Analysis Übungsstunde 12

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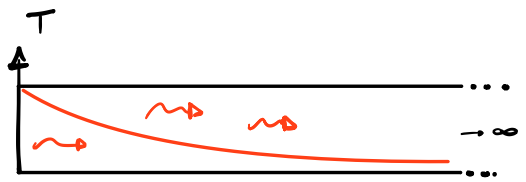
letzte Woche $t \to \infty$, heute $x \to \infty$

Wärmeleitgleichung auf einem unendlichen Stab

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = f(x) \\ x \in \mathbb{R},\ t > 0 \end{cases}$$

neu: keine RB in x $[u(0,t) = u(L,t)]$ $L \to \infty$

$u(\infty, t) = ??$



Herleitung: Skript S.81

(1) $$u(x,t) = \int_0^{\infty} (A(p)cos(px) + B(p)sin(px))e^{-c^2p^2t}dp$$

(2) $$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)cos(pv)dv$$ $\parallel$ $A(p) = 0$, falls $f(x)$ ungerade

(3) $$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)sin(pv)dv$$ $\parallel$ $B(p) = 0$, falls $f(x)$ gerade

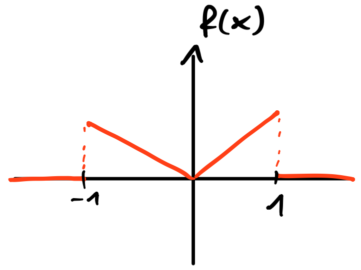
$\to$ wenn man (2)&(3) in (1) einsetzt, erhält man: (Herleitung Skript S.82)

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v) \exp\left[-\left(\frac{x-v}{2c\sqrt{t}}\right)^2\right] dv$$

Bsp

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = f(x) \end{cases}, \quad f(x) = \begin{cases} |x|, & |x| < 1 \\ 0, & \text{else} \end{cases}$$

$x \in \mathbb{R},\ t > 0$



$f(x) \to$

$A(p) =$

$=$

partiell integrieren

$\vdots \quad [\;]_0^1$

$$= \frac{2}{\pi p^2}\left(p\sin(p) + \cos(p) - 1\right)$$

$$u(x,t) = \int_0^\infty \left(A(p)\cos(px) + B(p)\sin(px)\right) e^{-c^2p^2t}\,dp$$

$$= \frac{2}{\pi}\int_0^\infty \frac{(p\sin(p)+\cos(p)-1)}{p^2}\cdot\cos(px)\cdot e^{-c^2p^2t}\,dp$$

WLG auf unendlicher Stab – Via Fourier Transformation

Gleiche Ausgangslage:

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = f(x) \\ x\in\mathbb{R},\ t>0 \end{cases}$$

Idee: $u(x,t) \xrightarrow{\mathcal{F}} \hat{u}(\omega,t)$

Remember: Woche 6: Fourier Transformation

Fourier Transformation

$$\mathcal{F}(f(t)) = \hat{f} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cdot e^{-i\omega t}\,dt$$

können auch x sein

$$\mathcal{F}^{-1}(g(\omega)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(\omega)\cdot e^{+i\omega t}\,d\omega$$

t ist ersetzbar: bsp $t^* = 4a + \frac{t^2}{2}$ oder auch mit x!

- $f(x) \underset{\mathcal{F}}{\overset{\mathcal{F}^{-1}}{\rightleftarrows}} \mathcal{F}(f(t))(\omega) = \hat{f}(\omega)$
- $\mathcal{F}^{-1}\cdot\mathcal{F} = \mathbb{I}$
- $\mathcal{F}^{-1}\left(\mathcal{F}(f(t))\right) = f(t)$

! Falls $\omega = 0$ nicht def: → $\hat{f}(0)$ separat berechnen (siehe bsp)

Eigenschaften (ZF)

- Linearität $\mathcal{F}(\alpha f + \beta g) = \alpha\mathcal{F}(f) + \beta\mathcal{F}(g)$
- Convolution $\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\cdot\mathcal{F}(g)$
- (II) Ableitungen im Zeitbereich $\mathcal{F}\left(\frac{df}{dt}\right) = i\omega\,\mathcal{F}(f(t))$

  $\mathcal{F}\left(\frac{df^2}{dt^2}\right) = -\omega^2\mathcal{F}(f(t))$ } pro weitere Ableitung je mal $i\omega$ dazu

