Analysis Übungsstunde 4

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10.10.24

Periodische Funktionen

$f(x)$ ist p-periodisch, wenn $\boxed{f(x) = f(x+p)}$

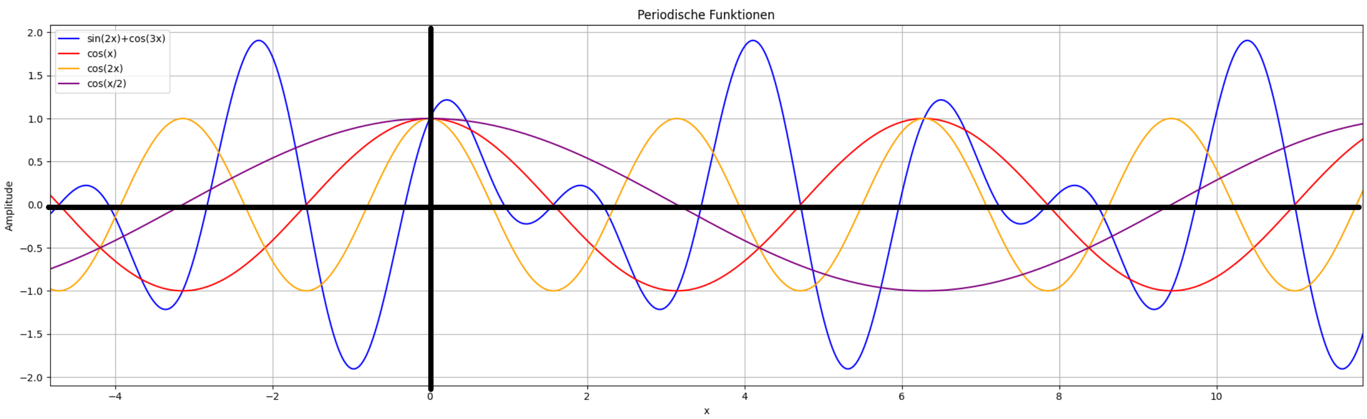
Eigenschaften

gleiche Periodizität p

def: $f(x) = f(x+p)$ $\quad h(x) = h(x+p)$ $\quad g(x) = g(x+d)$

- $f(ax)$ ist $\frac{p}{a}$ - periodisch
- $a \cdot f(x) + b \cdot h(x)$ p-periodisch (beide p-periodisch)
- f und h sind beide $n \cdot p$ - periodisch, $n \in \mathbb{Z}$
- $a \cdot f(x) + c \cdot g(x)$ sind periodisch, falls $\frac{p}{d} \in \mathbb{Q}$
  - ↳ Mit Periode $p^* = \text{kgV}(p, d)$

$$\boxed{\text{bei } \sin(kx) \text{ oder } \cos(kx) \Rightarrow p = \frac{2\pi}{k}}$$



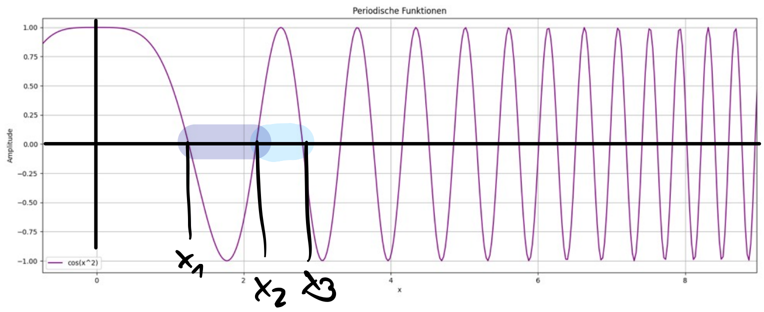
Bsp
- $\cos(x) \longrightarrow 2\pi$ def
- $\cos\left(\frac{x}{2}\right) \rightarrow \frac{x+p}{2} = \frac{x}{2} + 2\pi \quad \frac{p}{2} = 2\pi \rightarrow p = 4\pi$
  - ↳ $\cos(kx) \sim \frac{2\pi}{k} \quad k = \frac{1}{2} \quad p = \frac{2\pi}{\frac{1}{2}} = \underline{\underline{4\pi}}$

- $\cos(2x) \rightarrow \quad k=2 \quad p=\frac{2\pi}{2}=\pi$
- $\sin(2x)+\cos(3x) \qquad p_1=\frac{2\pi}{2} \qquad p_2=\frac{2\pi}{3}$

  $p_1=p \qquad p_2\ d=3$

$\rightarrow$ check $\frac{p}{d}=\frac{2}{3}\in\mathbb{Q}$ ✓ $\quad p^*=\text{kgV}\left(\pi,\frac{2}{3}\pi\right)=\frac{2\pi}{3}\cdot 3 = p^*=2\pi$

Bsp: Zeige, dass $\cos(x^2)$ nicht periodisch ist.



