Analysis Übungsstunde 10

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1-Dimensionale Wellengleichung – Dalambert

Problem in Form:

Herleitung in der VL (Skript S.70)

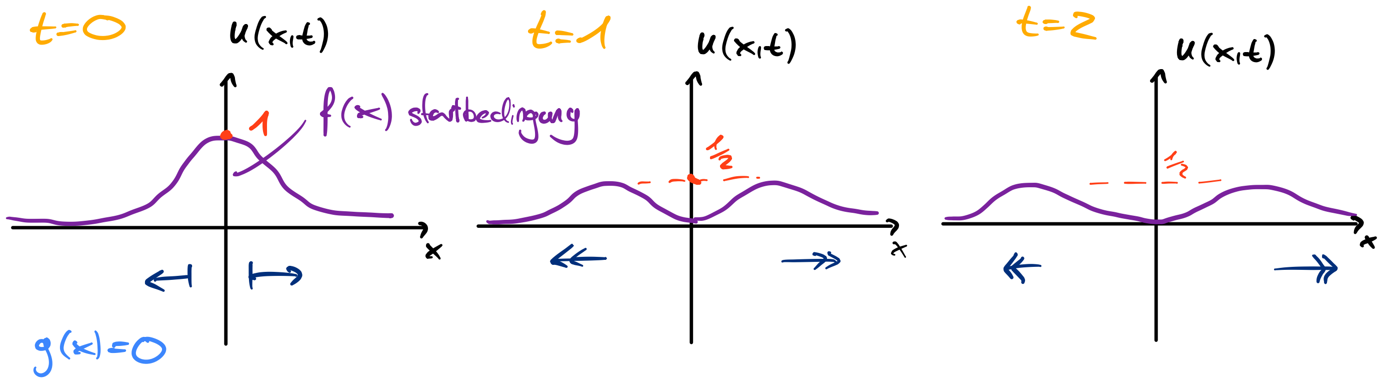
$$\begin{cases} u_{tt} = c^2 u_{xx} & x \in [-\infty, \infty] \text{ oder } x \in \mathbb{R} \\ u(x,0) = f(x) & t > 0 \\ u_t(x,0) = g(s) & \end{cases}$$

Man kann damit direkt die Lösung finden oder der Wert an einer bestimmten Stelle $x=a,\ t=b$ ...

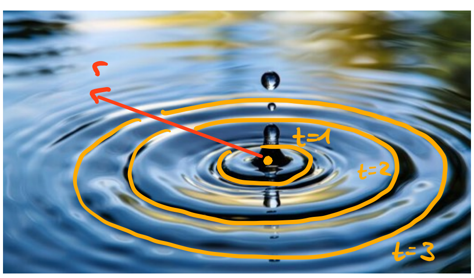
Lösung der PDE via Dalambert:

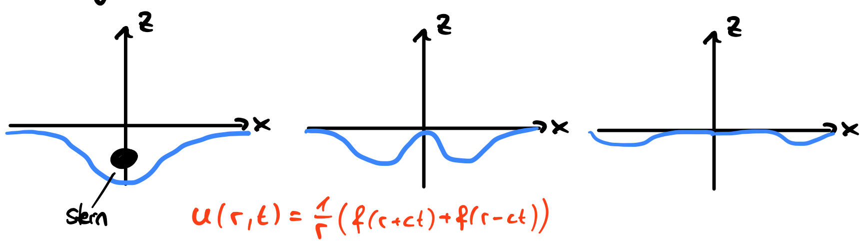
$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds}$$

Bsp Seil: Wie breitet sich eine Welle auf einem unendlichen Seil aus?



Bsp Stein im Wasser 2D Kugelwelle (nicht Prüfungsrelevant)





$u(r,t) = \frac{1}{r}\left(f(r+ct) + f(r-ct)\right)$

$g(x) = 0$

Ausbreitung ähnlich aber Amplitude nimmt ab
($\Rightarrow$ nicht mit 1D Dalambert Lösbar)

Bsp. Winter 2018

3. Wave Equation (*8 Points*)
Let $c > 0$. Consider the following initial value problem:

$$\begin{cases} u_{tt} = c^2 u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x,0) = e^{-x^2}\sin^2(x) + x = f(x) & x \in \mathbb{R} \\ u_t(x,0) = xe^{-x^2} = g(x) & x \in \mathbb{R}. \end{cases} \quad (2)$$

a) Use D'Alembert's formula to find the solution $u(x,t)$ of (2).

