Analysis Übungsstunde 7

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This Serie covers a new topic: the discrete Fourier transform (DFT) and the fast Fourier transform (FFT). This topic has not been taught in the last few years, so feel free to ask if you have a question.

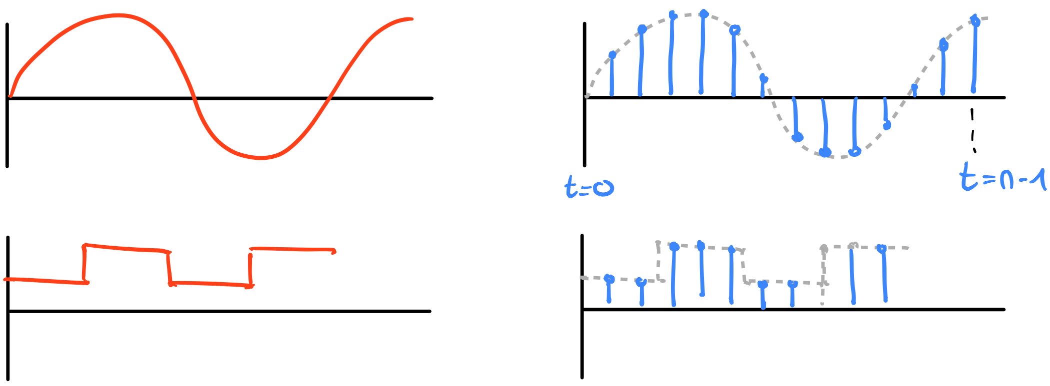
aber!

I'll have to talk with Prof. Iozzi to give you a precise answer. But I guess it was more about providing them insight into the DFT and FFT. The multiple-choice question might have a short question about the DFT or FFT in the exam (but I'm not even sure), but I don't think there'll be an open question about this topic.

Organisatorisches: Nächste Woche 7.11 keine Übungsstunde
→ Ich würde Recording auf Youtube laden

Discrete time (man kann nur an einzelnen punkten messen)

Continuous time | Discrete time



nicht für jeden Zeitwert definiert

Sampling time: wie oft gemessen wird

- Integrale werden zu Summen
- Zeit von 0 bis n-1 (das n-te Sample)

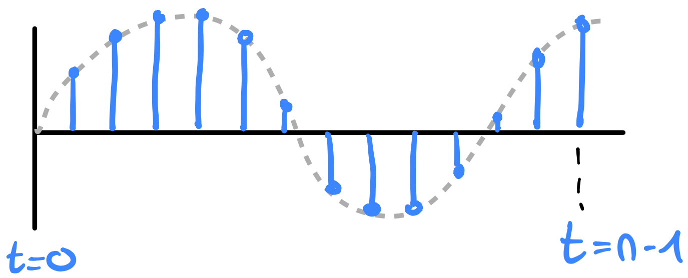
$$t_0 = 0$$
$$t_1 = \frac{2\pi}{n}$$
$$\vdots$$
$$t_{n-1} = (n-1)\frac{2\pi}{n}$$

$$\left.\begin{matrix} t_0 = 0 \\ \vdots \\ t_{n-1} \end{matrix}\right\} \; t_j = j \cdot \frac{2\pi}{n} \quad j \in [0, n-1]$$

$$f_j = f\left(j\frac{2\pi}{N}\right) \qquad t_j = j\frac{2\pi}{N}$$

⇒ gute Videos hier: (Steve Brunton)
DFT https://youtu.be/nl9TZanwbBk?si=b2lpDKE8KJ55uKzu
FFT https://youtu.be/toj_IoCQE-4?si=2iikxfd9X7aVWSzn

Discrete Fourier Transform



$$\underline{F} := \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix} \qquad \underline{C} := \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}$$

Vektor von Datenpunkten

endliche fourier Serie

$$f(t) = c_0 + c_1 e^{it} + c_2 e^{2it} + \dots + c_{n-1} e^{(n-1)it} =$$

für jeden sample Punkt t_j : $f_j = \sum_{k=0}^{n-1} c_k e^{kit_j}$ $\qquad \boxed{t_j = j\frac{2\pi}{n}}$

