Analysis Übungsstunde 12

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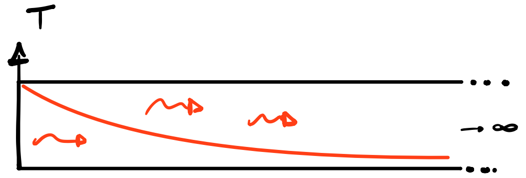
letzte Woche $t \to \infty$, heute $x \to \infty$

Wärmeleitgleichung auf einem unendlichen Stab

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = f(x) \\ x \in \mathbb{R},\ t > 0 \end{cases}$$

neu: keine RB in x $[u(0,t) = u(L,t)]$ $L \to \infty$

$u(\infty, t) = ??$



Herleitung: Skript S.81

(1) $$u(x,t) = \int_0^\infty (A(p)cos(px) + B(p)sin(px))e^{-c^2p^2t}dp$$

(2) $$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)cos(pv)dv$$ $\parallel$ $A(p) = 0$, falls $f(x)$ ungerade

(3) $$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)sin(pv)dv$$ $\parallel$ $B(p) = 0$, falls $f(x)$ gerade

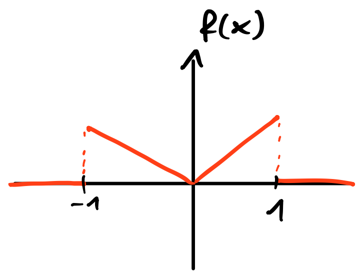
$\to$ wenn man (2)&(3) in (1) einsetzt, erhält man: (Herleitung Skript S.82)

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\exp\left[-\left(\frac{x-v}{2c\sqrt{t}}\right)^2\right] dv$$

Bsp

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = f(x) \end{cases}, \quad f(x) = \begin{cases} |x|, & |x| < 1 \\ 0, & \text{else} \end{cases}$$

$x \in \mathbb{R},\ t > 0$



$f(x)$ gerade $\to B(p) = 0$ $\to f(x)$ gerade

$$A(p) = \frac{1}{\pi}\int_0^\infty f(v)\cos(pv)\,dv = \frac{1}{\pi}\int_{-1}^{1} |x|\cos(px)\,dx$$

$$= \frac{2}{\pi}\int_0^1 x\cos(px)\,dx$$

partiell integrieren

$$\vdots \quad [\;]_0^1$$

$$= \frac{2}{\pi p^2}\left(p\sin(p)+\cos(p)-1\right)$$

$$u(x,t) = \int_0^\infty \left(A(p)\cos(px)+\cancel{B(p)}\sin(px)\right)e^{-c^2p^2t}\,dp$$

$$= \frac{2}{\pi}\int_0^\infty \frac{(p\sin(p)+\cos(p)-1)}{p^2}\cdot\cos(px)\cdot e^{-c^2p^2t}\,dp$$

WLG auf unendlicher Stab – Via Fourier Transformation

Gleiche Ausgangslage:

$$\begin{cases} u_t = c^2u_{xx} \\ u(x,0) = f(x) \\ x\in\mathbb{R},\ t>0 \end{cases}$$

Idee: $u(x,t) \xrightarrow{\mathcal{F}} \hat{u}(\omega,t)$

Remember: Woche 6: Fourier Transformation

Fourier Transformation

$$\mathcal{F}(f(t)) = \hat{f} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cdot e^{-i\omega t}\,dt$$

können auch x sein

$$\mathcal{F}^{-1}(g(\omega)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(\omega)\cdot e^{+i\omega t}\,d\omega$$

t ist ersetzbar: bsp $t^* = 4a + \frac{t^2}{2}$ oder auch mit x!

