Analysis Übungsstunde 2

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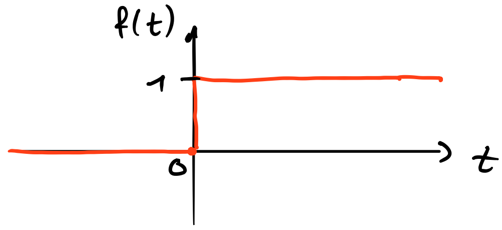
Heavyside funktion (nochmals)

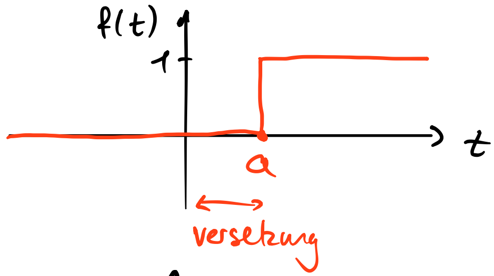
- Basic Heavyside

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



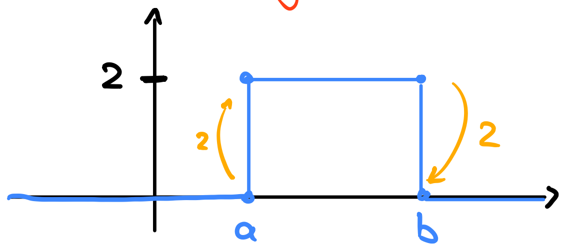
- Mit Versetzung

$$u(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < 0 \end{cases}$$

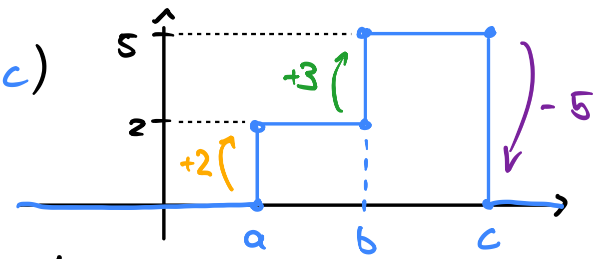


- Mehrere Heavysides

$$f(t) = 2u(t-2) - 2u(t-b)$$



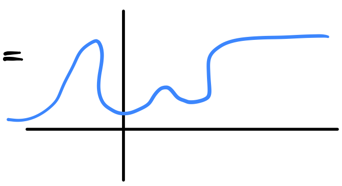
$$f(t) = 2u(t-a) + 3u(t-b) - 5u(t-c)$$



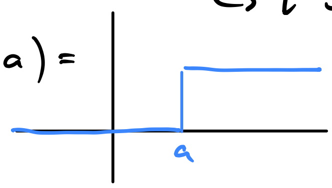
- Heavyside mit Funktion multipliziert

↳ t-shifting !!

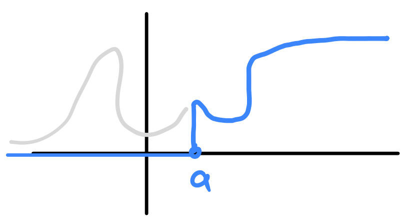
$f(t) =$



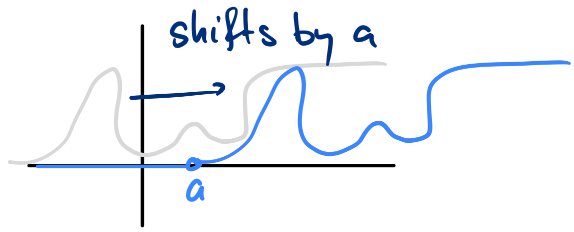
$u(t-a) =$



$f(t) \cdot u(t-a)$



$f(t-a) \cdot u(t-a)$



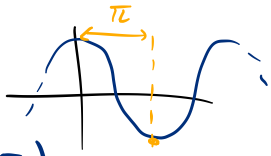
Tipps für T-Shifting

$$\mathscr{L}(f(t-a)u(t-a)) = e^{-as}F(s)$$
$$\mathscr{L}(f(t)u(t-a)) = e^{-as}\mathscr{L}(f(t+a))$$

Ziel: $f(t)$ umformen, in die Form $f(t-a)\cdot u(t-a)$

- **Trigonometrische Funktionen**

  Nutze Periodizität & Additionstheorem



$$\text{Bsp)}\quad u(t-\pi)\cdot\cos(t) = u(t-\pi)\cdot(-1)\cos(t-\pi)$$

11.1.3 Additionstheoreme 99

- $\sin(x)^2+\cos(x)^2=1 \Rightarrow \sin(x)=\sqrt{1-\cos(x)^2}$
- $\sin(\alpha\pm\beta)=\sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta)$
- $\cos(\alpha\pm\beta)=\cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)$
- $\sin(\arccos(x))=\cos(\arcsin(x))=\sqrt{1-x^2}$
- $\cos(x)=2\cos(\frac{x}{2})^2-1=\cos(\frac{x}{2})^2-\sin(\frac{x}{2})^2$
- $\tan(a\pm b)=\frac{\tan(a)\pm\tan(b)}{1\mp\tan(a)\tan(b)}$

$$\text{Bsp)}\quad u(t-3)\cdot\sin(t-2) =$$
$$= u(t-3)\cdot\sin(\underbrace{(t-3)+1}_{=t-2})$$
$$= u(t-3)\left[\sin(t-3)\cos(1)+\cos(t-3)\sin(1)\right]$$

