

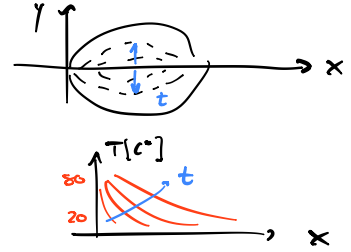
Analysis Übungsstunde 9

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- 1D Wellengleichung $u_{tt} = c^2 u_{xx}$
- 1D Wärmeleitungsgleichung $u_t = c^2 u_{xx}$
- 2D Poissonsgleichung $u_{xx} + u_{yy} = f(x, y)$
- 2D Wellengleichung $u_{tt} = c^2 (u_{xx} + u_{yy})$
- 2D Wärmeleitungsgleichung $u_t = c^2 (u_{xx} + u_{yy})$
- 3D Laplacegleichung $u_{xx} + u_{yy} + u_{zz} = 0 = \nabla^2 u$



1D Wellengleichung Allgemeiner Ansatz: "show your work"
"separation of variables"

$$\left\{ \begin{array}{l} u_{tt} = c^2 u_{xx} \rightarrow u_{tt} - c^2 u_{xx} = 0 \\ u(0, t) = 0 \\ u(L, t) = 0 \\ u(x, 0) = \sin\left(\frac{\pi}{L}x\right) \\ u_t(x, 0) = 0 \end{array} \right. \quad \text{RB}$$

1) Ansatz $u(x, t) = F(x)G(t)$

$$u_{tt} := F(x)\ddot{G}(t)$$

2) Einsetzen $u_{tt} = c^2 u_{xx}$
 $F\ddot{G} = c^2 F''G$

$$u_{xx} := F''(x)G(t)$$

→ bringe in verhältnis:

$$\boxed{\frac{F''}{F} = \frac{\ddot{G}}{c^2 G} := k}$$

$k \in \mathbb{R}$

3) Separieren

$$\frac{F''}{F} = k$$

$$\frac{\ddot{G}}{c^2 G} = k$$

4) Nutze Randbedingungen

$$u(0,t) = 0 \rightarrow F(0) \cdot G(t) = 0 \rightarrow F(0) = 0$$

$$u(L,t) = 0 \rightarrow F(L) \cdot G(t) = 0 \rightarrow F(L) = 0$$

$$u(x,0) = \sin\left(\frac{\pi}{L}x\right)$$

$$u_t(x,0) = 0 \rightarrow F(x) \cdot \dot{G}(0) = 0 \quad \dot{G}(0) = 0$$

5) Lösen von DGL mit Fallunterscheidung: für welches k gilt was?

$$\frac{F''}{F} = k$$

$k = 0$

$$F'' = 0 \rightarrow F(x) = Ax + B$$

$$\text{RB: } F(0) = 0 = 0 + B \quad B = 0$$

$$F(L) = 0 = AL + 0 \quad A = 0$$

$$F(x) = 0$$

$k > 0$

$$F'' = kF \rightarrow F'' - kF = 0 \quad \text{mit Ansatz } y = e^{\lambda x}$$

$$\lambda^2 - k = 0 \quad \lambda = \pm \sqrt{k}$$

$$\Rightarrow F(x) = C e^{\sqrt{k}x} + D e^{-\sqrt{k}x}$$

$$\text{RB: } F(0) = 0 = C + D \rightarrow D = -C$$

$$F(L) = 0 = C e^{\sqrt{k}L} - C e^{-\sqrt{k}L} = \begin{cases} \sqrt{k}L = -\sqrt{k}L & k > 0 \\ C = 0 \end{cases}$$

$$F(x) = 0$$

$k < 0$

$$F'' - kF = 0 \quad k := -\alpha$$

$$F'' + \alpha F = 0 \rightarrow \lambda^2 + \alpha = 0 \rightarrow \lambda = \pm \sqrt{-\alpha} = \pm i\sqrt{\alpha}$$