$\omega_n = e^{\frac{2\pi i}{n}}$

$$\triangleright \quad f_j = \sum_{k=0}^{n-1} c_k \cdot e^{kit_j} = \sum_{k=0}^{n-1} c_k \cdot \underbrace{\omega_n^{kj}}_{\to M_n} \quad \forall j \in [0, n-1]$$

$$\boxed{\triangleright\, \omega_n = e^{\frac{2\pi i}{n}}}$$

$$\triangleright \quad c_k = \frac{1}{n} \sum_{j=0}^{n-1} e^{-kit_j} \cdot f_j = \frac{1}{n} \sum_{j=0}^{n-1} \underbrace{\omega_n^{-kj}}_{\to\, = M_n^{-1}} \cdot f_j \quad \forall k \in [0, n-1] \Rightarrow \text{schreibe als Matrix}$$

(inverse)

$$\boxed{F = M_n C \quad \Longleftrightarrow \quad C = M_n^{-1} F}$$

$$M_n = (\omega_n^{hk})_{h,k=0\dots n-1}$$

$$\underbrace{\begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{n-1} \end{pmatrix}}_{F} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_{n-1}^{(n-1)^2} \end{bmatrix}}_{M_n} \underbrace{\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix}}_{C}$$

$$M_n^{-1} = \frac{1}{n}(\omega_n^{-hk})_{h,k=0\dots n-1}$$

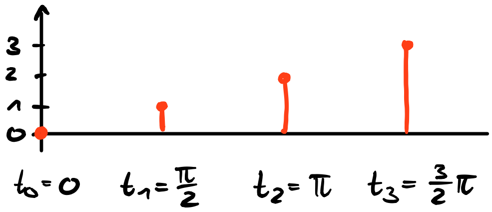
$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix} = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^{-2} & \cdots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-4} & \cdots & \omega_n^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \cdots & \omega_{n-1}^{-(n-1)^2} \end{bmatrix} \begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{n-1} \end{pmatrix}$$

(normalisation)

Vgl:

continous time	$\hat{f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t} dt$	$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(\omega) \cdot e^{+i\omega t} d\omega$
discrete time	$f_j = \sum_{n=0}^{n-1} c_k \cdot e^{kit_j}$ ↑ k-te frequenz coeff	$c_k = \frac{1}{n} \sum_{j=0}^{n-1} e^{-kit_j} \cdot f_j$

Bsp: sample:



finde trigonometrische representation

$t_j = \frac{2\pi}{n} j$

$\#$messungen $= 4$ $\quad n=4$

$F = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \quad C = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$f(0)=0, \; f(\frac{2\pi}{N})=1, \; f(2\frac{2\pi}{N})=2, \; f(3\frac{2\pi}{N})=3$

$f(t) = c_0 + c_1 e^{it} + c_2 e^{2it} + \dots + c_{n-1} e^{(n-1)it}$ // suche coeff c

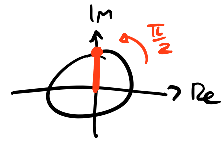
$= c_0 + c_1 e^{it} + c_2 e^{2it} + c_3 e^{3it}$

$C = M_n^{-1} F$

$$M_n^{-1} := \frac{1}{n}(w_n^{-hk})_{h,k=0,\dots,n-1} = \frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_n & w_n^{-2} & \cdots & w_n^{-(n-1)} \\ 1 & w_n^{-2} & w_n^{-4} & \cdots & w_n^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w_n^{-(n-1)} & w_n^{-2(n-1)} & \cdots & w_n^{-(n-1)^2} \end{pmatrix}$$

