Analysis Übungsstunde 5

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Komplexe Fourier Reihe

normale Fourier-Reihe: $a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$

ersetze durch cplx Schreibweise $\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \frac{1}{2}\left(e^{i\frac{n\pi}{L}x} + e^{-i\frac{n\pi}{L}x}\right) + b_n \frac{1}{2i}\left(e^{i\frac{n\pi}{L}} - e^{-i\frac{n\pi}{L}}\right)$$

$\boxed{\frac{1}{i} = -i}$

$\left(\frac{1}{i} \cdot \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1}\right)$

$$= a_0 + \sum_{n=1}^{\infty} \underbrace{\frac{a_n - ib_n}{2}}_{c_n} e^{i\frac{n\pi}{L}x} + \underbrace{\frac{a_n + ib_n}{2}}_{c_{-n}} e^{-i\frac{n\pi}{L}x}$$

$$= a_0 + \sum_{n=1}^{\infty} c_n e^{i\frac{n\pi}{L}x} + c_{-n} e^{-i\frac{n\pi}{L}x}$$

passe einfach auf dass $c_n \ \forall n \in \mathbb{Z}$ geht
bsp: $\frac{1}{n} \rightarrow c_0$ separat

$$= c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n \cdot e^{\frac{in\pi}{L}x} \quad \text{oder} \quad \sum_{n=-\infty}^{\infty} c_n \cdot e^{\frac{in\pi}{L}x}$$

(I) $\longleftrightarrow$ (II)

$$f(x) = c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n \cdot e^{\frac{i\pi n}{L}x}$$

! Wie immer, $f(x)$ muss $2L$-Periodisch sein!

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) \cdot e^{-\frac{i\pi n}{L}x} \quad ; \quad c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$c_n = \frac{1}{2}(a_n - ib_n)$

$c_{-n} = \frac{1}{2}(a_n + ib_n)$

$a_0 = c_0$; $a_n = c_n + c_{-n}$; $b_n = i(c_n - c_{-n})$

$e^{ix} + e^{-ix} = 2\cos(x)$; $e^{ix} - e^{-ix} = 2i\sin(x)$

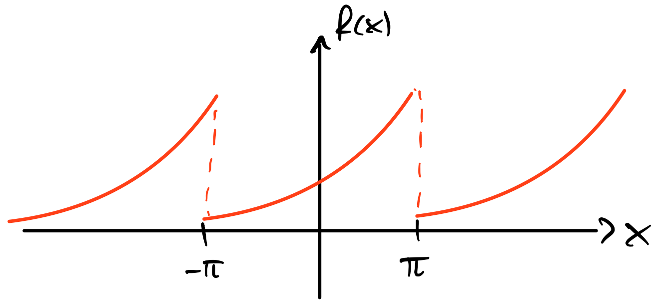
Remember:

- $e^{\pm ix} = \cos(x) \pm i\sin(x)$
- $e^{i\pi} = -1 \longrightarrow$ • $e^{\pm in\pi} = (-1)^n, \ n \in \mathbb{Z}$

Bsp: Kplx Fourier-Reihe

Berechne Fourier Reihe von $f(x) = e^x$, $x \in [-\pi, \pi]$
mit Periode $f(x) = f(x+2\pi)$

$$\Rightarrow p = 2\pi \overset{!}{=} 2L \qquad L = \pi$$



$$f(x) = c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n \cdot e^{\frac{in\pi}{L}x}$$

$$\Rightarrow c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{2\pi}\left(e^{\pi}-e^{-\pi}\right) = \frac{\sinh(\pi)}{\pi}$$

$$\Rightarrow c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi}{L}x} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \cdot e^{-inx}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,dx = \frac{1}{2\pi}\left[\frac{1}{(1-in)}\, e^{(1-in)x}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{(1-in)}\left(e^{\pi-in\pi} - e^{\pi+in\pi}\right) = \frac{1}{2\pi(1-in)}\left(e^{\pi}\cdot e^{-in\pi} - e^{\pi}\cdot e^{+in\pi}\right)$$

$$= \frac{1}{2\pi(1-in)}\left(e^{\pi}(-1)^n - e^{-\pi}(-1)^n\right) = \frac{(-1)^n}{\cancel{2}\pi(1-in)}\cdot\underbrace{\frac{\cancel{2}}{2}\cdot\left(e^{\pi}-e^{-\pi}\right)}_{\sinh(\pi)}$$