$$F(x) = e^{0x} (A \sin(\sqrt{\alpha}x) + B \cos(\sqrt{\alpha}x)) \quad \checkmark$$

$$\text{RB: } F(0) = 0 = A \cdot 0 + B \cdot 1 \quad B = 0$$

$$F(L) = 0 = A \sin(\sqrt{\alpha}L) + 0 \rightarrow A \neq 0$$

$$F(x) = A \sin\left(\frac{n\pi}{L}x\right)$$

brauchen eine nicht triviale Lsg

$$\sin(\cdot) = 0 \quad \sqrt{\alpha}L = n\pi$$

$$\boxed{\sqrt{\alpha} = \frac{n\pi}{L}}$$

Weil für Fälle $k=0, k>0$ $F(x)=0$ gilt, muss $G(t)$ für diese nicht mehr berechnet werden!

dh für $G(t)$ gilt

$$\ddot{G} = kc^2G \xrightarrow{k=-\alpha} \ddot{G} + \alpha c^2G = 0$$

($k < 0$)

$$0 = \lambda^2 + \alpha c^2 \rightarrow \lambda = \pm i\sqrt{\alpha}c$$

$$G(t) = e^{0t} (C \sin(\sqrt{\alpha}ct) + D \cos(\sqrt{\alpha}ct))$$

$\frac{d}{dt}$

$$\text{RB: } \dot{G}(0) = 0 = c\sqrt{\alpha} C \cos(c\sqrt{\alpha} \cdot 0) - D c\sqrt{\alpha} \sin(c\sqrt{\alpha} \cdot 0)$$

$$0 = c\sqrt{\alpha} C \cdot 1 - 0 \rightarrow C = 0$$

$$G(t) = D \cos(c\sqrt{\alpha} t)$$

6) Zusammensetzen $u(x, t) = F(x) \cdot G(t)$, $\sqrt{\alpha} = \frac{n\pi}{L}$

$$u_n(x, t) = A_n \sin\left(\frac{n\pi}{L} x\right) \cdot D_n \cos\left(c \frac{n\pi}{L} t\right) \parallel A_n \cdot D_n = B_n$$

7) Superposition

$$u_n(x, t) = \sum B_n \sin\left(\frac{n\pi}{L} x\right) \cdot \cos\left(\frac{n\pi}{L} ct\right)$$

8) letzte RB: Coeff vgl

$$u(x, 0) = \sin\left(\frac{1\pi}{L} x\right) = \sum B_n \sin\left(\frac{n\pi}{L} x\right) \cdot \overbrace{\cos(0)}^1$$

$$n=1 \quad B_1 = 1$$

$$\forall n \neq 1 \quad B_n = 0$$

9) Spezifische Lösung

$$\Rightarrow u(x, t) = 1 \cdot \sin\left(\frac{1\pi}{L} x\right) \cdot \cos\left(\frac{\pi}{L} ct\right)$$

1D Wellengleichung feste Lösung mit Fourier $(c = \sqrt{\frac{T}{\mu}})$ ^{Kraft} _{Massenverteilung}

Für eine eindimensionale Wellengleichung der Form $u_{tt} = c^2 u_{xx}$ und den Randbedingungen, $x \in [0, L]$

$$\begin{cases} u(0, t) = u(L, t) = 0 \\ u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases}$$

$$\lambda_n = \frac{c n \pi}{L} \quad (2)$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n \pi}{L} x\right) \quad (3)$$

$$g(x) = \sum_{n=1}^{\infty} B_n^* \lambda_n \sin\left(\frac{n \pi}{L} x\right) \quad (4)$$

finden wir eine allgemeine Lösung:

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)) \sin\left(\frac{n \pi}{L} x\right) \quad (1)$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n \pi}{L} x\right) dx \quad (5)$$

$$B_n^* = \frac{2}{L \lambda_n} \int_0^L g(x) \sin\left(\frac{n \pi}{L} x\right) dx \quad (6)$$

