Analysis Übungsstunde 1

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Laplace Transformation

Definition

$$f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathcal{L}(f)(s) = F(s) \quad ; \quad F(s) = \int_0^\infty f(t) \cdot e^{-st} \, dt$$

Regeln:

→ Linearität:
Konstanten
Addition/Subtraktion

$$\mathcal{L}(a \cdot f(t)) = a \cdot \mathcal{L}(f(t))$$
$$\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$$

→ Multiplikation !

$$\mathcal{L}(f \cdot g) \neq \mathcal{L}(f) \cdot \mathcal{L}(g)$$

→ S-Shifting

$$\mathcal{L}(e^{at} \cdot f(t)) = F(s-a)$$

→ t-shifting

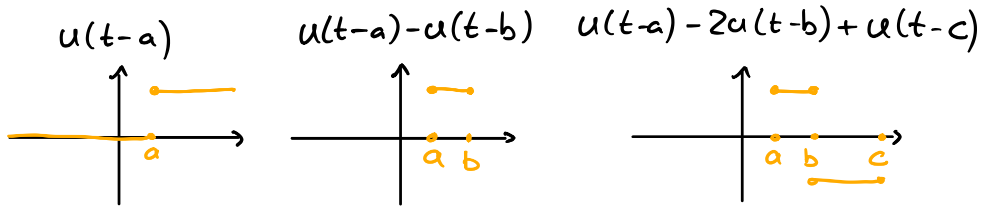
$$\mathcal{L}(f(t-a) \cdot u(t-a)) = e^{-as} F(s)$$
$$\mathcal{L}(f(t)\, u(t-a)) = e^{-as} \mathcal{L}(f(t+a))$$

→ Was ist u?

Heavyside function: $u(t)$ (step function)

$$u(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$$

step bei a (nicht stetig)



Lösen & Rezept $\mathcal{L}$, $\mathcal{L}^{-1}$

Ziel → so in Form bringen, damit es mit Tabellierten $\mathcal{L}$ lösbar ist.

Anwenden von
- Linearität
- Partialbruchzerlegung (PBZ)
- Erweitern
- t- / s-shifts

$f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}}$	$F(s)$
$f(t)$	$\mathscr{L}(f(t)) = F(s)$
1	$\frac{1}{s}$
$e^{at}f(t)$	$F(s-a)$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$\frac{d^n}{dt^n}\delta(t)$	s^n
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{n-1}f(0) - ... - f^{n-1}(0)$
$f * g(t)$	$F(s) \cdot G(s)$
$t^n \quad (n = 0, 1, 2, ...)$	$\frac{n!}{s^{n+1}}$
$sin(kt)$	$\frac{k}{s^2+k^2}$
$sin^2(kt)$	$\frac{2k^2}{s(s^2+4k^2)}$
$cos(kt)$	$\frac{s}{s^2+k^2}$
$cos^2(kt)$	$\frac{s^2+2k^2}{s(s^2+4k^2)}$
e^{at}	$\frac{1}{s-a}$
$ln(at)$	$-\frac{1}{s}\left(ln\left(\frac{s}{a}\right) + \gamma\right)$
$sinh(kt)$	$\frac{k}{s^2-k^2}$
$cosh(kt)$	$\frac{s}{s^2-k^2}$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$
$\frac{ae^{at}-be^{at}}{a-b}$	$\frac{s}{(s-a)(s-b)}$