$$= \frac{(-1)^n}{\pi}\cdot\sinh(\pi)\cdot\underbrace{\frac{1}{(1-in)}\cdot\frac{(1+in)}{(1+in)}}_{=1+n^2} =$$

$$\Rightarrow f(x) = \frac{\sinh(\pi)}{\pi} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{n=\infty}\frac{(-1)^n}{1+n^2}\cdot\frac{\sinh(\pi)}{\pi} \qquad (\mathrm{I})$$

or

$$\frac{\sinh(\pi)}{\pi}\cdot\sum_{n=-\infty}^{n=\infty}\frac{(-1)^n}{1+n^2} \qquad (\mathrm{II})$$

Bsp. Prüfung Sommer 2017 A2

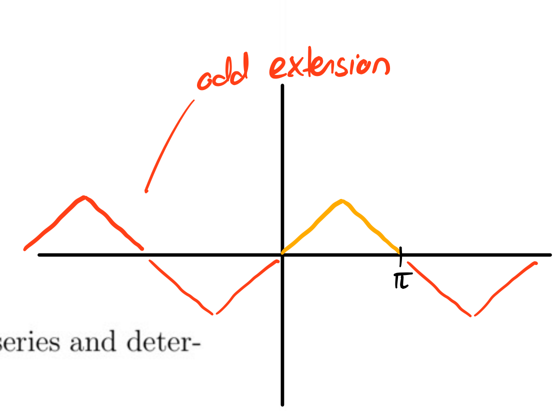
2. Fourier Series (*6+1=7 Points*)

a) Given the odd, 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ -(x-\pi) & \frac{\pi}{2} \le x \le \pi, \end{cases}$$

determine the real Fourier series of f, i.e. state it's Fourier series and determine the coefficients.

b) Does the Fourier series converge pointwise to the function f? Please justify your answer.



$\Rightarrow$ odd: $a_0 = 0\,,\ a_n = 0$

$\Rightarrow$ 2π periodic $\rightarrow 2L = 2\pi \rightarrow L = \pi$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx + \int_{\frac{\pi}{2}}^{\pi} -(x-\pi)\sin(nx)\,dx\right)$$

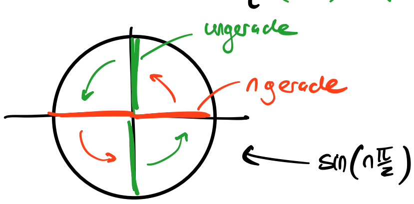
$\vdots$

$$\int \mathbf{x\,sin(nx)} = -\frac{x}{n}cos(nx) + \frac{1}{n}\int cos(nx)dx$$
$$= -\frac{x}{n}cos(nx) + \frac{1}{n^2}sin(nx) = \left[\frac{sin(nx) - nx\,cos(nx)}{n^2}\right]$$

Grenzen einsetzen

$$= \frac{4}{\pi n^2}\cdot\sin\left(n\frac{\pi}{2}\right)$$

$$\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & n = 2k \\ (-1)^k & n = 2k+1 \end{cases}$$



$$b_n = \frac{4}{\pi(2k+1)^2}\cdot(-1)^k$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) = \sum_{k=0}^{\infty} \frac{4\cdot(-1)^k}{\pi(2k+1)^2}\cdot\sin((2k+1)x)$$

$n = 1 = 2k+1 \Leftrightarrow k = 0$

FALLS:

(hypotetisch)

$$= \begin{cases} \frac{1}{k} & n = 2k \\ (-1)^k & n = 2k+1 \end{cases} \qquad L = \pi$$

$$f(x) = \underbrace{\sum_{k=1}^{\infty} \frac{4}{\pi(2k)^2} \cdot \frac{1}{k} \cdot \sin(2kx)}_{\text{Alle geraden}} + \underbrace{\sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)^2} \cdot (-1)^k \cdot \sin((2k+1)x)}_{\text{ungeraden Terme}}$$

$n = 2 = 2k \quad k = 1$

$n = 2k+1 \quad k = 0$

Für Serie: Brauche auch Skript letzte Woche!