Analysis Übungsstunde 3

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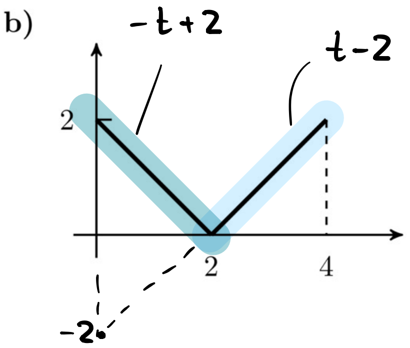
Kurze Nachbesprechung Serie 1, Aufg. 1b)

b)

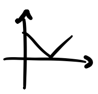


$$\rightarrow f(t) = \begin{cases} -t+2 & t \in [0,2] \\ t-2 & t \in [2,4] \\ 0 & \text{else} \end{cases} \quad \Rightarrow \mathcal{L} = ?$$

$$\mathscr{L}(u(t-a)) = \frac{e^{-as}}{s}$$

$$\mathscr{L}(f(t-a)u(t-a)) = e^{-as}F(s)$$

Anstatt Integral, Drücke $f(t)$ mit der Heavyside aus!



$$= g(t)\big(u(t-0) - u(t-2)\big) + h(t)\big(u(t-2) - u(t-4)\big)$$

$$f(t) = (2-t)u(t-0) - (2-t)u(t-2) + (t-2)u(t-2) - (t-2)u(t-4)$$

$$-(t-0-2)u(t-0) + (t-2)u(t-2) + (t-2)u(t-2) - (t-4+2)u(t-4)$$

$$-\big((t-0)u(t-0) - 2u(t-0)\big) + (t-2)u(t-2) + (t-2)u(t-2) - \big((t-4)u(t-4) + 2u(t-4)\big)$$

$\downarrow \mathcal{L}$

$$-\left(\frac{e^0}{s^2} - 2\frac{e^0}{s}\right) + \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s^2} - \left[\frac{e^{-4s}}{s^2} + 2\frac{e^{-4s}}{s}\right]$$

t^1

$a^2 - 2ab + b^2$

$$= \frac{2}{s} - \frac{(1 - 2e^{-2s} + e^{-4s})}{s^2} - 2\frac{e^{-4s}}{s}$$

$$= \frac{2}{s} - \frac{(1-e^{-2s})^2}{s^2} - 2\frac{e^{-4s}}{s}$$

Dirac Funktion $\delta(t-a)$

$$f(t) = \delta(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{else} \end{cases}$$

$$\int_0^\infty \mathcal{L}(f(t)) \cdot e^{-st} dt = \int_0^\infty \delta(t-a) \cdot e^{-st} dt \overset{(II)}{=} e^{-sa}$$

Regeln:

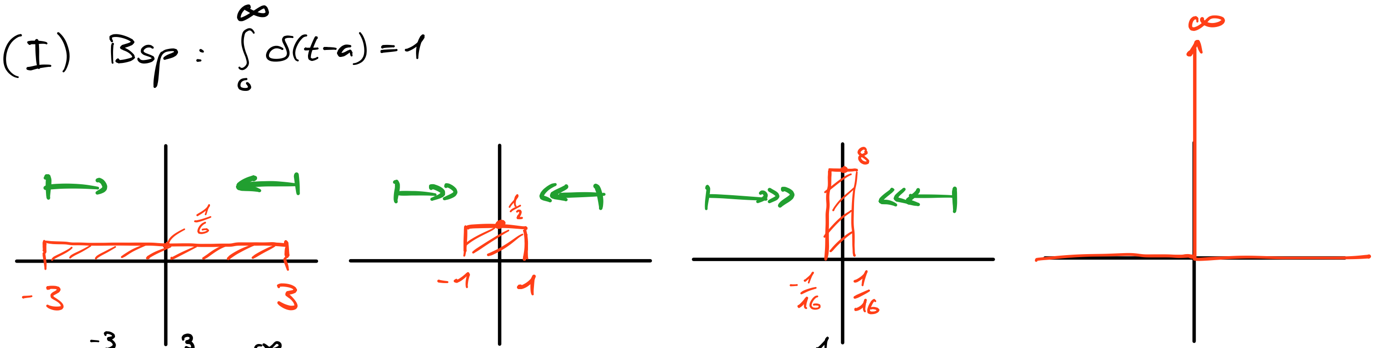
I) $\int_{-\infty}^{\infty} \delta(t-a) = 1$