$f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}}$	$F(s)$
te^{at}	$\frac{1}{(s-a)^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$-tf(t)$	$F'(s)$
$t^2 f(t)$	$F''(s)$
$(-t)^n f(t)$	$F^{(n)}(s)$
$\int\limits_0^t f(u)du$	$\frac{1}{s}F(s)$
$\int\limits_0^t \frac{(t-q)^{n-1}f(q)}{(n-1)!}dq, \; n \leq 1$	$\frac{1}{s^n}F(s), \; n \geq 1$
$\frac{1}{t}f(t)$	$\int\limits_s^\infty F(u)du$
$1 - e^{-at}$	$\frac{a}{s(s+a)}$
$e^{at}sin(kt)$	$\frac{k}{(s-a)^2+k^2}$
$e^{at}cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}sinh(kt)$	$\frac{k}{(s-a)^2-k^2}$
$e^{at}cosh(kt)$	$\frac{(s-a)}{(s-a)^2-k^2}$
$tsin(kt)$	$\frac{2ks}{(s^2+k^2)^2}$
$tcos(kt)$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$tsin(t)cos(t)$	$\frac{2s}{(s^2+4)^2}$
$tsinh(kt)$	$\frac{2ks}{(s^2-k^2)^2}$
$tcosh(kt)$	$\frac{s^2-k^2}{(s^2-k^2)^2}$
$sin(at) \cdot f(t)$	$\frac{1}{2i} \cdot (F(s-ia) - F(s+ia))$
$cos(at) \cdot f(t)$	$\frac{1}{2} \cdot (F(s-ia) + F(s+ia))$
$sinh(at) \cdot f(t)$	$\frac{1}{2} \cdot (F(s-a) - F(s+a))$
$cosh(at) \cdot f(t)$	$\frac{1}{2} \cdot (F(s-a) + F(s+a))$
$\frac{1}{\sqrt{\pi t}e^{-a^2/4t}}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$

Bsp 1 Linearität finde $\mathcal{L}$

$f(t) = 1 + 2t^2$

$$\mathcal{L}(1+2t^2)(s) = \mathcal{L}(1) + \mathcal{L}(2t^2)$$
$$= \mathcal{L}(1) + 2\,\mathcal{L}(t^2)$$
$$= \frac{1}{s} + 2\,\frac{2}{s^{2+1}}$$
$$= \frac{1}{s} + \frac{4}{s^3}$$

$t^n \quad (n = 0, 1, 2, ...) \qquad \frac{n!}{s^{n+1}}$

$t^2 = \frac{2!}{s^{2+1}}$

Bsp 2 PBZ, t-shift, Linearität finde $\mathcal{L}^{-1}$

$$F(s) = \frac{5s-3}{\underbrace{s^2-3s+2}} = \frac{5s-3}{(s-1)(s-2)} \overset{!}{=} \frac{A}{s-1} + \frac{B}{s-2}$$

$$= \frac{A(s-2)+B(s-1)}{(s-1)(s-2)} = \frac{As+Bs-2A-B}{''}$$

coeff vgl: $s^1 \quad As+Bs \overset{!}{=} 5s$

$s^0 \quad -2A-B \overset{!}{=} -3$

$A = -2$

$B = 7$

$$F(s) = -2\,\frac{1}{s-1} + 7\,\frac{1}{s-2} \qquad \mathcal{L}^{-1}: \text{ZF} \quad e^{at} \xleftarrow{\mathcal{L}^{-1}} \frac{1}{s-a}$$

$$f(t) \; -2e^{+1t} + 7e^{+2t}$$

Bsp 3 Erweitern: finde $\mathcal{L}^{-1}$

$$F(s) = \frac{1}{\underbrace{s^2+6s}_{(s+\ldots)^2}+\underbrace{13}_{9\;\;4}} = \frac{1}{\underbrace{s^2+6s+9}_{(s+3)^2}+4} = \frac{1}{(s+3)^2+4} = \frac{2}{\underbrace{(s+3)}_{\text{s-Shift}}{}^2+2^2}\cdot\frac{1}{2}$$