$w_n = e^{\frac{2\pi i}{n}}$ // $n=4$

$w_4 = e^{\frac{2\pi i}{4}} = e^{\frac{\pi}{2} i} = i$



$$M_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w_4^{-1} & w_4^{-2} & w_4^{-3} \\ 1 & w_4^{-2} & w_4^{-4} & w_4^{-6} \\ 1 & w_4^{-3} & w_4^{-6} & w_4^{-9} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & 1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & 1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{(-1+i)}{2} \\ -\frac{1}{2} \\ \frac{(-1-i)}{2} \end{pmatrix}$$

braucha Eulers form:

$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$

$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$

$$f(t) = \frac{1}{2}(\underbrace{3}_{c_0} + \underbrace{(-1+i)}_{c_1} e^{it} \underbrace{-1}_{c_2} e^{2it} \underbrace{-(1+i)}_{c_3} e^{3it})$$

$$f(t) = \underbrace{\frac{3}{2} - \frac{1}{2}\cos t - \frac{1}{2}\cos 2t - \frac{1}{2}\cos 3t - \frac{1}{2}\sin t + \frac{1}{2}\sin 3t}_{Re(f(t))} + i\underbrace{\left(\frac{1}{2}\cos t - \frac{1}{2}\cos 3t - \frac{1}{2}\sin t - \frac{1}{2}\sin 2t - \frac{1}{2}\sin 3t\right)}_{Im(f(t))}$$

- überprüfen: für $t = 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi$ würde funktion $0, 1, 2, 3$ werden
- da f nur die Reellen werte bei unseren sample points, nimm nur Realteil

$\Rightarrow$ Lösung $= Re(f(t)) = \frac{3}{2} - \frac{1}{2}\cos t - \frac{1}{2}\cos 2t - \frac{1}{2}\cos 3t - \frac{1}{2}\sin t + \frac{1}{2}\sin 3t$

$\Rightarrow$ n Reihen, n Spalten $\rightarrow O(n^2)$

Fast Fourier Transform (FFT)

Discrete Fourier transform $O(n^2)$

Fast Fourier Transform $O(n\log(n))$!

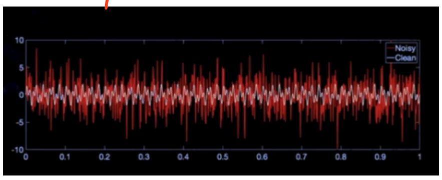
- Denoise Data
- Analysis of Data
- Compression of audio & Images

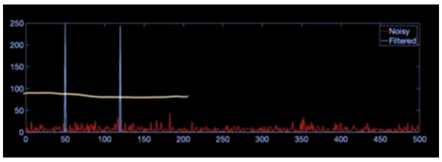
	DFT	FFT	
$N = 10^3$	$N^2 = 10^6$	$N\log_2 N \approx 10^4$	100 times smaller
$N = 10^6$	$N^2 = 10^{12}$	$N\log_2 N \approx 2\times 10^7$	50,000 times smaller

$$\underbrace{\begin{pmatrix}\hat f_0\\ \hat f_1\\ \hat f_2\\ \vdots\\ \hat f_{n-1}\end{pmatrix}}_{F} = \underbrace{\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1}\\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_{n-1}^{(n-1)^2}\end{bmatrix}}_{M_n} \underbrace{\begin{pmatrix}c_0\\ c_1\\ c_2\\ \vdots\\ c_{n-1}\end{pmatrix}}_{C}$$

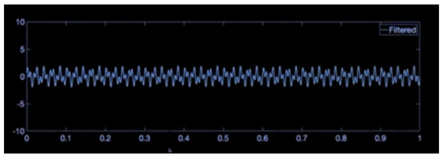
noisy Data

FFT →





cut →



bsp $n = 2^{10} = 1024$

$\Rightarrow$ $\underline{F = \underline{M_n} C}$

$F = M_{1024} \cdot C$ aufteilen von M-Matrix: gerade & ungerade reihen

$$F = \begin{bmatrix} I_{512} & -D_{512}\\ I_{512} & -D_{512}\end{bmatrix}\begin{bmatrix} M_{512} & 0\\ 0 & M_{512}\end{bmatrix}\begin{pmatrix} c_{even}\\ c_{odd}\end{pmatrix}$$

c_{even}: $\begin{pmatrix}c_0\\ c_2\\ c_4\\ c_6\end{pmatrix}$ → c_0, c_4, c_2, c_6

c_{odd}: $\begin{pmatrix}c_1\\ c_3\\ c_5\\ c_7\\ \vdots\end{pmatrix}$ → c_1, c_5, c_3, c_7