II) $\int_0^\infty g(t) \cdot \delta(t-a) = g(a)$

III) $\mathcal{L}(\delta(t-a)) = e^{-sa}$

$\mathcal{L}(\delta(t)) = 1 \quad = e^{-s\cdot 0} = 1$

(I) Bsp: $\int_0^\infty \delta(t-a) = 1$



$A = \int_{-\infty}^{-3} \cancel{0} + \int_{-3}^{3} \frac{1}{6}\,dx + \int_3^\infty \cancel{0}$ $= \frac{1}{6}(3-(-3)) = \frac{6}{6} = 1$

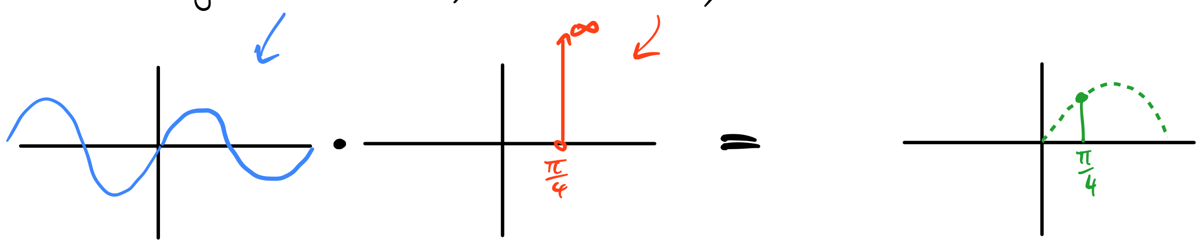
$A = \int_{-1}^{1} \frac{1}{2}\,dx$ $= \frac{1}{2}(1+1) = 1$

$A = \int_{-\frac{1}{16}}^{\frac{1}{16}} 8\,dx$ $= 8\left(\frac{2}{16}\right) = 1$

$A = \int_{-\infty}^{\infty} \delta(t-0)$ $= 1$

(II) Bsp $\int_{-\infty}^{\infty} g(t) \cdot \delta(t-a) = g(a)$

zB: $g(t) = \sin(t)$, $\delta = \delta(t - \frac{\pi}{4})$



$$\int_{-\infty}^{\infty} \sin(t) \cdot \delta\left(t - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

(Anhang) $\int_{-\infty}^{\infty} \underbrace{\delta(t)}_{\hat{=}\, g(t)} \cdot \underbrace{\delta(t-a)}_{\hat{=}\, \delta(t-a)} \overset{!}{=} \delta(a) = \infty$