Nullstellen:

(1) $x^2=\frac{\pi}{2} \qquad x_1=\sqrt{\frac{\pi}{2}}$

(2) $x^2=\frac{3\pi}{2} \qquad x_2=\sqrt{\frac{3\pi}{2}}$

(3) $x^2=\frac{5\pi}{2} \qquad x_3=\sqrt{\frac{5\pi}{2}}$

1-2) $\sqrt{\frac{3\pi}{2}}-\sqrt{\frac{\pi}{2}}=0{,}91$

2-3) $\sqrt{\frac{5\pi}{2}}-\sqrt{\frac{3\pi}{2}}=0{,}63$

$\neq$

Eigenschaften Periodizität & Begrenztheit

Es gilt: ist $f(x)$ periodisch & stetig $\Rightarrow$ Begrenzt

↳ dann ist $f''(x)$ (Ableitungen) auch periodisch & stetig $\Rightarrow$ Begrenzt

$\cos(x^2)$

Begrenzt ✓

Periodisch ✗

$\frac{d}{dx}\cos(x^2) = -2x\sin(x^2)$

$\lim_{x\to\infty} \rightarrow \infty$

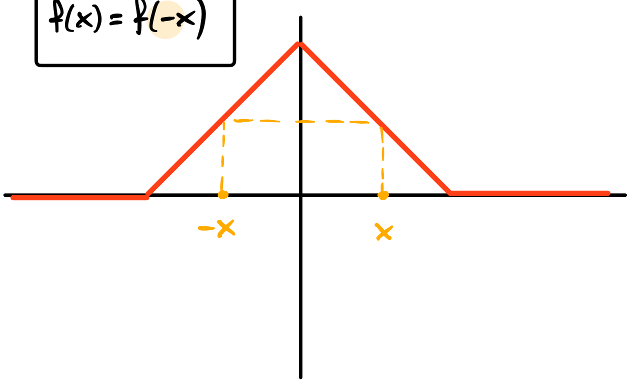
$\rightarrow$ nicht Begrenzt

$\rightarrow$ nicht periodisch

Gerade & Ungerade Funktionen

Gerade

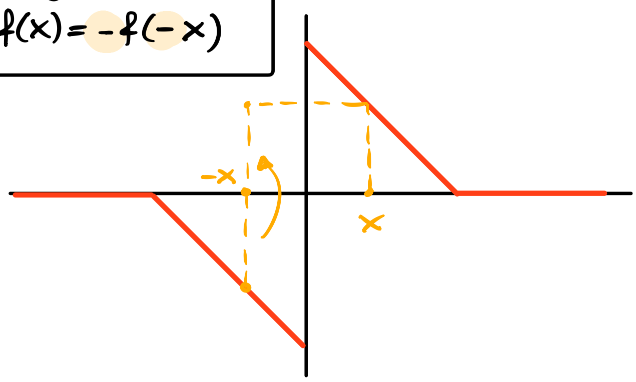
$f(x)=f(-x)$



Bsp: $\cos(x)$

Ungerade

$f(x)=-f(-x)$



Bsp: $\sin(x)$, $\tan(x)$

Eigenschaften

$g_1(x)$ & $g_2(x)$ $\rightarrow$ gerade

$u_1(x)$ & $u_2(x)$ $\rightarrow$ ungerade

- $g_1 \cdot g_2$: gerade
- $u_1 \cdot u_2$: gerade
- $u_1 \cdot g_1$: ungerade
- $\int_{-a}^{a} g_1(x) = 2\int_0^a g_1(x)\,dx$
- $\int_{-a}^{a} u_1(x)\,dx = 0$

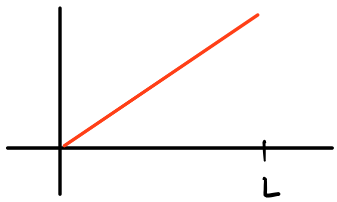
- $u_1 + u_2$: ungerade
- $g_1 + g_2$: gerade
- $\frac{d}{dx}(g(x))$: ungerade
- $\frac{d}{dx}(u(x))$: gerade

$\Rightarrow$ Gerade oder ungerade? $\rightarrow$ Bestimme $f(-x)$ $\rightarrow$ $f(x)$ g ; $\rightarrow$ $-f(x)$ ung.