! def $f^{(o)} := f_{odd}$

$f^{(e)} := f_{even}$

(same for c)

$$D_{512} = \begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & \omega & 0 & \cdots & 0\\ 0 & 0 & \omega^2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & & \omega^{511}\end{bmatrix}$$

einfach zum rechnen für computer: Diagonalmatrixen & Nulleinträge

weiters aufteilen $M_{1024} \to M_{512} \to M_{256} \to M_{128} \cdots M_4 \to M_2$!!

$$T_n := \begin{bmatrix}[M_2] & 0 & 0 & \cdots & 0\\ 0 & [M_2] & 0 & \cdots & 0\\ 0 & 0 & [M_2] & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & & [M_2]\end{bmatrix}$$

Bsp:

	$c^{(o)}$	c^{oo} c^{oe}
c_0	0	0
c_1	2	4
c_2	4	2
c_3	6	6 ...etc
c_4	1	1
c_5	3	5
c_6	5	3
c_7	7	7
	$c^{(e)}$	c^{ee} c^{eo}

$\Rightarrow O(n\log(n))$

→ code: f_hat = fft(f, n)

signal (mit noise) ↓, länge ↙

Formeln für Serie 7:

$$M = \frac{n}{2} \qquad c^{(o)} = \begin{pmatrix}c_0^{(o)}\\ c_2^{(o)}\end{pmatrix} = M_2^{-1} f^{(o)} \qquad c^{(e)} = \begin{pmatrix}c_0^{(e)}\\ c_2^{(e)}\end{pmatrix} \cdot M_2^{-1} f^{(e)}$$

$K < M$:

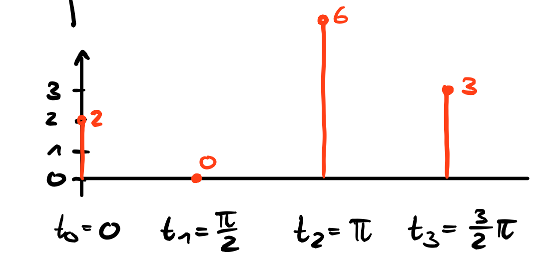
$$c_k = \frac{1}{2}\left(c_k^{(o)} + \omega_N^k c_k^{(e)}\right)$$

$K \geq M$:

$$c_{k+M} = \frac{1}{2}\left(c_k^{(o)} - \omega_N^k c_k^{(e)}\right)$$

Bsp mit FFT

finde trigonometrische repesentation



$t_j = \frac{2\pi}{n} j$

$\#$ Messungen $= 4$ $\quad n = 4$

3. Fast Fourier Transform (FFT)

Compute the Fast Fourier Transform (FFT) of the same function given in exercise 1. Check that you get the same result.

Steps:

1) Find the value of w_M, where $M = \frac{N}{2}$.
2) Compute the even and odd coefficients $\mathbf{C}^{(o)}$ and $\mathbf{C}^{(e)}$ using the formula

$$\mathbf{C}^{(o)} = \begin{bmatrix} c_0^{(o)} \\ c_2^{(o)} \end{bmatrix} = \mathbf{M}_2^{-1}\mathbf{f}^{(o)}, \qquad \text{and} \qquad \mathbf{C}^{(e)} = \begin{bmatrix} c_0^{(e)} \\ c_2^{(e)} \end{bmatrix} = \mathbf{M}_2^{-1}\mathbf{f}^{(e)}$$

3) Find the value of w_N.
4) Compute the coefficient c_k using the formulas for $k < M$