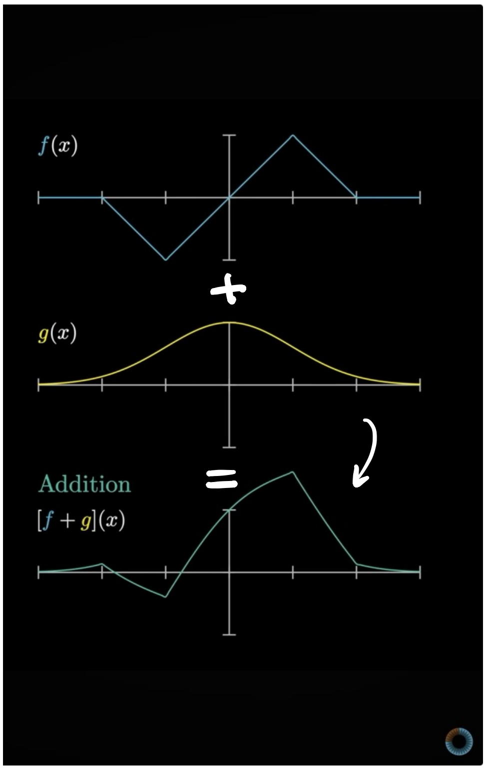
Faltung / Convolution

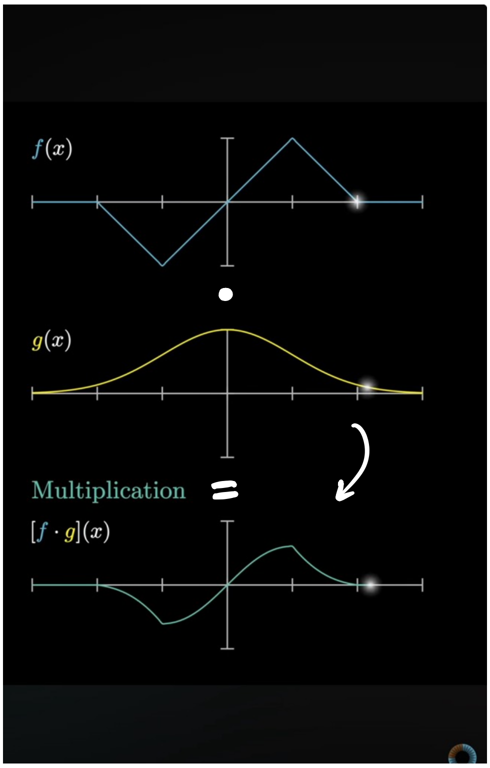
Wie interagieren zwei funktionen miteinander?

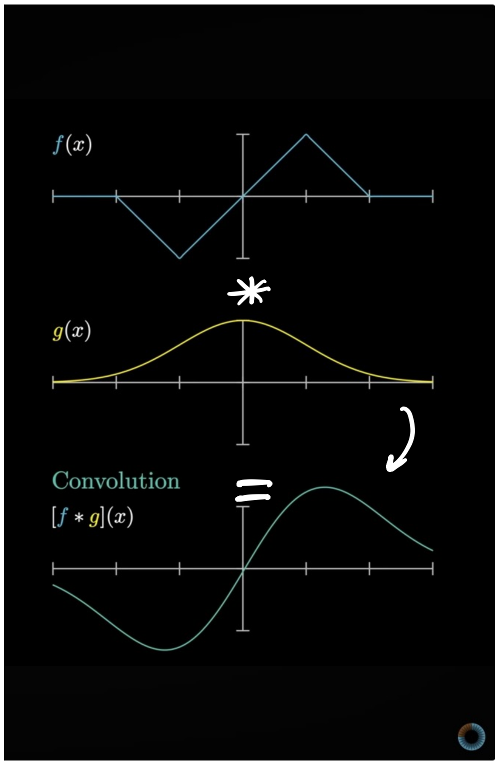
1 Addition ±

2 Multiplikation •, ÷

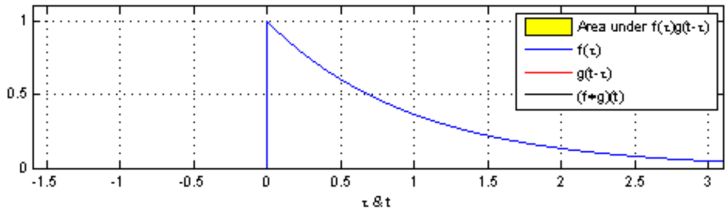
3 Convolution



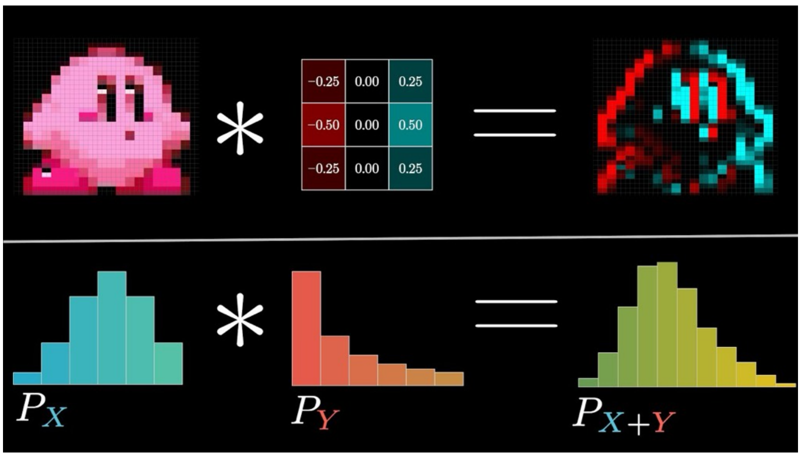




" Interaktion von zwei funktionen über die Zeit "



3B1B :



Info 2

Analysis 3

Theory https://www.youtube.com/watch?v=KuXjwB4LzSA

Analogie : https://www.youtube.com/watch?v=QmcoPYUfbJ8

Faltung / Convolution

Remember? $\mathcal{L}(f \cdot g) \neq \mathcal{L}(g \cdot f)$ neu: Faltung

Kombinieren von funkt: $f \pm g$, $f \cdot / g$
neu dritte operation $f * g$

Def: $$f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) \, d\tau$$

Convolution

Eigenschaften

(I) $f * g = g * f$ (kommutiert!)