$\sin(kt) \xleftarrow{\mathcal{L}^{-1}} \frac{k}{s^2+k^2}$

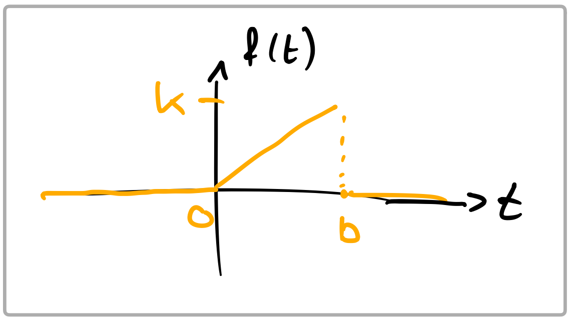
$$\rightarrow f(t) = \frac{1}{2}\cdot e^{-3t}\cdot \sin(2t)$$

Bsp 4 esw. $\mathcal{L}^{-1}$? $\quad t^n \Rightarrow \frac{n!}{s^{n+1}}$

$$F(s) = \frac{1}{s^4} = \frac{1}{s^{3+1}} = \frac{1}{3!} \cdot \frac{3!}{s^{3+1}}$$

$$f(t) = \frac{1}{6} \cdot t^3$$

Bsp 5 rechnen ohne Tabellen, $\mathcal{L}$



$$f(t) = \begin{cases} \frac{k}{b} \cdot t & , t \in [0, b] \\ 0 & \text{else} \end{cases}$$

$$F(s) = \int_0^\infty f(t) \cdot e^{-st} dt = \int_0^b \frac{k}{b} t \cdot e^{-st} dt + \cancel{\int_b^\infty 0 \cdot e^{-st} dt}$$

$$= \frac{k}{b} \int_0^b t \cdot e^{-st} dt = \underbrace{\ldots}_{\text{partieller Integration}} = \frac{k}{b}\left(-\frac{b}{s} e^{-sb} + \frac{1}{s^2}\left(e^{-sb} - 1\right)\right)$$

$$F(t) \left[-\frac{t}{s} e^{-st} \right]_0^\infty$$

Bsp 6 $\mathcal{L}$

$$f(t) = \sin(\omega t) \qquad F(s) = \int_0^\infty \underbrace{\sin(\omega t)}_{u} \cdot \underbrace{e^{-st}}_{v'} dt \quad = uv - \int u'v$$

$$= \underbrace{\sin(\omega t)}_{u} \cdot \underbrace{\left(-\frac{1}{s}\right) e^{-st}}_{v} \Big|_0^\infty - \int_0^\infty \underbrace{\omega\cos(\omega t)}_{u'} \underbrace{\left(-\frac{1}{s}\right) e^{-st}}_{v} dt$$

$$\cancel{\sin(\infty) \cdot -\frac{1}{s} e^{-\infty}} - \underbrace{\cancel{\sin(0)} \cdot \frac{1}{s} e^0}_{=0}$$

$$= -\int_0^\infty \omega\cos(\omega t)\left(-\frac{1}{s}\right) e^{-st} dt$$

$$= -\frac{\omega}{s} \int_0^\infty \underbrace{e^{-st}}_{v'} \underbrace{\cos(\omega t)}_{u} dt = \underbrace{-\frac{\omega}{s^2} e^{-st} \cos(\omega t) \Big|_0^\infty}_{\left(0 - \left(-\frac{\omega}{s^2} 1 \cdot 1\right)\right)} - \frac{\omega^2}{s^2} \int_0^\infty e^{-st} \sin(\omega t) dt$$

$$F(s) = \frac{\omega}{s^2} - \frac{\omega^2}{s^2} \cdot F(s)$$

$$F(s)\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2}$$

$$F(s) = \frac{\omega}{s^2} \cdot \frac{1}{\left(1 + \frac{\omega^2}{s^2}\right)} = \frac{\omega}{\cancel{s^2}} \cdot \frac{\cancel{s^2}}{s^2 + \omega^2}$$

$$\left(1 + \frac{\omega^2}{s^2}\right) \to \frac{s^2 + \omega^2}{s^2}$$

$$= \frac{\omega}{s^2 + \omega^2}$$