Analysis Übungsstunde 9

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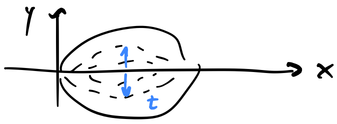
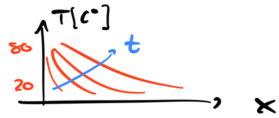
- 1D Wellengleichung $u_{tt} = c^2 u_{xx}$
- 1D Wärmeleitgleichung $u_t = c^2 u_{xx}$
- 2D Poissongleichung $u_{xx} + u_{yy} = f(x,y)$
- 2D Wellengleichung $u_{tt} = c^2(u_{xx} + u_{yy})$
- 2D Wärmeleitgleichung $u_t = c^2(u_{xx} + u_{yy})$
- 3D Laplacegleichung $u_{xx} + u_{yy} + u_{zz} = 0 = \nabla^2 u$





1D Wellengleichung

Allgemeiner Ansatz: "show your work"

"separation of variables"

$$\begin{cases} u_{tt} = c^2 u_{xx} \rightarrow u_{tt} - c^2 u_{xx} = 0 \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = \sin(\frac{\pi}{L}x) \\ u_t(x,0) = 0 \end{cases}$$

RB

1) Ansatz $u(x,t) = F(x) \cdot G(t)$

$u_{tt} := F(x) \cdot \ddot{G}(t)$

$u_{xx} := F''(x) \cdot G(t)$

2) Einsetzen $u_{tt} = c^2 u_{xx}$

$$F\ddot{G} = c^2 F'' G$$

→ bringe in verhältnis: $\boxed{\frac{F''}{F} = \frac{\ddot{G}}{c^2 G} := k}$ $k \in \mathbb{R}$

3) Separieren

$$\frac{F''}{F} = k \qquad \frac{\ddot{G}}{c^2 G} = k$$

4) Nutze Randbedingungen

$u(0,t) = 0 \rightarrow F(0)\cdot G(t) = 0 \rightarrow F(0) \overset{!}{=} 0$

$u(L,t) = 0 \rightarrow F(L)\cdot G(t) = 0 \rightarrow F(L) \overset{!}{=} 0$

$u(x,0) = \sin(\frac{\pi}{L}x)$

$u_t(x,0) = 0 \rightarrow F(x)\cdot \dot{G}(0) = 0 \rightarrow \dot{G}(0) = 0$

5) Lösen von DGL mit Fallunterscheidung: für welches k gilt was?

$$\frac{F''}{F} = k$$

$k = 0$

$F'' = 0 \rightarrow F(x) = Ax + B$

RB: $F(0) = 0 = 0 + B \rightarrow B = 0$

$F(L) = 0 = AL + 0 \rightarrow A = 0$

$\underline{F(x) = 0}$

$k > 0$

$F'' = kF \rightarrow F'' - kF = 0$ mit Ansatz $y = e^{\lambda x}$

$\lambda^2 - k = 0 \qquad \lambda = \pm\sqrt{k}$

$\Rightarrow F(x) = Ce^{\sqrt{k}x} + De^{-\sqrt{k}x}$

RB: $F(0) = 0 = C\cdot 1 + D\cdot 1 \qquad D = -C$

$F(L) = 0 = Ce^{\sqrt{k}L} - Ce^{-\sqrt{k}L} \rightarrow \cancel{\sqrt{k}L \overset{!}{=} -\sqrt{k}L} \quad k \overset{!}{>} 0$

$\rightarrow C = 0$

$\underline{F(x) = 0}$

$k < 0$

$F'' - kF = 0 \qquad k := -\alpha$

$F'' + \alpha F = 0 \rightarrow \lambda^2 + \alpha = 0 \qquad \lambda = \pm\sqrt{-\alpha} = \pm i\sqrt{\alpha}$

$$F(x) = \underbrace{e^{0x}}_{=1}\left(A\sin(\sqrt{\alpha}x) + B\cos(\sqrt{\alpha}x)\right)$$

RB: $F(0) = 0 = A\cdot 0 + B\cdot 1 \qquad B = 0$

$F(L) = 0 = A\sin(\sqrt{\alpha}L) + 0 \rightarrow \cancel{A = 0}$ brauchen eine nicht triviale Lsg

$\rightarrow \sin(\;) = 0 \rightarrow \sqrt{\alpha}L = n\cdot\pi$

$\sqrt{\alpha} = \frac{n\pi}{L}$

$F(x) = A\sin(\frac{n\pi}{L}x)$

$\boxed{\text{Weil für Fälle } k=0, k>0 \; F(x)=0 \text{ gilt, muss } G(t) \text{ für diese nicht mehr berechnet werden!}}$

dh für $G(t)$ gilt

$(k<0)$

$\ddot{G} = kc^2G \xrightarrow{k=-\alpha} \ddot{G} + \alpha c^2 G = 0$

$0 = \lambda^2 + \alpha c^2 \rightarrow \lambda = \pm i\sqrt{\alpha}c$

$$G(t) = \underbrace{e^{0x}}_{=1}\left(C\sin(\sqrt{\alpha}ct) + D\cos(c\sqrt{\alpha}t)\right)$$

