Analysis Übungsstunde 2

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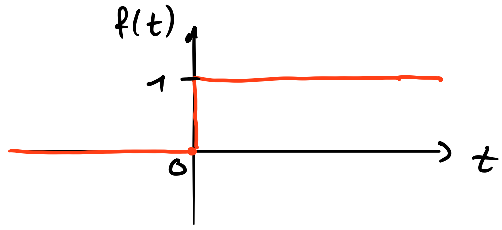
Heavyside funktion (nochmals)

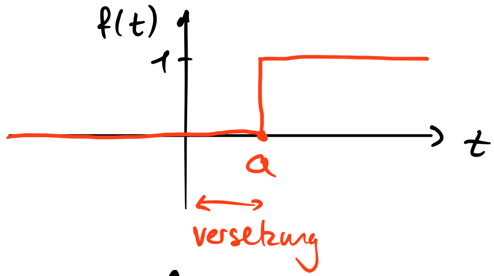
- Basic Heavyside

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



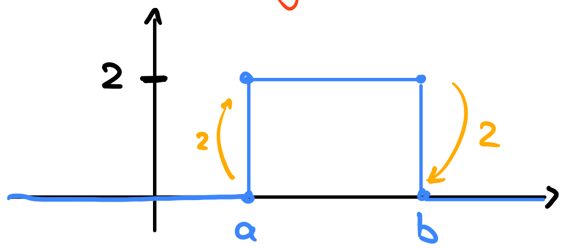
- Mit Versetzung

$$u(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < 0 \end{cases}$$

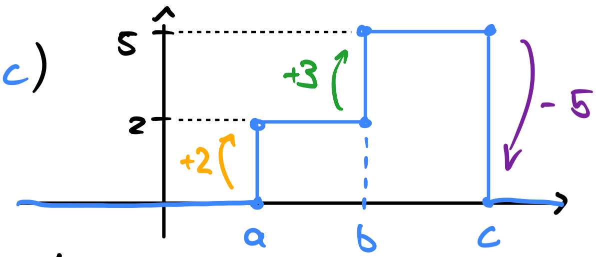


- Mehrere Heavysides

$$f(t) = 2u(t-a) - 2u(t-b)$$



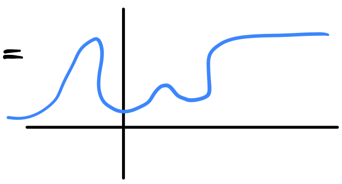
$$f(t) = 2u(t-a) + 3u(t-b) - 5u(t-c)$$



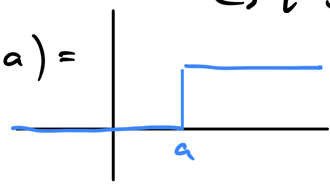
- Heavyside mit Funktion multipliziert

↳ t-shifting !!

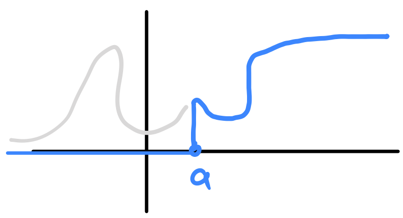
$f(t) =$



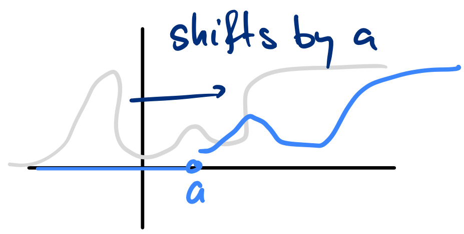
$u(t-a) =$



$f(t) \cdot u(t-a)$



$f(t-a) \cdot u(t-a)$



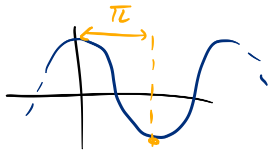
Tipps für T-Shifting

$$\mathscr{L}(f(t-a)u(t-a)) = e^{-as}F(s)$$
$$\mathscr{L}(f(t)u(t-a)) = e^{-as}\mathscr{L}(f(t+a))$$

Ziel: $f(t)$ umformen, in die Form $f(t-a)\cdot u(t-a)$!