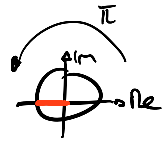
$$c_k = \frac{1}{2}\left(c_k^{(o)} + w_N^k c_k^{(e)}\right).$$

And for the coefficient c_k with $k \geq M$,

$$c_k = \frac{1}{2}\left(c_k^{(o)} - w_N^k c_k^{(e)}\right).$$

$\to \omega_M = ? \quad \to M = \frac{n}{2} = \frac{4}{2} = 2$

$\omega_M = e^{\frac{2\pi i}{2}} = e^{\pi i} = -1$



$C = \frac{1}{n} M_M^{-1} F, \quad F = \begin{pmatrix} 2 \\ 0 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$ $\quad f^{(o)} = \begin{matrix} f_0 \\ f_2 \end{matrix}$ $\quad f^{(e)} = \begin{pmatrix} f_1 \\ f_3 \end{pmatrix}$

Anstatt 4x4 Matrix nun zwei einzelne 2x2

$c^{(o)} = M_2^{-1} \cdot f^{(o)}$

$$\begin{pmatrix} c_0^{(o)} \\ c_1^{(o)} \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} f_0 \\ f_2 \end{pmatrix}$$

$$\begin{pmatrix} c_0^{(o)} \\ c_1^{(o)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} f_0 + f_2 \\ f_0 - f_2 \end{pmatrix}$$

$c^{(e)} = M_2^{-1} f^{(e)}$

$$\begin{pmatrix} c_0^{(e)} \\ c_1^{(e)} \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} f_1 \\ f_3 \end{pmatrix}$$

$$\begin{pmatrix} c_0^{(e)} \\ c_1^{(e)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} f_1 + f_3 \\ f_1 - f_3 \end{pmatrix}$$

since $N = 4 \to \underline{\omega_N = e^{\frac{2\pi i}{4}} = e^{i\frac{\pi}{2}} = i}$ $\qquad k = 0, 1, 2, 3$

$k < \overset{=2}{M}$: $\quad k = 0, 1$

$$c_k = \frac{1}{2}\left(c_k^{(o)} + \omega_N^k c_k^{(e)}\right)$$

$$c_0 = \frac{1}{2}\left(c_0^{(o)} + i^0 c_0^{(e)}\right)$$
$$= \frac{1}{4}(f_0 + f_1 + f_2 + f_3)$$

$$c_1 = \frac{1}{2}\left(c_1^{(o)} + i^1 c_1^{(e)}\right)$$
$$= \frac{1}{4}(f_0 - if_1 - f_2 + if_3)$$

$k \geq \overset{=2}{M}$ $\quad k = 2, 3$

$$c_{k+M} = \frac{1}{2}\left(c_k^{(o)} - \omega_N^k c_k^{(e)}\right)$$

$$\underset{0+2}{c_2} = \frac{1}{2}\left(c_0^{(o)} - i^2 c_0^{(e)}\right)$$
$$= \frac{1}{4}(f_0 - f_1 + f_2 - f_3)$$

$$\underset{1+2}{c_3} = \frac{1}{2}\left(c_1^{(o)} - i^3 c_1^{(e)}\right)$$
$$= \frac{1}{4}(f_0 + if_1 - f_2 - if_3)$$

brauche Eulers form:

$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$

$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$

$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \\ 3 \end{pmatrix} \to$ einsetzen

$c_0 = \frac{11}{4} \qquad c_2 = \frac{5}{4}$

$c_1 = \frac{-4+3i}{4} \qquad c_3 = \frac{-4-3i}{4}$

$f(t) = c_0 + c_1 e^{it} + c_2 e^{2it} + c_3 e^{3it}$

$$f(t) = \frac{1}{4}\underbrace{\left(11 - 4\cos(t) - 3\sin(t) + 5\cos(2t) - 4\cos(3t) + 3\sin(3t)\right)}_{Re(\,)} + \frac{i}{4}\underbrace{\left(-4\sin(t) + 3\cos(t) + 5\sin(2t) - 4\sin(3t) - 3\cos(3t)\right)}_{Im(\,)}$$