(II) $f * (g * h) = (f * g) * h$ (assoziativ)

(III) $f * (g + h) = (f * g) + (f * h)$ (distributiv)

(IV) $f * 0 = 0$

(V) $f * 1 \neq f$

Beweis:

$$f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) \, d\tau$$

$u = t - \tau$ $\quad du = -d\tau$ $\quad \tau = 0 \;\; u = t$ $\quad \tau = t \;\; u = 0$

$$-\int_t^0 f(t-u)\, g(u)\, du$$

$$+\int_0^t f(t-u) \cdot g(u)\, du$$

$$= \int_0^t g(u) \cdot f(t-u)\, du$$

$$= g(t) * f(t)$$

Wie in Laplace anwenden?

inverse

- $\mathcal{L}(f * g) = \mathcal{L}(\overbrace{f(t)}^{F(s)}) \cdot \mathcal{L}(\overbrace{g(t)}^{G(s)})$
- $\mathcal{L}^{-1}(F(s) \cdot G(s)) = (f * g) = \int_0^t f(\tau) g(t-\tau) \, d\tau$
- $\mathcal{L}(f * 1) = \int_0^t f(\tau) \, d\tau$

Anmerkungen T-Shifting:

- T-Shift & S-Shift gemischt → Reihenfolge!

t-shift .... $\mathcal{L}^{-1}$... $f(t-2)$

↳ def: $t^* = t-2$

Bsp. $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s+3)^2}\right)(t)$

s-shift : ... e^{-3t}

$$\mathcal{L}^{-1}\left(\frac{1}{(s+3)^2}\right)(t^*) = e^{-3t^*}\cdot(t^*)^1\cdot u(t^*) \quad \Big\downarrow\; t^* = t-2$$

$$= e^{-3(t-2)}\cdot(t-2)^1\cdot u(t-2)$$

- Ansatz $\mathscr{L}(f(t)u(t-a)) = e^{-as}\mathscr{L}(f(t+a))$

↳ geht, aber $f(t+a)$ muss ausmultipliziert sein

Bsp: $\mathcal{L}\left(t^2\cdot u(t-2)\right)(s)$ $\quad$ $f(t) = t^2$ $\quad$ $a = 2$

↓ $\quad$ $f(t+a) = (t+2)^2 = t^2 + \cancel{2}4t + 4$

$$\mathcal{L}\left(t^2\cdot u(t-2)\right)(s) = e^{-2s}\mathcal{L}\left(t^2 + 2t + 4\right) \qquad t^n \xrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}}$$

$\mathcal{L}(f) = F$

$$= e^{-2s}\left(\frac{2}{s^3} + 2\frac{1!}{s^2} + \frac{4}{s}\right)$$

Prüfung Winter 2018:

2. Laplace Trasform (*7 Points*)
Solve the following integral equation using the Laplace transform:

$$y(t) + 2e^t \int_0^t y(\tau)e^{-\tau}d\tau = te^t,\ t > 0. \qquad (1)$$

Konzept gleich wie DGL: ① Funktion

② Transformiere

$$\mathcal{L}\left(y(t) + 2e^t \int_0^t y(\tau)\cdot e^{-\tau} d\tau\right) = \mathcal{L}(t\cdot e^t) \quad \hat{=} \text{ convolution(?)}$$

$$\mathcal{L}\left[(y(t)) + 2\int_0^t y(\tau)\cdot \underbrace{e^{(t-\tau)}}_{=f(t-\tau)} d\tau\right] = \mathcal{L}(t^1 \cdot e^{+1t})$$