- **Trigonometrische Funktionen**

  Nutze Periodizität & Additionstheorem



Bsp) $u(t-\pi)\cdot\cos(t) = u(t-\pi)\cdot\cos(t-\pi)(-1)$

11.1.3 Additionstheoreme 99

- $\sin(x)^2+\cos(x)^2 = 1 \Rightarrow \sin(x) = \sqrt{1-\cos(x)^2}$
- $\sin(\alpha\pm\beta) = \sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta)$
- $\cos(\alpha\pm\beta) = \cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)$
- $\sin(\arccos(x)) = \cos(\arcsin(x)) = \sqrt{1-x^2}$
- $\cos(x) = 2\cos(\frac{x}{2})^2-1 = \cos(\frac{x}{2})^2-\sin(\frac{x}{2})^2$
- $\tan(a\pm b) = \frac{\tan(a)\pm\tan(b)}{1\mp\tan(a)\tan(b)}$

Bsp) $u(t-3)\cdot\sin(t-2) =$

$$u(t-3)\cdot\sin(\underbrace{(t-3)+1}_{=t-2})$$

$$= u(t-3)\left[\sin(t-3)\cos(1)+\cos(t-3)\sin(1)\right]$$

- **Ergänzen & erweitern**

Bsp $u(t+3)\cdot t^2 = u(t+3)\big((\overset{t^2+6t+9}{(t+3)^2}) - 6t - 9\big)$

$$= u(t+3)(t+3)^2 - u(t+3)6t - u(t+3)9$$

$$= u(t+3)(t+3)^2 - 6((t+3)-3)u(t+3) - 9u(t+3)$$

$$= u(t+3)(t+3)^2 - 6(t+3)u(t+3) + 18u(t+3) - 9u(t+3)$$

$\vdots$

Ableitungen der Laplacetransformation

Zeitbereich

#ableitungen

$$\mathcal{L}(f^{n}(t))(s) = s^{n}\mathcal{L}(f) - \sum_{j=0}^{n-1} s^{n-1-j} \cdot f^{j}(0)$$

oft gebraucht:

F(s) = freq. Bereich

Zeitbereich

$$\mathcal{L}(f'(t))(s) = s^{1}\underbrace{\mathcal{L}(f(t))}_{} - f(0)$$

$$\mathcal{L}(f''(t))(s) = s^{2}\mathcal{L}(f) - sf(0) - f'(0)$$

$$\mathcal{L}(f'''(t))(s) = s^{3}\mathcal{L}(f) - s^{2}f(0) - sf'(0) - f''(0)$$

Frequenzbereich

$$\mathcal{L}(t^{n}f(t)) = (-1)^{n}\frac{d^{n}}{ds^{n}} \cdot F(s) \quad , n = 1,2,3\ldots$$

Integration der Laplacetransformation

$$\mathcal{L}\left(\int_{0}^{t} f(x)\,dx\right) = \frac{1}{s} \cdot F(s)$$

⇒ Rezept Lösen DGL

1) DGL finden (eigentlich immer gegeben)
2) Beide Seiten Transformieren $f(t) \to \mathcal{L}$
3) Nach $Y(s)$ auflösen
4) Inverse Laplacetransformation $\mathcal{L}^{-1} \to f(t)$

Bsp: ① $\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

$$\boxed{\mathcal{L}(y(t)) = Y(s)}$$

$\mathscr{L}(f') = s\mathscr{L}(f) - f(0)$
$\mathscr{L}(f'') = s^2\mathscr{L}(f) - sf(0) - f'(0)$
$\mathscr{L}(f^n) = s^n\mathscr{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$

② transformiere:

$$\mathcal{L}(y''-y'-2y) = \mathcal{L}(y'') - \mathcal{L}(y') - 2\mathcal{L}(y) = 0$$