Bsp $f(x) = \cos(2x) + \cos(x^2)$

$$f(-x) = \underline{\cos(-2x)} + \cos\left((-x)^2\right) = \cos(2x) + \cos(x^2) = \underbrace{f(x)}_{\text{gerade}}$$

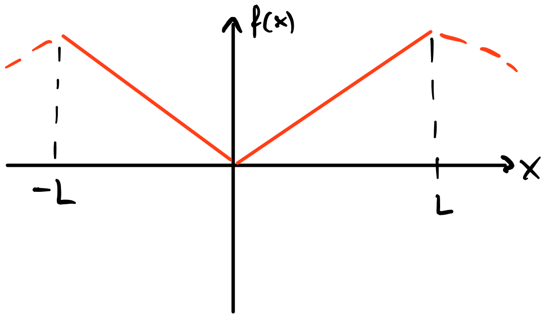
Periodische Fortsetzung



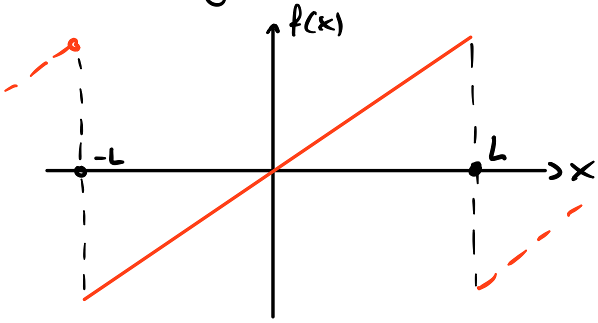
$f(x)$ ist gegeben für $x \in [0, L]$

Fortsetzung soll $2L$-periodisch sein

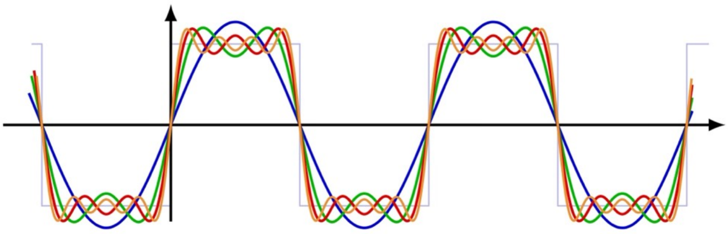
gerade

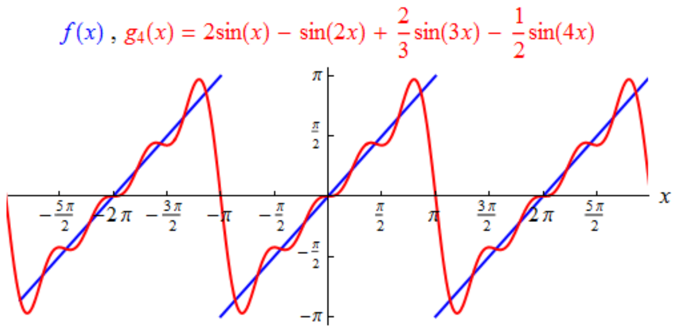


ungerade



Fourier-Reihe

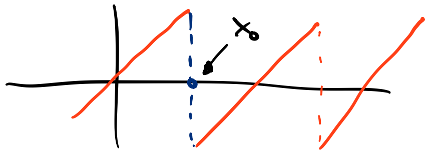




Ziel: Eine Funktion $f(x)$ welche $2L$ periodisch ist, als Summe von Sinus & Cosinus schreiben

$f(x)$ kann unstetig sein, muss aber das Drichlet Theorem erfüllen

Drichlet Theorem: $\boxed{\frac{1}{2}\left(f(x_0^-) + f(x_0^+)\right) = f(x_0)}$



$\frac{1}{2}(-1+1) = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

$\cos\left(\frac{n\pi}{L}x\right)$ & $\sin\left(\frac{n\pi}{L}x\right)$ sind beide $2L$-Periodisch

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

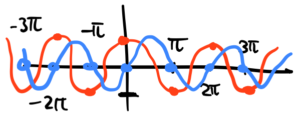
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Wichtige Vereinfachungen

• $f(x)$ gerade	• $f(x)$ ungerade
$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$	$a_0 = 0$
$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$	$a_n = 0$
$b_n = 0$	$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

Wichtige Integrale & Geometrie (ZF S.9/10)



$$\int_{-L}^{L} cos(\frac{n\pi x}{L}) \cdot cos(\frac{m\pi x}{L}) dx = \begin{cases} 0 & \text{für } n \neq m \\ L & \text{für } n = m \\ 2L & \text{für } n = m = 0 \end{cases}$$

$$\int_{-L}^{L} sin(\frac{n\pi x}{L}) \cdot sin(\frac{m\pi x}{L}) dx = \begin{cases} 0 & \text{für } n \neq m \\ L & \text{für } n = m \neq 0 \end{cases}$$