▷ λ_n bestimmen = $\frac{c n \pi}{L}$: c & L aus Aufgabe lesen

▷ ! versuche zuerst B_n & B_n^* mit Koeff vgl zu bestimmen !
(3), (4) Zeit sparen

▷ sonst B_n & B_n^* mit den Integralen finden (5), (6)

Bsp: Winkler 2017 A4

4. (10 Points)

Consider the string of length $L = \pi$ and $c^2 = 1$ with zero initial velocity, initial deflection $u(x, 0) = x(x - \pi)$ and fixed endpoints. The deflection $u(x, t)$ is a solution of the following PDE

$$\begin{cases} u_{tt} = u_{xx}, & t > 0, x \in (0, \pi) \\ u(0, t) = 0 = u(\pi, t), & t \geq 0 \\ f(x) = u(x, 0) = x(x - \pi), & 0 \leq x \leq \pi \\ g(x) = u_t(x, 0) = 0, & 0 \leq x \leq \pi \end{cases}$$

Find $u(x, t)$.

$$\rightarrow L = \pi$$

$$\rightarrow c = 1$$

$$\lambda_n = \frac{c n \pi}{L} = n$$

▷ B_n & B_n^* : $g(x) = 0 = \sum B_n^* \sin\left(\frac{n \pi}{L} x\right)$ $B_n^* = 0 \quad \forall n$

$$f(x) = x(x - \pi) = \sum B_n \cdot \sin\left(\frac{n \pi}{L} x\right) = \sum B_n \cdot \sin(nx) \dots ?$$

B_n nicht mit Koeff vgl bestimmbar

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n \pi}{L} x\right) dx = \frac{2}{\pi} \int_0^\pi x(x - \pi) \sin(nx) dx$$

∴ partielle integration (nutzt ZF)

$$= -\frac{4}{n^3 \pi} \cos(nx) \Big|_0^\pi = -\frac{4}{n^3 \pi} \left(\underbrace{\cos(n\pi)}_{(-1)^n} - 1 \right) = 0, -2, 0, -2$$

n : gerade
ungerade

$$B_n = \begin{cases} 0 & \text{wenn } n = 2j \text{ (gerade)} \\ \frac{8}{n^3\pi} & \text{wenn } n = 2j+1 \text{ (ungerade)} \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} [B_n \cos(\lambda_n t) + \cancel{B_n^*} \sin(\lambda_n t)] \sin\left(\frac{n\pi}{L}x\right)$$

$$= \sum \frac{8}{(2j+1)^3\pi} \cancel{\sin} \cos((2j+1)t) \cdot \sin((2j+1)x)$$

$$\begin{aligned} n=1 &= 2j+1 \\ j &= 0 \\ \lambda_n &= n = 2j+1 \\ B_n &= \frac{8}{n^3\pi} \end{aligned}$$

Bsp: Mit Coeff vgl sehr schnell!

$$\left\{ \begin{array}{l} u_{xx} = u_{tt} \quad x \in [0, \pi] \\ u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = \sin(x) - 2\sin(3x) = f(x) \\ u_t(x,0) = 0 = g(x) \end{array} \right. \rightarrow \left. \begin{array}{l} L = \pi \\ c = 1 \end{array} \right\} \lambda_n = n$$

$$\rightarrow g(x) = B_n^* = 0$$

$$\rightarrow B_n \text{ durch coeff vgl!}$$

$$\underline{1 \cdot \sin(1x) - 2 \sin(3x)} = \sum B_n \sin\left(\frac{n\pi}{\pi}x\right) \quad B_n = \begin{cases} 1 & n=1 \\ -2 & n=3 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow u(x,t) = \cos(1t)\sin(1x) - 2\cos(3t)\sin(3x)$$

1D - Wärmeleitungsgleichung Fourier Lösung

$$c = \frac{\kappa}{\rho c_p} \quad \begin{array}{l} \text{Leitfähigkeit} \\ \text{spez. Wärme} \\ \text{dicke} \end{array}$$

$$\text{Wenn } \left\{ \begin{array}{l} u_t = c^2 u_{xx} \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = f(x) \end{array} \right. \quad \begin{array}{l} x \in [0, L] \\ t > 0 \end{array}$$

$$\text{allg. Lösung: } u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}$$

aus aufgabe

$$\lambda_n = \frac{cn\pi}{L}; \quad B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$\Rightarrow$ Manchmal kann man B_n auch über Koeffizientenvergleich bestimmen!