RB: $\dot{G}(0) = 0 = c\sqrt{\alpha}\, C\cos(c\sqrt{\alpha}t) - Dc\sqrt{\alpha}\sin(c\sqrt{\alpha}t)$

$0 = c\sqrt{\alpha}C - 0 \quad \rightarrow C = 0$

$G(t) = D\cos(c\sqrt{\alpha}t)$

6) Zusammensetzen $u(x,t) = F(x)\cdot G(t)$, $\sqrt{\alpha} = \frac{n\pi}{L}$

$u_n(x,t) = A_n\sin\left(\frac{n\pi}{L}x\right)\cdot D_n\cos\left(c\,\frac{n\pi}{L}t\right) \quad / \; A_n\cdot D_n := B_n$

7) Superposition

$u_n(x,t) = \sum B_n\sin\left(\frac{n\pi}{L}x\right)\cdot\cos\left(\frac{n\pi}{L}ct\right)$

8) letzte RB: Coeff vgl

$u(x,0) = \sin\left(\frac{1\pi}{L}x\right) = \sum B_n\sin\left(\frac{n\pi}{L}x\right)\cdot\overbrace{\cos(0)}^{1}$

$n = 1 \quad \rightarrow B_1 = 1$

$\forall n\setminus 1 \quad \rightarrow B_n = 0$

9) Spezifische Lösung

$\Rightarrow u(x,t) = 1\cdot\sin\left(\frac{\pi}{L}x\right)\cos\left(\frac{\pi}{L}ct\right)$ ($n=1$)

1D Wellengleichung festige Lösung mit Fourier $\left(c = \sqrt{\frac{T}{\mu}}\right)$ Kraft Massenverteilung

Für eine eindimensionale Wellengleichung der Form $u_{tt} = c^2 u_{xx}$ und den Randbedingungen, $x \in [0, L]$

$$\begin{cases} u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

finden wir eine allgemeine Lösung:

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n cos(\lambda_n t) + B_n^* sin(\lambda_n t)\right) sin(\frac{\pi n}{L} x) \quad (1)$$

$$\lambda_n = \frac{c\, n\, \pi}{L} \quad (2)$$

$$f(x) = \sum_{n=1}^{\infty} B_n sin(\frac{n\,\pi}{L} x) \quad (3)$$

$$g(x) = \sum_{n=1}^{\infty} B_n^* \lambda_n sin(\frac{n\,\pi}{L} x) \quad (4)$$

$$B_n = \frac{2}{L} \int_0^L f(x) sin(\frac{n\,\pi}{L} x) dx \quad (5)$$

$$B_n^* = \frac{2}{L\lambda_n} \int_0^L g(x) sin(\frac{n\,\pi}{L} x) dx \quad (6)$$

- λ_n bestimmen $= \frac{cn\pi}{L}$: c & L aus Aufgabe lesen
- ! versuche zuerst B_n & B_n^* mit Koeff vgl zu bestimmen !
  (3), (4) Zeit sparen
- sonst B_n & B_n^* mit den Integralen finden (5), (6)

Bsp: Winter 2017 A4

4. (*10 Points*)
Consider the string of length $L = \pi$ and $c^2 = 1$ with zero initial velocity, initial deflection $u(x,0) = x(x-\pi)$ and fixed endpoints. The deflection $u(x,t)$ is a solution of the following PDE

$$\begin{cases} u_{tt} = u_{xx}, & t > 0,\ x \in (0,\pi) \\ u(0,t) = 0 = u(\pi,t), & t \geq 0 \\ u(x,0) = x(x-\pi), & 0 \leq x \leq \pi \\ u_t(x,0) = 0, & 0 \leq x \leq \pi \end{cases}$$

$f(x)$
$g(x)$

Find $u(x,t)$.

→ $L = \pi$
→ $c = 1$
$\lambda_n = \frac{cn\pi}{L} = n$

- B_n & B_n^* : $g(x) = 0 = \sum B_n^* sin(\frac{n\pi}{L}x)$ $\quad B_n^* = 0 \ \forall n$

$f(x) = x(x-\pi) = \sum B_n \lambda sin(\frac{n\pi}{L}x) = \sum B_n \cdot n \cdot sin(nx) \ldots ?$

B_n nicht mit coeff vgl bestimmbar

$$B_n = \frac{2}{L}\int_0^L g(x) sin(\frac{n\pi}{L}x)\, dx = \frac{2}{\pi}\int_0^{\pi} x(x-\pi) sin(nx)$$