$\hookrightarrow$ s-shift mit t^n, $n=1$

$$Y(s) + 2\mathcal{L}\left(\int_0^t y(\tau)\cdot e^{1(t-\tau)} d\tau\right) = \frac{1!}{(s-1)^{1+1}}$$

(I) $\downarrow$

$$Y(s) + 2\mathcal{L}(y(t) * e^{1t})$$

II, III $\downarrow$

$$Y(s) + 2(\mathcal{L}(y(t)) \cdot \mathcal{L}(e^t))$$

$$Y(s) + 2Y(s)\cdot \frac{1}{s-1}$$

(I) $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$

(II) $\boxed{\mathscr{L}(f * g) = \mathscr{L}(f)\cdot\mathscr{L}(g)}$

(III) $e^{at} \xrightarrow[\mathcal{L}]{} \frac{1}{s-a}$

③ Nach Y(s) auflösen

$$Y(s)\left(1 + 2\cdot\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

$$Y(s) = \frac{1}{(s-1)^2}\cdot\frac{1}{\left(1+\frac{2}{s-1}\right)} = \frac{1}{(s-1)^2}\cdot\frac{1}{\frac{\underbrace{(s-1)+2}_{=s+1}}{s-1}} = \frac{1}{(s-1)^{\cancel{2}}}\cdot\frac{\cancel{(s-1)}}{s+1}$$

$$Y(s) = \frac{1}{(s-1)(s+1)} = \frac{1}{s^2 - 1^2}$$

④ Rücktransform

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2 - 1^2}\right) \to \underline{\underline{y(t) = \sinh(1t)}}$$

$$\sinh(kt) \xleftarrow[\mathcal{L}^{-1}]{} \frac{k}{s^2 - k^2}$$

Aufgabe 2

$$\mathcal{L}^{-1}\left(F(s)\cdot G(s)\right) = \left(f^{(t)} * g^{(t)}\right) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$= \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1}(G) = f * g$$

$1 \quad \frac{1}{s} \qquad sin(kt) \quad \frac{k}{s^2+k^2}$

$$\mathcal{L}^{-1}\left(\frac{2}{s(s^2+4)}\right) =$$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\cdot\frac{2}{(s^2+4)}\right) = \mathcal{L}^{-1}\left(\underbrace{\frac{1}{s}}_{F(s)}\right) * \mathcal{L}^{-1}\left(\frac{2}{(s^2+2^2)}\right)$$

$$= \underbrace{1}_{f(t)} * \sin(2t) = \int_0^t 1\cdot\sin(2(t-\tau))\,d\tau$$

$$= +\frac{1}{2}\cos(2(t-\tau))\Big|_0^t$$

$$\int_0^t 1\cdot\sin(2\tau)\,d\tau = \frac{1}{2}\left|-\cos(2\tau)\right|_0^t$$

$u = 2(t-\tau)$

$du = -2d\tau$

$$= +\frac{1}{2}\left(\cos(2(\underbrace{t-t}_{0})) - \cos(2(\underbrace{t-0}_{t}))\right)$$

$$= -\frac{1}{2}\left(\cos(2t) + \underbrace{\cos(0)}_{=1}\right)$$

$$=$$

$$= -\frac{1}{2}\cos(2t) + \frac{1}{2}$$

⇒ Lösen DGL

1) DGL finden (eigentlich immer gegeben)
2) Beide Seiten Transformieren $f(t) \to \mathcal{L}$
3) Nach $Y(s)$ auflösen
4) Inverse Laplacetransformation $\mathcal{L}^{-1} \to f(t)$

MC Winter 2023

1.MC1 [3 Points] Let f be a solution of the following ordinary differential equation (ODE),

$$\begin{cases} f''(t) + \omega^2 f(t) = 0, & t > 0 \\ f(0) = 1, \quad f'(0) = 2\omega, \end{cases}$$ ①

where $\omega > 0$ is a positive constant. Find the Laplace transform $\mathcal{L}(f) = F$ of the function f.

