D-MAVT
D-MATL

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Analysis III

Exam Solutions

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1. a) 2. order, non-linear

b) 2. order, linear, inhomogeneous, hyperbolic

$AC-B^2 = (4xy^3)(xy)-(2x\sqrt{y})^2 = 4x^2y(y^3-1) < 0$ for $\{(x,y) \in \mathbb{R}^2 : x > 0, 0 < y < 1\}$

c) 3. order, linear, inhomogeneous

2. We Laplace transform both sides of the integral equation

$$\mathcal{F}\left[y(t) + 2\int_0^t y(\tau)e^{(t-\tau)}d\tau\right] = Y(s) + 2Y(s)\frac{1}{s-1} = Y(s)\frac{s+1}{s-1},$$

$$\mathcal{F}\left[te^t\right] = \frac{1}{(s-1)^2},$$

where we have used the convolution property on the left side and the s-shift on the right hand side. Solving for $Y(s)$ leads to

$$Y(s) = \frac{1}{(s+1)(s-1)} = \frac{1}{(s^2-1)}.$$

We transform back and get

$$y(t) = \sinh(t).$$

3. a) D'Alembert's formula for the solution of the wave equation is given by

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy.$$

With the given initial conditions we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(e^{-(x+ct)^2}\sin^2(x+ct) + e^{-(x-ct)^2}\sin^2(x-ct) + (x+ct+(x-ct))\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} ye^{-y^2}dy \\ &= \frac{1}{2}\left(e^{-(x+ct)^2}\sin^2(x+ct) + e^{-(x-ct)^2}\sin^2(x-ct) + 2x\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} -\frac{1}{2}\frac{d}{dy}e^{-y^2}dy \\ &= \frac{1}{2}\left(e^{-(x+ct)^2}\sin^2(x+ct) + e^{-(x-ct)^2}\sin^2(x-ct) + 2x\right) - \frac{1}{4c}\left(e^{-(x+ct)^2} - e^{-(x-ct)^2}\right) \end{aligned}$$

Please turn!

b)

$$\lim_{t\to\infty} u(a,t) = \lim_{t\to\infty} \frac{1}{2}\left(e^{-(a+ct)^2}\sin^2(a+ct) + e^{-(a-ct)^2}\sin^2(a-ct) + 2a\right) - \frac{1}{4c}\left(e^{-(a+ct)^2} - e^{-(a-ct)^2}\right)$$
$$= a$$

4. a) (i)

$$\int_{\mathbb{R}} e^{-ax^2}dx = \sqrt{2\pi}\mathcal{F}\left[e^{-ax^2}\right](0) = \sqrt{\frac{\pi}{a}}$$

(ii)

$$\int_{\mathbb{R}} xe^{-ax^2}dx = \sqrt{2\pi}i\frac{d}{d\omega}\mathcal{F}\left[e^{-ax^2}\right](0) = i\frac{d}{d\omega}\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}\Big|_{\omega=0} = 0$$

(iii)

$$\begin{aligned}\int_{\mathbb{R}} x^2e^{-ax^2}dx &= \sqrt{2\pi}i^2\frac{d^2}{d\omega^2}\mathcal{F}\left[e^{-ax^2}\right](0)\\ &= -\frac{d^2}{d\omega^2}\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}\Big|_{\omega=0}\\ &= \left(\frac{d}{d\omega}\frac{\omega}{2a}\right)\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}\Big|_{\omega=0}\\ &= \sqrt{\frac{\pi}{4a^3}}\end{aligned}$$

b) With $f(x) = x^2$ and a transformation of variables $y_{new} = x - y$ we get

$$u(x,t) = \int_{-\infty}^{\infty} y^2K(x-y,t)dy = \int_{-\infty}^{\infty}(x-y)^2K(y,t)dy = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\left(x^2-2xy+y^2\right)e^{-\frac{y^2}{4t}}dy.$$

Now we use the results from the first part with $a = \frac{1}{4t}$ to calculate the integral:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\left(x^2\sqrt{\frac{\pi}{a}} + 0 + \sqrt{\frac{\pi}{4a^3}}\right)\Big|_{a=\frac{1}{4t}} = x^2 + 2t.$$