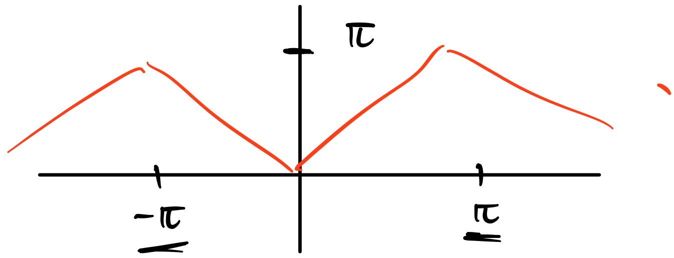
$$\int_{-L}^{L} sin(\frac{n\pi x}{L}) \cdot cos(\frac{m\pi x}{L}) dx = 0 \;\; \forall \, n, m$$

$sin(\pi n) = 0$; $cos(\pi n) = (-1)^n$

$$cos(\frac{\pi}{2} n) = \left(\frac{1 + (-1)^n}{2}\right) (-1)^{\frac{n}{2}} = \begin{cases} 0, & n = 2j+1 \\ (-1)^j, & n = 2j \end{cases}$$

$$sin(\frac{\pi}{2} n) = \left(\frac{1 + (-1)^n}{2}\right) (-1)^{\frac{n+2}{2}} = \begin{cases} 0, & n = 2j \\ (-1)^j, & n = 2j+1 \end{cases}$$

Bsp $f(x) = |x|$, $x \in [-\pi, \pi]$



2π-periodisch

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi}{L} x) + b_n sin(\frac{n\pi}{L} x)$$

$\rightarrow b_n = 0$ da $f(x)$ gerade !

$\rightarrow 2\pi \overset{!}{=} 2L \rightarrow L = \pi$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx = \frac{1}{\pi} \int_0^{\pi} x \, dx = \frac{1}{\pi} \left[\frac{1}{2} x^2 \right]_0^{\pi}$$

$$a_0 = \frac{1}{\pi} \left(\frac{\pi^2}{2} - 0 \right) = \underline{\underline{\frac{\pi}{2}}}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot cos\left(\frac{n\pi}{L} x\right) dx \underset{\substack{L=\pi \\ f(x)=|x|}}{=} \underline{\frac{2}{\pi}} \int_0^{\pi} x \cdot cos(nx)$$

ZF :
$$\int \mathbf{x \, cos(nx)} = \frac{x}{n} sin(nx) - \frac{1}{n} \int sin(nx) dx$$
$$= \frac{x}{n} sin(nx) + \frac{1}{n^2} cos(nx) = \frac{nx \; sin(nx) + cos(nx)}{n^2}$$

$$= \frac{2}{\pi} \left[\frac{nx \cdot sin(nx) + cos(nx)}{n^2} \right]_0^{\pi} = \frac{2}{\pi} \left(\frac{n\pi \cdot sin(n\pi) + cos(n\pi)}{n^2} - \frac{0 + cos(0)}{n^2} \right)$$

$$a_n = \frac{2}{\pi} \left(\frac{\cancel{n\pi \cdot sin(n\pi)}^{\,0} + \overbrace{cos(n\pi)}^{(-1)^n} - 1}{n^2} \right) = \frac{2}{\pi n^2} \left((-1)^n - 1 \right)$$

$$\underline{\underline{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi} \left((-1)^n - 1 \right) \cdot cos(nx) + \cancel{b_n}^{\,0}}}$$

Summen mit Fourier berechnen

→ umformen bis es passt und wähle x_0 entsprechend

Gegeben: $$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2L}{\pi n} \sin\left(\frac{n\pi}{L} x\right)$$

Gefragt: $$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = ?$$

Wähle: $2k+1 = n$ $\quad k = \frac{n-1}{2}$

! Anpassen der Summe! $\sum_{n=1}^{\infty} (\;)$ $\quad n=1 \rightarrow k = \frac{1-1}{2} = 0$

$2(k+1)$ immer gerade $(-1)^{(\;)} = +1$

neu: $$x = \frac{2L}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{(2k+1+1)}}{(2k+1)} \cdot \sin\left(\frac{(2k+1)\pi}{L} x\right)$$

$$= \frac{2L}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \cdot \sin\left(\frac{(2k+1)\pi}{L} x\right)$$

soll $(-1)^k$ geben → Wähle x

bei $k=0$ $\quad \underbrace{(-1)^0}_{=1} = \sin\left(\frac{\pi}{L} x_0\right)$ → true wenn $\sin\left(\frac{\pi}{2}\right)$

$$\frac{\pi}{2} \overset{!}{=} \frac{\pi}{L} x_0 \qquad \underline{x_0 = \frac{L}{2}}$$

$$x_0 = \frac{L}{2} \longrightarrow \sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k$$

$$x = x_0 = \frac{L}{2} = \frac{2L}{\pi}\left(\underbrace{\sum_{k=0}^{\infty} \frac{1}{2k+1} \cdot (-1)^k}_{a}\right)$$

$\Leftrightarrow$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{L}{2} \cdot \frac{\pi}{2L} = \underline{\underline{\frac{\pi}{4}}}$$

$$a = \frac{\pi}{4}$$