⋮ partielle integration (nutzt ZF)

$$= -\frac{4}{n^3\pi} cos(nx)\Big|_0^{\pi} = -\frac{4}{n^3\pi}\big(\underbrace{cos(n\pi)}_{(-1)^n} - 1\big) = 0, -2, 0, -2 \ldots$$

n=0 n=1 n=2 n=3

n: gerade
ungerade

$$B_n = \begin{cases} 0 & \text{wenn } n = 2j \quad (\text{gerade}) \\ \frac{8}{n^3\pi} & \text{wenn } n = 2j+1 \quad (\text{ungerade}) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)\right] \sin\left(\frac{n\pi}{L}x\right)$$

$$= \sum_{j=0}^{\infty} \frac{8}{(2j+1)^3\pi} \sin((2j+1)t) \sin((2j+1)x)$$

$n=1 = 2j+1$
$j = 0$
$\lambda_n = n = 2j+1$
$B_n^* = 0$
$B_n = \frac{8}{n^3\pi}$

Bsp: Mit Coef vgl sehr schnell!

$$\begin{cases} u_{xx} = u_{tt} & x \in [0,\pi] \\ u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = \sin(x) - 2\sin(3x) = f(x) \\ u_t(x,0) = 0 = g(x) \end{cases}$$

$\to L = \pi$, $\to c = 1$ $\}$ $\lambda_n = n$

$\to g(x) = 0 \to B_n^* = 0$

$\to B_n$ durch coef vgl!

$$1 \cdot \sin(1x) - 2\sin(3x) = \sum B_n \sin\left(\frac{n\pi}{\pi}x\right) \qquad B_n = \begin{cases} 1 & n=1 \\ -2 & n=3 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow u(x,t) = \cos(1t)\sin(t) - 2\cos(3t)\sin(3x)$$

1D - Wärmeleitgleichung Fourier Lösung

$c = \frac{K}{\rho\sigma}$ — Leitfähigkeit; spez. Wärme; dichte

Wenn
$$\begin{cases} u_t = c^2 u_{xx} & \\ u(0,t) = 0 & x \in [0,L] \\ u(L,t) = 0 & t > 0 \\ u(x,0) = f(x) \end{cases}$$

allg. Lösung: $u(x,t) = \sum_{n=1}^{\infty} B_n sin(\frac{n\pi}{L}x)e^{-\lambda_n^2 t}$

aus aufgabe

$\lambda_n = \frac{cn\pi}{L}$; $B_n = \frac{2}{L}\int_0^L f(x)sin(\frac{n\pi}{L}x)dx$

$\Rightarrow$ ~~Manchmal~~ kann man B_n auch über Koeffizientenvergleich bestimmen!

Probiere immer zuerst den Coeff vgl!

1D - Wärmeleitgleichung Allgemeiner Ansatz:

$\Rightarrow$ Genau gleich wie bei Wellengleichung, einfach $\begin{cases} u_t = F\dot{G} \\ u_{xx} = F''G \end{cases}$

Inhomogene Randbedingungen (WLG & WG)

Bsp Sommer 2022 / Serie 9

5. Wave Equation with inhomogeneous boundary conditions (*15 Points*)

Find the solution of the following wave equation (**with inhomogeneous boundary conditions**) on the interval $[0, \pi]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t \geq 0,\ x \in [0, \pi] \\ u(0,t) = 3\pi^2, & t \geq 0 \\ u(\pi, t) = 7\pi, & t \geq 0 \\ u(x,0) = 2\sin(5x) + \sin(4x) + (7-3\pi)x + 3\pi^2, & x \in [0,\pi] \\ u_t(x,0) = 0. & x \in [0,\pi] \end{cases} \quad (1)$$

coeff vgl!

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w''(x) = 0$, $w(0) = 3\pi^2$, and $w(\pi) = 7\pi$.

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

c) (i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (1).

Idee:

- Forme so um dass homogen wird
- Löse Homogenes Problem
- Rückformung

$\longrightarrow$ Mit $w(x)$ wird dann RB

$v(0,t) = 0$

$v(\pi,t) = 0$

a) $w''(x) = 0 \qquad w(x) = Ax + B \qquad w(x) = (7-3\pi)x + 3\pi^2$

$w(0) = 3\pi^2 = 0 + B \quad \rightarrow B = 3\pi^2$

$w(\pi) = 7\pi = A\pi + 3\pi^2 \quad \rightarrow A = 7 - 3\pi$

b) $v(x,t) := u(x,t) - w(x)$: schreibe Problem neu:

$v = u(x,t) - w(x)$

$w_t = 0 \quad w_{xx} = 0$

$w_{tt} = 0$

$$\begin{cases} v_{tt} = c^2 v_{tt} \\ v(0,t) = u(0,t) - w(0) = 3\pi^2 - 3\pi^2 = 0 \\ v(\pi,t) = u(\pi,t) - w(\pi) = 7\pi - 7\pi = 0 \\ v(x,0) = u(x,0) - w(x) = 2\sin(5x) + \sin(4x) + \cancel{(7-3\pi)x + 3\pi^2} - \cancel{((7-3\pi)x + 3\pi^2)} \\ v_t(x,0) = u_t - w_t = 0 - w_t = 0 \end{cases}$$

$\downarrow$ Löse gleich wie Bsp 1 heute sep of var oder fourier coeff vgl etc

$v(x,t) = \ldots$

$\downarrow$ Zurückformen: $v(x,t) := u(x,t) - w(x) \iff$

$\rightarrow u(x,t) = v(x,t) + w(x)$

vom Anfang