b) For a fixed $a \in \mathbb{R}$, determine the limit

$$\lim_{t\to\infty} u(a,t). \quad (3)$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$$= \frac{1}{2}\left(e^{-(x+ct)^2} \cdot \sin^2(x+ct) + (x+\cancel{ct}) + e^{-(x-ct)^2} \cdot \sin^2(x-ct) + (x-\cancel{ct}) \right) + \frac{1}{2c}\int_{x-ct}^{x+ct} se^{-s^2}\,ds$$

$$\left(\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2} + C \right)$$

$$= \frac{1}{2}\left(e^{-(x+ct)^2} \cdot \sin^2(x+ct) + e^{-(x-ct)^2} \cdot \sin^2(x-ct) + 2x + \frac{1}{2c}\left[-\frac{1}{2}e^{-x^2} \right]_{x-ct}^{x+ct}\right.$$

$$u(x,t) = \frac{1}{2}\left(e^{-(x+ct)^2} \cdot \sin^2(x+ct) + e^{-(x-ct)^2} \cdot \sin^2(x-ct) + 2x - \frac{1}{4c}\left[e^{-(x+ct)} - e^{-(x-ct)} \right]\right.$$

b) $\lim\limits_{t\to\infty} u(a,t)$:

$$\lim_{t\to\infty} \frac{1}{2}\left(\overbrace{e^{-(a+ct)^2} \cdot \sin^2(a+ct)}^{\to e^{-\infty} = 0} + \overbrace{e^{-(a-ct)^2} \cdot \sin^2(a-ct)}^{\to e^{-\infty} = 0} + 2a - \frac{1}{4c}\left[\overset{0}{\cancel{e^{-(a+ct)}}} - \overset{0}{\cancel{e^{-(a-ct)}}} \right] \right]$$

$$= \frac{1}{2} \cdot 2a - 0 = \underline{\underline{a}}$$

Wenn nur ein Wert gesucht ist: einfach einsetzen!

Bsp. Sommer 2018

2. Wave Equation (*7 Points*)

Let $u(x,t)$ be the solution of the following initial value problem

$c = c^2 = 4$

$$\begin{cases} u_{tt} = 4u_{xx}, & x \in \mathbb{R},\ t>0 \\ u(x,0) = f(x) = \begin{cases} 3, & |x| \le 2 \\ 0, & |x| > 2 \end{cases}, & x \in \mathbb{R} \\ u_t(x,0) = g(x) = \begin{cases} 1, & |x| \le 2 \\ 0, & |x| > 2 \end{cases}, & x \in \mathbb{R} \end{cases}$$

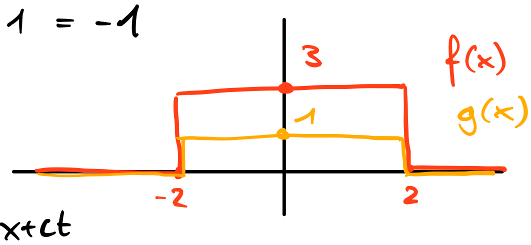
a) Find $u(1,1)$ by using D'Alembert's formula.

b) Find $\lim\limits_{t\to\infty} u(1,t)$.

$c = 2$

$x + ct = 1 + 2 \cdot 1 = 3$

$x - ct = 1 - 2 \cdot 1 = -1$



$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\,ds$$

$$= \frac{1}{2}\Big(\underbrace{f(3)}_{\substack{|3|>2 \\ 0}} + \underbrace{f(-1)}_{\substack{|-1|<2 \\ 3}} \Big) + \frac{1}{2\cdot 2} \int_{-1}^{3} g(s)\,ds$$

$$= \frac{1}{2}(0+3) + \frac{1}{4}\left(\int_{-1}^{2} 1\,dx + \cancel{\int_{2}^{3} 0\,dx} \right)$$

$$= \frac{1}{2}\cdot 3 + \frac{1}{4}\left[2-(-1)\right] = \frac{3}{2} + \frac{3}{4} = \frac{6}{4} + \frac{3}{4} = \frac{9}{4}$$

b) $$\lim_{t\to\infty} u(1,t) : \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\,ds$$

$$= \cancel{\frac{1}{2}\Big(\underbrace{f(1+2t)}_{\substack{|\infty|>2 \\ 0}} + \underbrace{f(1-2t)}_{\substack{|-\infty|>2 \\ 0}} \Big)} + \frac{1}{4} \int_{1-2t \to \infty}^{1+2t \to \infty} 1\,dx$$

$x = 1$

$|1| < 2$

$$= \frac{1}{4} \int_{-\infty}^{-2} 0 + \int_{-2}^{2} 1 + \int_{2}^{\infty} 0 = \frac{1}{4}(2+2) = 1$$

$|-2| \le 2$, $|2| \le 2$

Bsp S2024

1.MC7 [3 Points] Wave equation with D'Alembert solution.

Consider the following wave equation:

$$\begin{array}{r} f(x) = \\ g(x) = \end{array} \begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\, t \geq 0, \\ u(x,0) = e^{2x}, & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Find the value of the solution u at position $x = 0$, i.e. $u(0,t)$

(A) $u(0,t) = \cos(2ct)$.

(B) $u(0,t) = \sin(2ct)$.

(C) $u(0,t) = \sinh(2ct)$.

(D) $u(0,t) = \cosh(2ct)$.

$$\frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$$u(0,t) = \frac{1}{2}\left(f(ct) + f(-ct) \right) + \cancel{\frac{1}{2c}\int_{-ct}^{ct} g(s)\,ds}$$

$$= \frac{1}{2}\left(e^{2ct} + e^{-2ct} \right) \quad \Rightarrow \text{euler form}$$

$$= \underline{\cosh(2ct)}$$