Analysis Übungsstunde 10

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1-Dimensionale Wellengleichung – Dalambert

Problem in Form:

Herleitung in der VL (Skript S.70)

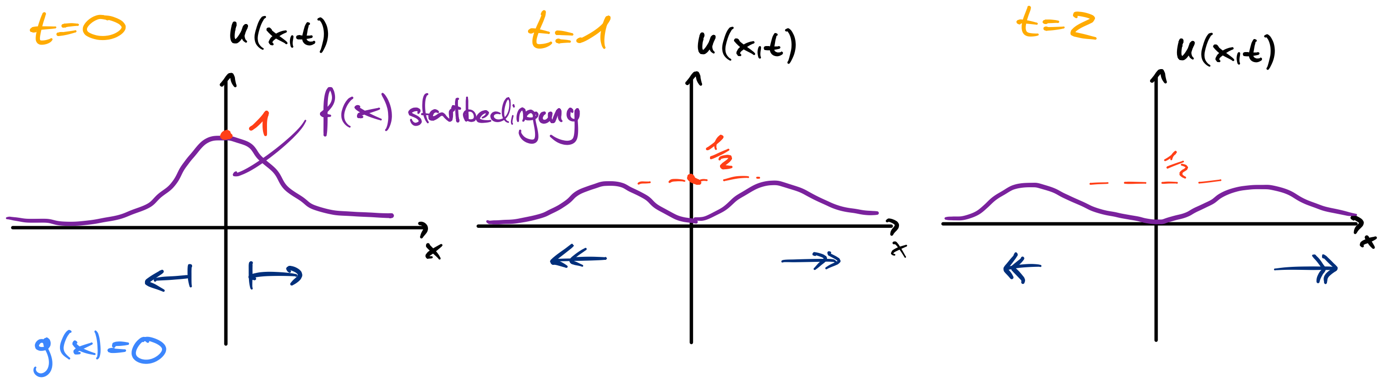
$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = g(s) \end{cases} \qquad x \in [-\infty, \infty] \text{ oder } x \in \mathbb{R}, \quad t > 0$$

Man kann damit direkt die Lösung finden oder den Wert an einer bestimmten Stelle $x=a$, $t=b$ ...

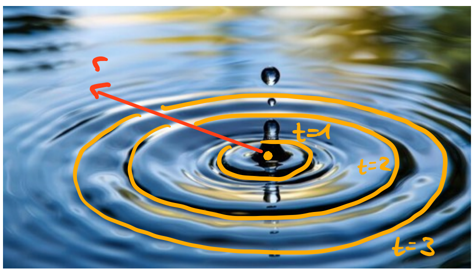
Lösung der PDE via Dalambert:

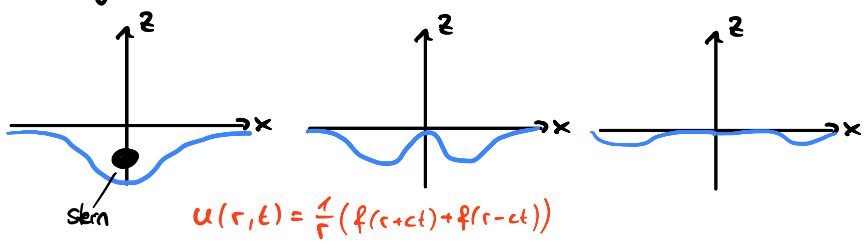
$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds}$$

Bsp Seil: Wie breitet sich eine Welle auf einem unendlichen Seil aus?



Bsp Stein im Wasser 2D Kugelwelle (nicht Prüfungsrelevant)





$u(r,t) = \frac{1}{r}\left(f(r+ct) + f(r-ct)\right)$

$g(x) = 0$

Ausbreitung ähnlich aber Amplitude nimmt ab

($\Rightarrow$ nicht mit 1D Dalambert lösbar)

Bsp. Winter 2018

3. Wave Equation (*8 Points*)
Let $c > 0$. Consider the following initial value problem:

$$\begin{cases} u_{tt} = c^2 u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x,0) = e^{-x^2}\sin^2(x) + x & x \in \mathbb{R} \\ u_t(x,0) = xe^{-x^2} & x \in \mathbb{R}. \end{cases} \tag{2}$$

a) Use D'Alembert's formula to find the solution $u(x,t)$ of (2).

b) For a fixed $a \in \mathbb{R}$, determine the limit

$$\lim_{t\to\infty} u(a,t). \tag{3}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

b) $\lim\limits_{t\to\infty} u(a,t)$:

$\lim\limits_{t\to\infty}$

Wenn nur ein Wert gesucht ist: einfach einsetzen!

Bsp. Sommer 2018

2. Wave Equation (*7 Points*)
Let $u(x,t)$ be the solution of the following initial value problem

$$\begin{cases} u_{tt} = 4u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) = \begin{cases} 3, & |x| \le 2 \\ 0, & |x| > 2 \end{cases}, & x \in \mathbb{R} \\ u_t(x,0) = g(x) = \begin{cases} 1, & |x| \le 2 \\ 0, & |x| > 2 \end{cases}, & x \in \mathbb{R} \end{cases}$$

a) Find $u(1,1)$ by using D'Alambert's formula.

b) Find $\lim_{t\to\infty} u(1,t)$.

$c =$

$x + ct =$

$x - ct =$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$$=$$

b) $\lim_{t\to\infty} u(1,t) : \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

Bsp S2024

1.MC7 **[3 Points]** Wave equation with D'Alembert solution.

Consider the following wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \geq 0, \\ u(x,0) = e^{2x}, & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Find the value of the solution u at position $x = 0$, i.e. $u(0,t)$

(A) $u(0,t) = \cos(2ct)$.

(B) $u(0,t) = \sin(2ct)$.

(C) $u(0,t) = \sinh(2ct)$.

(D) $u(0,t) = \cosh(2ct)$.

$$\frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds$$