- $f(x) \underset{\mathcal{F}}{\overset{\mathcal{F}^{-1}}{\rightleftarrows}} \mathcal{F}(f(t))(\omega) = \hat{f}(\omega)$
- $\mathcal{F}^{-1}\cdot\mathcal{F} = \mathbb{I}$
- $\mathcal{F}^{-1}\left(\mathcal{F}(f(t))\right) = f(t)$

! Falls $\omega = 0$ nicht def: → $\hat{f}(0)$ separat berechnen (siehe bsp)

Eigenschaften (ZF)

- Linearität: $\mathcal{F}(\alpha f+\beta g) = \alpha\mathcal{F}(f)+\beta\mathcal{F}(g)$
- Convolution: $\mathcal{F}(f*g) = \sqrt{2\pi}\,\mathcal{F}(f)\cdot\mathcal{F}(g)$
- (II) Ableitungen im Zeitbereich: $\mathcal{F}\left(\frac{df}{dt}\right) = i\omega\,\mathcal{F}(f(t))$

  $\mathcal{F}\left(\frac{df^2}{dt^2}\right) = -\omega^2\mathcal{F}(f(t))$ } pro weitere Ableitung je mal $i\omega$ dazu

  $\vdots$
- Ableitungen im Frequenzbereich: $\frac{d\hat{f}(\omega)}{d\omega} = -i\,\mathcal{F}(t\cdot f(t)) \to$ Allg: $\mathcal{F}(x^k f) = i^k\frac{d^k\hat{f}(\omega)}{d\omega^k}$
- shifts: $\mathcal{F}(f(t-a)) = e^{-ia\omega}\cdot\mathcal{F}(f(t))$

  $\mathcal{F}(\omega-b) = \mathcal{F}(e^{ib\omega}f(t))$

① Fourier Transform. nach x

$u_t = c^2u_{xx}$ // Fourier-Transform

$$\mathcal{F}(u_t) = \frac{\partial}{\partial t}\cdot\hat{u}(\omega,t)$$

$$\mathcal{F}(u_{xx}) \overset{(II)}{=} -\omega^2\hat{u}(\omega,t)$$

$$\mathcal{F}(f(x)) = \hat{f}(\omega)$$

schreibe Problem neu:

$$\begin{cases} \frac{\partial}{\partial t}\hat{u}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \mathcal{F}(f(x)) \end{cases}$$

⇒ haben "normale" DGL!

② Löse mit sep. of var.

$$\frac{d\hat{u}}{dt} = -c^2\omega^2\hat{u} \quad \rightarrow \int \frac{1}{\hat{u}}\, d\hat{u} = \int -c^2\omega^2 dt$$

$$\ln(\hat{u}) = -c^2\omega^2 t + C(\omega) \quad // \exp$$

$$\boxed{\hat{u}(\omega,t) = C(\omega)\cdot e^{-c^2\omega^2 t}}$$

③ Nutze Transformierte RB

$$\hat{u}(\omega,0) = C(\omega)\cdot e^0 = \mathcal{F}(f(x)) \quad \rightarrow \text{finde so } C(\omega)$$

④ Rücktransformation

$$u(x,t) = \mathcal{F}^{-1}\left[\mathcal{F}(f(x))\cdot e^{-c^2\omega^2 t}\right]$$

Wichtige Fourier & Fourier$^{(-1)}$ transformationen:

fehler auf ZF

	Generell:	Vorgefertigt	Wichtige Integrale, nicht auf ZF
$\mathcal{F}$	$\hat{f} = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$	$\mathcal{F}\left(xe^{-ax^2}\right)(\omega) = \frac{-i\omega}{(2a)^{3/2}}e^{-\frac{\omega^2}{4a}}$ $\mathcal{F}\left(x^2 e^{-ax^2}\right) = \left(1-\frac{\omega^2}{2a}\right)\cdot\frac{1}{\sqrt{8a^3}}\, e^{-\frac{\omega^2}{4a}}$	▷ $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$ ▷ $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \pi$ ▷ $\int_{-\infty}^{\infty} e^{-ax^2}\cdot e^{-ikx}dx = e^{\frac{k^2}{4a}}\sqrt{\frac{\pi}{a}}$ ▷ $\int_{-\infty}^{\infty} e^{-(ak^2+bk+c)}dk = e^{\frac{b^2}{4a}-c}\cdot\sqrt{\frac{\pi}{a}}$
$\mathcal{F}^{-1}$	$\mathcal{F}^{-1}(g)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}d\omega$	$\mathcal{F}^{-1}\left(i\omega e^{-b\omega^2}\right) = \frac{x}{(2b)^{3/2}}e^{-\frac{x^2}{4b}}$	

