frac{n\pi}{L}x\right) - \left(\frac{n\pi}{L}\right) x \cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}$ (+ constant)
4)	$\displaystyle\int x^2 \sin\left(\frac{n\pi}{L}x\right)\, dx = \frac{\left(2 - \left(\frac{n\pi}{L}\right)^2 x^2\right) \cos\left(\frac{n\pi}{L}x\right) + 2\left(\frac{n\pi}{L}\right) x \sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^3}$ (+ constant)
5)	$\displaystyle\int \frac{1}{1+x^2}\, dx = \arctan(x)$ (+ constant)

You can use these formulas without justification.

Please turn!

1. Periodicity and Even/Odd functions (*8 Points*)

Definition : A function is

- *even* if $f(x) = f(-x)$
- *odd* if $f(x) = -f(-x)$

Determine which of the following functions are even, odd, or neither. And determine which of the following functions is periodic and which is not. For the periodic ones, determine their fundamental period[1].
Write the answer in the box.

(e.g. $\sin(x)$ is a periodic function of period 2π and it's an odd function.)

a) $\cos(\frac{2\pi x}{L})$, where $L > 0$ is a constant.

b) $\sin(2x) + x^3$

c) $\sin(x^2)$

d) $4\cos(3x) + 16\sin(4x)$

[*Hint:* *Recall that every periodic, continuous function is bounded, and that every periodic, differentiable functions has periodic derivative.*]

[1] A periodic function of period $P > 0$ is a function f such that $f(x + P) = f(x)$ for all $x \in \mathbb{R}$. The *fundamental period* of a periodic function is the smallest period P.

2. Laplace Transform (*10 Points*)

Find the solution $f(t)$ of the following initial value problem:

$$\begin{cases} f''(t) - a^2 f(t) = a, & t > 0, \\ f(0) = 2, \quad f'(0) = a, \end{cases}$$

where $a > 0$ is a positive constant.

[*Hint:* $\mathcal{L}^{-1}\left(\frac{a}{s(s^2-a^2)}\right) = \mathcal{L}^{-1}\Big(\mathcal{L}(1)\mathcal{L}(\sinh(at))\Big)$ and then use the convolution for Laplace transform.]

Write the intermediate answer of $F = \mathcal{L}(f)$ in the box.

$F = \mathcal{L}(f) =$

Write the final answer in the box.

$f(t) =$

3. Fourier transform (*8 Points*)

Compute the Fourier transform of the following function (you don't need to compute the case $w = 0$):

$$f(x) = \begin{cases} \sqrt{2\pi}(1+x), & 0 \le x \le \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that the Fourier transform of f is given by

$$\mathcal{F}(f)(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx.$$

Write the final answer in the box.

$\mathcal{F}(f)(w) =$

4. Wave Equation with D'Alembert solution (*12 Points*)

Let $c > 0$. Consider the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = \frac{1}{c}\arctan(x), & x \in \mathbb{R} \\ u_t(x,0) = \frac{1}{1+x^2}, & x \in \mathbb{R}. \end{cases}$$

Find the solution $u(x,t)$. You may use D'Alembert formula.
[Simplify the expression as much as possible: no unsolved integrals].

5. Wave Equation with inhomogeneous boundary conditions (*15 Points*)

Find the solution of the following wave equation (**with inhomogeneous boundary conditions**) on the interval $[0,\pi]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t \geq 0,\ x \in [0,\pi] \\ u(0,t) = 3\pi^2, & t \geq 0 \\ u(\pi,t) = 7\pi, & t \geq 0 \\ u(x,0) = 2\sin(5x) + \sin(4x) + (7-3\pi)x + 3\pi^2, & x \in [0,\pi] \\ u_t(x,0) = 0. & x \in [0,\pi] \end{cases} \tag{1}$$

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w''(x) = 0$, $w(0) = 3\pi^2$, and $w(\pi) = 7\pi$.

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

c) (i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (1).

6. Separation of variables for the Heat equation (*15 Points*)

Consider the following time-dependent version of the heat equation on the interval $[0, L]$. We also impose boundary conditions and we look for a solution $u = u(x, t)$ such that:

$$\begin{cases} u_t = (1+t)u_{xx}, & x \in [0, L], t \in [0, +\infty), \\ u(0, t) = 0, & t \in [0, +\infty), \\ u(L, t) = 0, & t \in [0, +\infty), \\ u(x, 0) = f(x), & x \in [0, L]. \end{cases}$$

Where f is a given function. The Fourier series of the $2L$ periodic odd extension of f is given by

$$f(x) := \sum_{n=1}^{\infty} \frac{\pi^2}{(8+n)^2} \sin\left(\frac{n\pi}{L}x\right).$$

Find the solution $u(x, t)$ using separation of variable. Proceed as in the lecture and adapt the steps if necessary.