5. a) We determine the Fourier integral:

$$\begin{aligned}\int_0^{\pi} x^2\sin(nx)dx &= x^2\frac{-1}{n}\cos(nx)\Big|_0^{\pi} - \int_0^{\pi} 2x\frac{-1}{n}\cos(nx)dx\\ &= \frac{-\pi^2}{n}(-1)^n + 2\int_0^{\pi}\frac{-1}{n^2}\sin(nx)dx\\ &= \frac{-\pi^2}{n}(-1)^n + 2\frac{1}{n^3}\cos(nx)\Big|_0^{\pi}\\ &= \frac{-\pi^2}{n}(-1)^n + 2\frac{1}{n^3}((-1)^n - 1).\end{aligned}$$

Furthermore, we have

$$\int_0^{\pi} x\sin(nx)dx = x\frac{-1}{n}\cos(nx)|_0^{\pi} - \int_0^{\pi}\frac{-1}{n}\cos(nx)dx$$
$$= \frac{-1\pi}{n}(-1)^n.$$

Therefore the real Fourier series of f is given by

$$\sum_{n=1}^{\infty} b_n \sin(nx),$$

with

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\ &= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)dx \\ &= 2\frac{-1\pi}{n}(-1)^n - \frac{2}{\pi}\left(\frac{-\pi^2}{n}(-1)^n + 2\frac{1}{n^3}((-1)^n - 1)\right) \\ &= \frac{4}{n^3\pi}(1-(-1)^n). \end{aligned}$$

b) Using separation of variables we set $u(x,y) = F(x)G(y)$ and obtain

$$u_{xx} = F''(x)G(y) \quad \text{and} \quad u_{yy} = F(x)G''(y),$$

which plugged into the PDE gives

$$F''(x)G(y) + F(x)G''(y) = 0 \quad \Leftrightarrow \quad (-1)\cdot\frac{G''(y)}{G(y)} = \frac{F''(x)}{F(x)} = k, \tag{1}$$

where k is a constant. The boundary conditions $u(0,y) = 0$ and $u(\pi,y) = 0$ (we are considering $G(y) \not\equiv 0$ as otherwise u would be trivial) translate into

$$F(0) = 0 \quad \text{and} \quad F(\pi) = 0.$$

Consequently we first need to solve the following initial value problem:

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(\pi) = 0. \end{cases}$$

One has to distinguish the following cases, depending on the sign of k:

$k = 0$: In this case the general solution is given by $F_0(x) = Ax + B$. Plugging in the boundary conditions gives the trivial solution $A = 0 = B$, i.e. $F_0(x) = Ax + B = 0$.

$k > 0$: In this case the general solution is given by $F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$. Plugging in the boundary conditions, gives again the trivial solution $A = 0 = B$.

$k < 0$: In this case the general solution is given by $F(x) = A\cos(\sqrt{-k}x) + B\sin(\sqrt{-k}x)$. From the first boundary condition we obtain the requirement that

$$F(x) = B\sin(px),$$

and from the second $\sqrt{-k} = n, \quad n \in \mathbb{N}$, i.e. all the solutions are of the form

$$F_n(x) = A_n \sin(nx).$$

Now we solve the second ODE coming from Equation (1), i.e we solve

$$\begin{cases} G_n(y) = n^2 G(y) \\ G_n(0) = 0. \end{cases}$$

For $n = 0$, the general solution is given by

$$G_0(y) = Cy + D,$$

and the boundary condition gives $B = 0$, thus

$$G_0(y) = Cy.$$

For $n \geq 1$, the solutions are of the form

$$G_n(y) = B_n \sinh(ny).$$

Consequently for $n = 0$ we obtain the solution

$$u_0(x, y) = F_0(x)G_0(y) = 0$$

and for any $n \geq 1$, we obtain the solution

$$u_n(x, y) = F_n(x) \cdot G_n(y) = A_n \sin(nx) \cdot B_n \sinh(ny) =: C_n \sin(nx) \sinh(ny).$$

In oder to fulfil the inhomogeneous boundary condition we use the superposition principle and get the following ansatz for a general solution

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} C_n \sin(nx) \sinh(ny).$$

We are left with the inhomogeneous boundary condition

$$u(x, \pi) = \sum_{n=1}^{\infty} C_n \sin(nx) \sinh(n\pi) = \sum_{n=1}^{\infty} b_n \sin(nx) = f(x).$$

Comparing the coefficients leads to

$$C_n = \frac{1}{\sinh(n\pi)} b_n.$$

Hence we get the solution

$$u(x, y) = \sum_{n=1}^{\infty} \frac{b_n}{\sinh(n\pi)} \sin(nx) \sinh(ny).$$