  $\vdots$
- Ableitungen im Frequenzbereich $\frac{d\hat{f}(\omega)}{d\omega} = -i\,\mathcal{F}(t\cdot f(t)) \to$ Allg: $\mathcal{F}(x^k f) = i^k\frac{d^k\hat{f}(\omega)}{d\omega^k}$
- shifts $\mathcal{F}(f(t-a)) = e^{-ia\omega}\cdot\mathcal{F}(f(t))$

  $\mathcal{F}(\omega - b) = \mathcal{F}(e^{ib\omega}f(t))$

① Fourier Transform. nach x

$u_t = c^2 u_{xx}$ // Fourier-Transform

$\mathcal{F}(u_t) =$

$\mathcal{F}(u_{xx}) =$

$\mathcal{F}(f(x)) =$

schreibe Problem neu:

$\Big\{$

⇒ haben "normale" DGL!

② Löse mit sep. of var.

$$\boxed{\hat{u}(\omega,t) = C(\omega)\cdot e^{-c^2\omega^2 t}}$$

③ Nutze Transformierte RB

$$\hat{u}(\omega,0) =$$

④ Rücktransformation

$$u(x,t) = \mathcal{F}^{-1}\left[\mathcal{F}(f(x))\cdot e^{-c^2\omega^2 t}\right]$$

Wichtige Fourier & Fourier$^{(-1)}$ transformationen:

	Generell:	Vorgefertigt ↓ fehler auf ZF	Wichtige Integrale, nicht auf ZF
$\mathcal{F}$	$\hat{f} = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$	$\mathcal{F}\left(xe^{-ax^2}\right)(\omega) = \frac{-i\omega}{(2a)^{3/2}}e^{-\frac{\omega^2}{4a}}$ $\mathcal{F}\left(x^2e^{-ax^2}\right) = \left(1-\frac{\omega^2}{2a}\right)\cdot\frac{1}{\sqrt{8a^3}}e^{-\frac{\omega^2}{4a}}$	▷ $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$ ▷ $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \pi$ ▷ $\int_{-\infty}^{\infty} e^{-ax^2}\cdot e^{-ikx}dx = e^{\frac{k^2}{4a}}\sqrt{\frac{\pi}{a}}$
$\mathcal{F}^{-1}$	$\mathcal{F}^{-1}(g)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}d\omega$	$\mathcal{F}^{-1}\left(i\omega e^{-b\omega^2}\right) = \frac{x}{(2b)^{3/2}}e^{-\frac{x^2}{4b}}$	▷ $\int_{-\infty}^{\infty} e^{-(ak^2+bk+c)}dk = e^{\frac{b^2}{4a}-c}\cdot\sqrt{\frac{\pi}{a}}$

$\hat{f}(\omega+a) = e^{-iax}\cdot\hat{f}(\omega)$

$\Rightarrow$ je nach Problem, passe a & b an

$\rightarrow$ Ziel: PDE mittels Fouriertransformation in normale DGL umformen

Bsp: W2022 A6

6. PDE with Fourier transform (*15 Points*)

Find the solution $u = u(x,t)$ of the following equation using the Fourier transform:

$$\begin{cases} u_x + u_t + u = 0\,, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x)\,, & x \in \mathbb{R}. \end{cases} \longrightarrow$$

[*Hint:* *You can proceed as follow:*

a) Take the Fourier transform with respect to the x variable of the PDE and the initial condition and transform them into an ODE.

b) Solve the ODE.

c) Take the inverse Fourier transform of the solution of the ODE to find the solution of the PDE.]

Woche 6

$$\left[\begin{aligned} \mathcal{F}\left(\tfrac{df}{dt}\right) &= i\omega\,\mathcal{F}(f(t)) \\ \mathcal{F}\left(\tfrac{df^2}{dt^2}\right) &= -\omega^2\,\mathcal{F}(f(t)) \\ &\vdots \end{aligned}\right]$$

pro weitere Ableitung je mal $i\omega$ dazu

With respect to x:

$$\mathcal{F}\left(\frac{df}{dx}\right) = i\omega\,\mathcal{F}(f(x))$$

① Fourier Transform. nach x

$$\begin{cases} u_x + u_t + u = 0 \\ u(x,0) = f(x) \end{cases} \quad /\!/\ \mathcal{F} \quad \longrightarrow \quad \begin{aligned} &\mathcal{F}\left(\tfrac{du}{dx}\right) + \mathcal{F}\left(\tfrac{du}{dt}\right) + \mathcal{F}(u) = 0 \\ &\mathcal{F}(u(x,0)) = \mathcal{F}(f(x)) \end{aligned}$$

$$\begin{cases} \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases} \Rightarrow \text{DGL}$$