(A) $F(s) = \boxed{\frac{s}{s^2+\omega^2} + \frac{2\omega}{s^2+\omega^2}.}$

(B) $F(s) = \frac{1}{s^2+\omega^2} + \frac{2s\omega}{s^2+\omega^2}.$

(C) $F(s) = \frac{2\omega}{s+\omega^2}.$

(D) $F(s) = \frac{2\omega}{s^2+\omega}.$

$\mathscr{L}(f') = s\mathscr{L}(f) - f(0)$
$\mathscr{L}(f'') = s^2\mathscr{L}(f) - sf(0) - f'(0)$
$\mathscr{L}(f^n) = s^n\mathscr{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$

② $s^2F(s) - s\overset{1}{f(0)} - \overset{2\omega}{f'(0)} + \omega^2 F(s) = 0$

$F(s)(s^2+\omega^2) - s - 2\omega = 0$

$F(s) = \frac{s}{s^2+\omega^2} - \frac{2\omega}{s^2+\omega}$

DGL was wenn AWP $y(a) = ..$ $y'(a) = ..$

Bsp: $\begin{cases} y''(t) + 4y'(t) = 4t \\ y(3) = 0 \\ y'(3) = 7 \end{cases}$

Wie vorgehen?

- Substitution mit $\eta = t - \underset{=3 \text{ im Bsp}}{a}$

neue Funktion $u(\eta) = y(\eta + a)$
$u'(\eta) = y'(\eta+a)$
$u''(\eta) = y''(\eta+a)$

⇒ Nach Substitution gleich wie bei normaler DGL rechnen (Rezept)

⇒ wenn $u(\eta)$ gefunden: Rücksubstitution

formuliere neu: $\eta = t - 3$, $t = \eta + 3$

$y''(\eta+3) + 4y'(\eta+3) = 4(\eta+3)$

$u(\eta) = y(\eta+3) \to u(0) = y(3)$
$u'(\eta) = y'(\eta+3) \to u'(0) = y'(3)$
$u''(\eta) = y''(\eta+3)$ ''

① neu $\begin{cases} u''(\eta) + 4u'(\eta) = 4(\eta+3) \\ u(0) = 0 \\ u'(0) = 7 \end{cases}$

Wende Rezept an: Transformiere

$\mathscr{L}(f') = s\mathscr{L}(f) - f(0)$
$\mathscr{L}(f'') = s^2\mathscr{L}(f) - sf(0) - f'(0)$

(2) $\mathcal{L}(u''(\eta)) + 4\mathcal{L}(u'(\eta)) = 4\mathcal{L}(\eta) + \mathcal{L}(12)$

$$s^2U(s) - s\underbrace{\cancel{u(0)}}_{0} - \underbrace{u'(0)}_{7} + 4sU(s) - \underbrace{\cancel{u(0)}}_{0} = \frac{4}{s^2} + \frac{12}{s}$$

$$U(s)(s^2+4s) - 7 = \frac{4+12s}{s^2}$$

$$U(s)(s^2+4s) = \frac{4+12s+7s^2}{s^2}$$

$$U(s) = \frac{4+12s+7s^2}{s^3(s+4)} \overset{\text{PBZ}}{=} \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+4}$$

⋮ kompliziertes Rechnen

$$U(s) = \frac{17}{16}\cdot\frac{1}{s} + \frac{11}{4}\cdot\frac{1}{s^2} + 1\cdot\frac{1}{s^3} - \frac{17}{16}\cdot\frac{1}{s+4}$$

$\downarrow \mathcal{L}^{-1}(s)(\eta)$

$$u(\eta) = \frac{17}{16} + \frac{11}{4}\eta + \frac{1}{2}\eta^2 - \frac{17}{16}e^{-4\eta}$$

Rücksubstitution $\eta = t-3$

$$u(t-3) := y(t) = \frac{17}{16} + \frac{11}{4}(t-3) + \frac{1}{2}(t-3)^2 - \frac{17}{16}e^{-4(t-3)}$$

Tipps für die Serie 3

① Basics – Wichtig

a-e) verschiedene t- und s-shifts

② Convolution: brauche die Formeln

a) brauche $f(t) * g(t) = \mathcal{L}^{-1}(F(s) \cdot G(s))$

b) trig. identity

c) brauche Laplace, nicht integral

③ DGL Rezept

④ DGL und Faltung : schreibe $\frac{G(s)}{s} = \left(G(s) \cdot \frac{1}{s}\right) \to \mathcal{L}^{-1}$ und Faltung

⑤ DGL Rezept mit $\delta(t-a)$

$\mathcal{L}(\delta(t-a)) = e^{-sa}$