- **Ergänzen & erweitern**

$$\text{Bsp}\quad u(t+3)\cdot t^2 = u(t+3)\left(\overbrace{(t+3)^2}^{t^2+6t+9}-6t-9\right)$$
$$= u(t+3)(t+3)^2-6t\,u(t+3)-9u(t+3)$$
$$= u(t+3)(t+3)^2-6((t+3)-3)u(t+3)-9u(t+3)$$
$$= u(t+3)(t+3)^2-6(t+3)u(t+3)+18u(t+3)-9u(t+3)$$

Ableitungen der Laplacetransformation

Zeitbereich

#ableitungen

$$\mathcal{L}(f^{n}(t))(s) = s^n \mathcal{L}(f) - \sum_{j=0}^{n-1} s^{n-1-j} \cdot f^{j}(0)$$

oft gebraucht:

$$\mathcal{L}(f'(t))(s) = s^1 \underbrace{\mathcal{L}(f)}_{F(s) = \text{Frequenzbereich}} - \underbrace{f(0)}_{\text{Zeitbereich}}$$

$$\mathcal{L}(f''(t))(s) = s^2 \mathcal{L}(f) - sf(0) - f'(0)$$

$$\mathcal{L}(f'''(t))(s) = s^3 \mathcal{L}(f) - s^2 f(0) - sf'(0) - f''(0)$$

Frequenzbereich

$$\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} \cdot F(s) \quad , n = 1, 2, 3 \ldots$$

Integration der Laplacetransformation

$$\mathcal{L}\left(\int_0^t f(x)\,dx \right) = \frac{1}{s} \cdot F(s)$$

⇒ Rezept Lösen DGL

1) DGL finden (eigentlich immer gegeben)
2) Beide Seiten Transformieren $f(t) \rightarrow \mathcal{L}$
3) Nach $Y(s)$ auflösen
4) Inverse Laplacetransformation $\mathcal{L}^{-1} \rightarrow f(t)$

Bsp: ① $\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

$$\boxed{\mathcal{L}(y(t)) = Y(s)}$$

② transformiere:

$$\mathcal{L}(y'') - \mathcal{L}(y') - 2\mathcal{L}(y) = 0$$

ZF: $\mathscr{L}(f') = s\mathscr{L}(f) - f(0)$
$\mathscr{L}(f'') = s^2\mathscr{L}(f) - sf(0) - f'(0)$

$$= s^2Y(s) - s\underbrace{y(0)}_{1} - \cancel{y'(0)}^{0} - sY(s) + \underbrace{y(0)}_{1} - 2Y(s) = 0$$

AWP

$$= s^2Y(s) - s - sY(s) + 1 - 2Y(s) = 0$$

③ Auflösen nach $Y(s)$

$$Y(s)\left(s^2 - s - 2\right) = s - 1$$

$$Y(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)} \overset{\text{PBZ}}{=} \frac{A}{s-2} + \frac{B}{s+1}$$

PBZ Gleichung: $s - 1 = A(s+1) + B(s-2)$

coeff vgl: $s^1: \; 1 = A + B$
$s^0: \; -1 = A - 2B$

$A = \frac{1}{3}$
$B = \frac{2}{3}$

$$Y(s) = \frac{1}{3}\cdot\frac{1}{s-2} + \frac{2}{3}\cdot\frac{1}{s+1}$$

④ Rücktransformation

$$\mathcal{L}^{-1}(Y(s)) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

ZF, s-shift

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-1t}$$

Bsp Inhomogene DGL:

① $\begin{cases} y'' + 4y' + 4y = t^3 \cdot e^{-2t} \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

② Transformiere beide Seiten

(I) $\mathcal{L}(y'' + 4y' + 4y) = \mathcal{L}(y'') + 4\mathcal{L}(y') + 4\mathcal{L}(y)$

$= s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 4(sY(s) - \cancel{y(0)}^{0}) + 4Y(s)$

$= s^2 Y(s) + 4sY(s) + 4Y(s)$

$= Y(s)(s^2 + 4s + 4) = Y(s)(s+2)^2$

(II) Inhomogenes teil : $\mathcal{L}(t^3 e^{-2t})$ ← $e^{at} \cdot f(t) = F(s-a)$, $a = -2$

$= \frac{3!}{(s+2)^{3+1}} = \frac{6}{(s+2)^4}$ $\quad t^n = \frac{n!}{s^{n+1}}$, $n = 3$

③ (I) = (II)

$Y(s)(s+2)^2 = \frac{6}{(s+2)^4}$

$\rightarrow Y(s) = \frac{6}{(s+2)^6} \underset{\text{erw.}}{=} \frac{6}{5!} \frac{5!}{(s+2)^{5+1}}$

④ Rücktransformation

$\mathcal{L}^{-1}\left(\frac{6}{5!} \frac{5!}{(s+2)^{5+1}}\right) = \underline{\underline{\frac{6}{5!} e^{-2t} t^5 = y(t)}}$

$a = -2$

$\mathscr{L}(e^{at} \cdot f(t)) = F(s-a)$