AWP

$$= s^2Y(s) - s\,\overset{1}{y(0)} - \cancel{y'(0)}^{\,0} - (sY(s) - \overset{1}{y(0)}) - 2Y(s)$$

$$= s^2Y(s) - s - sY(s) + 1 - 2Y(s) = 0$$

③ Auflösen nach $Y(s)$

$$Y(s)(s^2 - s - 2) = s - 1$$

$$Y(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)} \overset{!}{=} \frac{A}{s-2} + \frac{B}{s+1}$$

PBZ Gleichungen: $s-1 = A(s+1) + B(s-2)$

coef vgl: $\left.\begin{aligned} s^1 &: 1 = A + B \\ s^0 & -1 = A - 2B \end{aligned}\right\} \begin{aligned} A &= \tfrac{1}{3} \\ B &= \tfrac{2}{3} \end{aligned}$

$$\Rightarrow Y(s) = \frac{1}{3}\cdot\frac{1}{(s-2)} + \frac{2}{3}\cdot\frac{1}{s+1}$$

④ Rücktransformation

$\mathscr{L}(e^{at}\cdot f(t)) = F(s-a)$

$$\mathcal{L}^{-1}(Y(s)) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= \frac{1}{3}\cdot e^{-(-2t)} + \frac{2}{3}e^{-1t} = \underline{\underline{\frac{1}{3}e^{2t} + \frac{2}{3}e^{-1t}}}$$

Bsp Inhomogene DGL:

① $\begin{cases} y'' + 4y' + 4y = t^3 \cdot e^{-2t} \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

$\mathscr{L}(f') = s\mathscr{L}(f) - f(0)$
$\mathscr{L}(f'') = s^2\mathscr{L}(f) - sf(0) - f'(0)$
$\mathscr{L}(f^n) = s^n\mathscr{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$

② Transformiere beide Seiten

(I) $\mathcal{L}(y'' + 4y' + 4y) = \mathcal{L}(y'') + 4\mathcal{L}(y') + 4\underline{\mathcal{L}(y)}$

$s^2 Y(s) - s\cancel{y(0)} - \cancel{y'(0)} + 4(s^1 Y(s) - \cancel{y(0)}) + 4Y(s)$

$s^2 Y(s) + 0.. + 4sY(s) + 4Y(s)$

$Y(s)(s^2 + 4s + 4) = Y(s)(s+2)^2$

(II) Inhomogener teil: $\mathcal{L}(t^3 \cdot e^{-2t})$ $\leftarrow \begin{cases} e^{at} f(t) = F(s-a) \\ t^n \to \frac{n!}{s^{n+1}} \end{cases}$

$= \frac{3!}{(s+2)^{3+1}} = \frac{6}{(s+2)^4}$

③ (I) = (II)

$Y(s) \cdot (s+2)^2 = \frac{6}{(s+2)^4} \longrightarrow Y(s) = \frac{6}{(s+2)^6}$ $!5$ $5+1$

$Y(s) = \frac{6}{5!} \cdot \frac{5!}{(s+2)^{5+1}}$

④ Rücktransformation

$\mathcal{L}^{-1}\left(\frac{6}{5!} \cdot \frac{5!}{(s+2)^{5+1}}\right)$

$\mathscr{L}(e^{at} \cdot f(t)) = F(s-a)$

$t^n \quad (n = 0, 1, 2, ...) \qquad \frac{n!}{s^{n+1}}$

$\frac{6}{5!} \cdot \mathcal{L}\left(\frac{5!}{(s+2)^{5+1}}\right)$
$a = -2$

$\Rightarrow \frac{6}{5!} \cdot e^{-2t} \cdot t^5 = y(t)$

Tipps für die Serie 2

1. Basics – wichtig
   a) erweitern
   b) $25 = 5^2 \to \dots$ ZF
   c–e) PBZ, teil etwas kompliziert

2. a) s-shift und erw.
   b) $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$
   c, d) nutze ableitungsregel der $\mathcal{L}$aplace transformation
   e, f) t-shifting
   g) t-shifting und geschickt erweitern

3. a, b) Schreibe es als multiplikation mit der Heavyside $u(t-a)$

4,5) DGL wie in der übungsstunde $\to$ Rezept

6. Proof, nicht wirklich relevant