Probiere immer zuerst den Coeff vgl!

1D - Wärmeleitungsgleichung Allgemeiner Ansatz:

$$\Rightarrow \text{Genau gleich wie bei Wellengleichung, einfach } \begin{cases} u_t = F \dot{G} \\ u_{xx} = F'' G \end{cases}$$

Inhomogene Randbedingungen (WLG & WG)

Bsp Sommer 2022 / Serie 3

5. Wave Equation with inhomogeneous boundary conditions (15 Points)

Find the solution of the following wave equation (with inhomogeneous boundary conditions) on the interval $[0, \pi]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t \geq 0, x \in [0, \pi] \\ u(0, t) = 3\pi^2, & t \geq 0 \\ u(\pi, t) = 7\pi, & t \geq 0 \\ u(x, 0) = 2 \sin(5x) + \sin(4x) + (7 - 3\pi)x + 3\pi^2, & x \in [0, \pi] \\ u_t(x, 0) = 0. & x \in [0, \pi] \end{cases} \quad (1)$$

↑ coeff vgl!

You must proceed as follows.

- Find the unique function $w = w(x)$ with $w''(x) = 0$, $w(0) = 3\pi^2$, and $w(\pi) = 7\pi$.
- Define $v(x, t) := u(x, t) - w(x)$. Formulate the corresponding problem for v , equivalent to (1).
- (i) Find, using the formula from the script, the solution $v(x, t)$ of the problem you have just formulated.
(ii) Write down explicitly the solution $u(x, t)$ of the original problem (1).

Idee:

- ▷ Forme so um dass homogen wird
- ▷ Löse Homogenes Problem
- ▷ Rückformung

→ Mit $w(x)$ wird dann RB

$$v(0, t) = 0$$

$$v(\pi, t) = 0$$

a) $w''(x) = 0$ $w(x) = Ax + B$ $w(x) = (7 - 3\pi)x + 3\pi^2$

$$\begin{cases} w(0) = 3\pi^2 = 0 + B \Rightarrow B = 3\pi^2 \\ w(\pi) = 7\pi = A\pi + 3\pi^2 \rightarrow A = 7 - 3\pi \end{cases}$$

b) $v(x, t) := u(x, t) - w(x)$: schreibe Problem neu:

$$v = u(x, t) - w(x)$$

$$\begin{aligned} w_t &= 0 & w_{xx} &= 0 \\ w_{tt} &= 0 \end{aligned}$$

$$\begin{cases} v_{tt} = c^2 v_{xx} \\ v(0, t) = u(0, t) - w(0) = 3\pi^2 - 3\pi^2 = 0 \\ v(\pi, t) = u(\pi, t) - w(\pi) = 7\pi - 7\pi = 0 \\ v(x, 0) = u(x, 0) - w(x) = \underline{2\sin(5x) + \sin(4x)} + (7 - 3\pi)x + 3\pi^2 - ((7 - 3\pi)x + 3\pi^2) \\ v_t(x, 0) = u_t - \underset{0}{w_t} = 0 = 0 = 0 \end{cases}$$

↓ Löse gleich wie Bsp 1 heute sep of var oder fourier
coeff vgl etc

$$v(x, t) = \dots$$

↓ Zurückformen: $v(x, t) := u(x, t) - w(x) \Leftrightarrow$

$$\rightarrow u(x, t) = v(x, t) + w(x)$$

↑ vom Anfang