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Autumn 2024

Analysis III

Serie 13

Topics: Laplace equation on a region with symmetries.

1. Find the solution of the following Laplace equation on the disk of radius 2:

$$\begin{cases} \Delta u = 0, & (x, y) \in D_2 \\ u(x, y) = x^3. & (x, y) \in \partial D_2 \end{cases}$$

Do as follows:

a) Write the boundary condition in polar coordinates.

b) Solve the problem in polar coordinates, using the methods/formulas from the Lecture Notes.

[*Hint:* It might be useful at some point to use the trigonometric formula

$$\cos^3(\vartheta) = \frac{3}{4}\cos(\vartheta) + \frac{1}{4}\cos(3\vartheta)\,]$$

c) Express the solution in the standard cartesian coordinates:

$$u(x, y) = ?$$

2. a) Find the solution $u(r, \vartheta)$ of the following Dirichlet problem on the disk of radius R in polar coordinates:

$$\begin{cases} \Delta u = 0, & 0 \le r \le R, 0 \le \vartheta \le 2\pi \\ u(R, \vartheta) = \sin^2(\vartheta). & 0 \le \vartheta \le 2\pi \end{cases}$$

[*Hint:* Remember the trigonometric formula

$$\sin^2(\vartheta) = \frac{1}{2} - \frac{1}{2}\cos(2\vartheta)\,]$$

b) Find the maximum of $u(r, \vartheta)$. In which point(s) is it reached?

c) Express the solution in the standard cartesian coordinates.

3. Prove, without computing explicitely the integrals, that for each $0 \le r < 1$ and for each $0 \le \vartheta \le 2\pi$:

a)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\vartheta-\varphi)+r^2}\,d\varphi = 1.$$

b)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\big(\cos^3(\varphi)\sin(\varphi)-\sin^3(\varphi)\cos(\varphi)\big)}{1-2r\cos(\vartheta-\varphi)+r^2}\,d\varphi = \frac{r^4}{4}\sin(4\vartheta).$$

4. Consider the following Neumann problem (Laplace equation with fixed normal derivative on the boundary):

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ \partial_n u(R,\vartheta) = \vartheta(2\pi-\vartheta)(\vartheta^2-12), & 0 \le \vartheta \le 2\pi \ \big(\text{ parametrising } \partial D_R\big) \end{cases}$$

with D_R the disk center in the origin and radius R.

Is there a solution?

5. Consider the Dirichlet problem for the Laplace equation,

$$R = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 2\},$$

$$\begin{cases} \Delta u = 0, & (x,y) \in R, \\ u(0,y) = u(1,y) = 0, & 0 \le y \le 2, \\ u(x,0) = 0, & 0 \le x \le 1, \\ u(x,2) = f(x) & 0 \le x \le 1, \end{cases}$$

where f is a given continuous function. Show that there exists a unique solution.

[*Hint:* Assume to have two solutions u_1 and u_2 and consider the Dirichlet problem for the Laplace equation for $v = u_1 - u_2$.]

Hand in on Moodle by: Wednesday 18 December 2024.