$\hat{f}(\omega + a) = e^{-iax}\cdot\hat{f}(\omega)$

⇒ je nach Problem, passe a & b an

→ Ziel: PDE mittels Fouriertransformation in normale DGL umformen

Bsp: W2022 A6

6. PDE with Fourier transform *(15 Points)*

Find the solution $u = u(x,t)$ of the following equation using the Fourier transform:

$$\begin{cases} u_x + u_t + u = 0\,, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x)\,, & x \in \mathbb{R}. \end{cases} \longrightarrow$$

[*Hint:* *You can proceed as follow:*

a) *Take the Fourier transform with respect to the x variable of the PDE and the initial condition and transform them into an ODE.*

b) *Solve the ODE.*

c) *Take the inverse Fourier transform of the solution of the ODE to find the solution of the PDE.*]

Woche 6

$$\left[\begin{array}{l} \mathcal{F}\left(\frac{df}{dt}\right) = i\omega \mathcal{F}(f(t)) \\ \mathcal{F}\left(\frac{df^2}{dt^2}\right) = -\omega^2 \mathcal{F}(f(t)) \end{array}\right]$$

pro weitere Ableitung je mal $i\omega$ dazu

With respect to x:

$$\mathcal{F}\left(\frac{df}{dx}\right) = i\omega \mathcal{F}(f(x))$$

① Fourier Transform. nach x

$$\begin{cases} u_x + u_t + u = 0 \\ u(x,0) = f(x) \end{cases} \quad // \mathcal{F} \quad \longrightarrow \quad \begin{array}{l} \mathcal{F}\left(\frac{du}{dx}\right) + \mathcal{F}\left(\frac{du}{dt}\right) + \mathcal{F}(u) = 0 \\ \mathcal{F}(u(x,0)) = \mathcal{F}(f(x)) \end{array}$$

$$\begin{cases} i\omega\, \hat{u}(\omega,t) + \frac{d}{dt}\hat{u}(\omega,t) + \hat{u}(\omega,t) = 0 \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases} \qquad \Rightarrow \text{DGL}$$

② Löse mit sep. of var.

$$\frac{d\hat{u}}{dt} = \hat{u}(-1-i\omega) \quad \longrightarrow \quad \int \frac{1}{\hat{u}}\, d\hat{u} = \int (-1-i\omega)\, dt$$

$$\ln(\hat{u}(\omega,t)) = (-1-i\omega)t + C(\omega) \quad // \exp$$

$$\hat{u}(\omega,t) = C(\omega)\cdot e^{(-1-i\omega)t}$$

③ Nutze Transformierte RB

$$\hat{u}(\omega,t) = C(\omega)\cdot e^{(-1-i\omega)t} \quad // \text{RB} \quad \hat{u}(\omega,0) = \hat{f}(\omega) = \mathcal{F}(f(x))$$

$$\hat{f}(\omega) = C(\omega)\cdot e^{0} \rightarrow C(\omega) = \hat{f}(\omega)$$

④ Rücktransformation

$$\underbrace{\mathcal{F}^{-1}\left(\hat{u}(\omega,t)\right)}_{u(x,t)} = \mathcal{F}^{-1}\left(\hat{f}(\omega)\cdot e^{(-1-i\omega)t}\right) \quad // \quad \mathcal{F}^{-1}(g)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}\,d\omega$$

x x

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)\cdot e^{(-1-i\omega)t}\cdot e^{i\omega x}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-t} \cdot e^{-i\omega t} \cdot e^{+i\omega x}\, d\omega$$