② Löse mit sep. of var.

$$\longrightarrow \quad \frac{1}{\hat{u}}\,d\hat{u} = (-1-i\omega)\,dt$$

$$\ln(\hat{u}(\omega,t)) = (-1-i\omega)t + C(\omega) \quad /\!/\ \exp$$

$$\hat{u}(\omega,t) = C(\omega)\cdot e^{(-1-i\omega)t}$$

③ Nutze Transformierte RB

$$\hat{u}(\omega,t) = C(\omega)\cdot e^{(-1-i\omega)t} \quad /\!/\ \text{RB} \quad \hat{u}(\omega,0) = \hat{f}(\omega) = \mathcal{F}(f(x))$$

④ Rücktransformation

$$\hat{u}(\omega,t) = \quad f(\omega)\cdot e^{(-1-i\omega)t} \quad /\!/\ \mathcal{F}^{-1}(g)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}\,d\omega$$

x x

$$u(x,t) =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-t} \cdot e^{-i\omega t} \cdot e^{+i\omega x}\, d\omega$$

$\neq f(\omega)$

$$= \frac{1}{\sqrt{2\pi}} e^{-t} \int_{-\infty}^{\infty} \hat{f}(\omega) \cdot e^{i\omega \overbrace{(x-t)}^{=\alpha}}\, d\omega \qquad \boxed{\text{now } (x-t) := \alpha}$$

$$= e^{-t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) \cdot e^{i\omega\alpha}\, d\omega \qquad \mathcal{F}^{-1}(g)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega$$

$x-t$ $\quad$ $x-t$

$g(\alpha)$

$=$

$u(x,t) =$

Coeff vgl mit Summe & Recap zu inhomogenen PDE

W2022 A5

5. Heat Equation with inhomogeneous boundary conditions (*15 Points*)

Find the general solution of the Heat equation (**with inhomogeneous boundary conditions**) for the following problem:

$$\begin{cases} u_t = c^2 u_{xx}, & 0 \le x \le L, t \ge 0, \\ u(0,t) = 0, & t \ge 0, \\ u(L,t) = L, & t \ge 0, \\ u(x,0) = f(x) + x, & 0 \le x \le L, \end{cases} \tag{1}$$

$f(0) = 0$

$f(L) = 0$

where $L > 0$ is a constant, and $f(x)$ is any (twice differentiable) function such that $f(0) = 0, f(L) = 0$.

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w'' = 0$, $w(0) = 0$, and $w(L) = L$.

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

c) The Fourier series of the $2L$ periodic odd extension of f is given by

$$f(x) := \sum_{n=1}^{+\infty} \frac{(4n+2)}{(n^2 + \pi n - 1)^3} \sin\left(\frac{n\pi}{L}x\right).$$

(i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (1).

a) $w(x)$, $w'' = 0$, $\rightsquigarrow w(x) = Ax + B$

b) $v = u - w$

$u = v + w$

$u_t =$

$u_{xx} =$

RB $\quad v(0,t) =$

$v(L,t) =$

$v(x,0) =$

$$\Rightarrow \boxed{\begin{cases} v_t = c^2 v_{xx}, & 0 \le x \le L, t \ge 0, \\ v(0,t) = 0, & t \ge 0, \\ v(\pi,t) = 0, & t \ge 0, \\ v(x,0) = f(x), & 0 \le x \le L. \end{cases}}$$

c) WLG: ZF

Sei $u_t = c^2 u_{xx}$ mit Randbediungungen $u(0,t) = u(L,t) = 0$ und $u(x,0) = f(x)$ auf $x \in [0,L]$. Via Fourier-Reihe erhalten wir die Lösung:

$$u(x,t) = \sum_{n=1}^{\infty} B_n sin(\frac{n\pi}{L}x) e^{-\lambda_n^2 t}$$

$$\lambda_n = \frac{cn\pi}{L} \quad ; \quad B_n = \frac{2}{L} \int_0^L f(x) sin(\frac{n\pi}{L}x) dx$$

RB: $f(x) = v(x,0) = \sum \frac{(4n+2)}{(n^2+\pi n-1)^3} \sin\left(\frac{n\pi}{L}x\right)$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^0 = \sum_{n=1}^{\infty} \frac{(4n+2)}{(n^2+\pi n-1)^3} \sin\left(\frac{n\pi}{L}x\right)$$

$u(x,t) =$