$\neq f(\omega)$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-t} \int_{-\infty}^{\infty} \hat{f}(\omega) \cdot e^{i\omega \overbrace{(x-t)}^{=\alpha}}\, d\omega$$

$\boxed{\text{now } (x-t) := \alpha}$

$$= e^{-t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) \cdot e^{i\omega\alpha}\, d\omega$$

$$\mathcal{F}^{-1}(g)(\overset{x-t}{t}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega \overset{x-t}{t}} d\omega$$

$\downarrow$ $g(\alpha)$

$$= e^{-t} \cdot f(\alpha) \qquad // \ \alpha = x-t$$

$$u(x,t) = e^{-t} \cdot f(x-t)$$

Coeff vgl mit Summe & Recap zu inhomogenen PDE

W2022 A5

5. Heat Equation with inhomogeneous boundary conditions (*15 Points*)

Find the general solution of the Heat equation (**with inhomogeneous boundary conditions**) for the following problem:

$$\begin{cases} u_t = c^2 u_{xx}, & 0 \le x \le L, t \ge 0, \\ u(0,t) = 0, & t \ge 0, \\ u(L,t) = L, & t \ge 0, \\ u(x,0) = f(x) + x, & 0 \le x \le L, \end{cases} \qquad (1)$$

$f(0) = 0$
$f(L) = 0$

where $L > 0$ is a constant, and $f(x)$ is any (twice differentiable) function such that $f(0) = 0, f(L) = 0$.

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w'' = 0$, $w(0) = 0$, and $w(L) = L$.

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

c) The Fourier series of the $2L$ periodic odd extension of f is given by

$$f(x) := \sum_{n=1}^{+\infty} \frac{(4n+2)}{(n^2 + \pi n - 1)^3} \sin\left(\frac{n\pi}{L}x\right).$$

(i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (1).

a) $w(x)$, $w'' = 0$, $\rightsquigarrow w(x) = Ax + B$

$w(0) = 0 = A \cdot 0 + B \qquad B = 0$

$w(L) = L = AL + 0 \qquad A = 1$

$w(x) = x$

b) $v = u - w$

$u = v + w$

$u_t = \frac{d}{dt}(v(x,t) + w(x)) = v_t + 0$

$u_{xx} = \frac{d^2}{dx^2}(v(x,t) + w(x)) = v_{xx} + 0$

RB
$v(0,t) = u(0,t) - w(0) = 0 - 0 = 0$
$v(L,t) = u(L,t) - w(L) = L - L = 0$
$v(x,0) = u(x,0) - w(x) = f(x) + x - x = f(x)$

$$\Rightarrow \boxed{\begin{cases} v_t = c^2 v_{xx}, & 0 \le x \le L, t \ge 0, \\ v(0,t) = 0, & t \ge 0, \\ v(\pi,t) = 0, & t \ge 0, \\ v(x,0) = f(x), & 0 \le x \le L. \end{cases}}$$

AB: $f(x) = v(x,0) = \sum \frac{(4n+2)}{(n^2+\pi n-1)^3} \sin\left(\frac{n\pi}{L}x\right)$

c) WLG: ZF

Sei $u_t = c^2 u_{xx}$ mit Randbediungungen $u(0,t) = u(L,t) = 0$ und $u(x,0) = f(x)$ auf $x \in [0, L]$. Via Fourier-Reihe erhalten wir die Lösung:

$$u(x,t) = \sum_{n=1}^{\infty} B_n sin(\frac{n\pi}{L}x) e^{-\lambda_n^2 t}$$

$$\lambda_n = \frac{cn\pi}{L} \quad ; \quad B_n = \frac{2}{L} \int_0^L f(x) sin(\frac{n\pi}{L}x) dx$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{0} = \sum_{n=1}^{\infty} \frac{(4n+2)}{(n^2+\pi n-1)^3} \sin\left(\frac{n\pi}{L}x\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4n+2}{(n^2+\pi n-1)^3} \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}$$