1 Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

1.1 Eigenschaften

Ähnlichkeit:	$\mathcal{L}\{\frac{1}{a} \cdot f(\frac{t}{a})\} = F(s \cdot a)$
s-shifting:	$\mathcal{L}\{f(t) \cdot e^{at}\} = F(s-a)$
t-shifting:	$\mathcal{L}\{f(t-a) \cdot u(t-a)\} = e^{-as} \cdot F(s)$
	$\mathcal{L}\{f(t) \cdot u(t-a)\} = e^{-as} \cdot F(s+a)$
Ableitung, t:	$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
	$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
	$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
Ableitung, s:	$\mathcal{L}\{t \cdot f(t)\} = -F'(s)$
	$\mathcal{L}\{t^n \cdot f(t)\} = (-1)^n \cdot F^{(n)}(s)$
Integration, t:	$\mathcal{L}\{\int_0^\infty f(\tau)d\tau\} = \frac{1}{s}F(s)$
Integration, s:	$\mathcal{L}\{\frac{1}{t}(t)\} = \int_0^\infty F(\tau)d\tau$
Faltung:	$(f*g)(t) = \int_0^t f(\tau) \cdot g(t-\tau)d\tau, \quad f*g = g*f$
	$f*(g*h) = (f*g)*h, \quad f*(g+h) = f*g + f*h$
	$f*0 = 0, \quad f*1 = \int_0^\infty f(t)dt$
	$\mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\} = \mathcal{L}\{f(t)*g(t)\}$
Anfangswert:	$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} s \cdot F(s)$
Endwert:	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s \cdot F(s)$

1.2 Wichtige Signaltransformationen

$u(t)$	$U(s)$
Impuls (dirac delta) δ	1
$\delta(t-a)$	e^{-as}
Schritt (heavyside)	$\frac{1}{s}$
(heavyside): $u_c(t) = u(t-c)$	$\frac{e^{-cs}}{s}$
ramp, t	$\frac{1}{s^2}$
$h(t) \cdot t^n \cdot e^{\alpha t}$	$\frac{n!}{(s-\alpha)^{n+1}}$
$h(t) \cdot \sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$h(t) \cdot \cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$h(t) \cdot \sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$h(t) \cdot \cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$h(t) \cdot (e^{\alpha t} - 1)$	$\frac{a}{s(s-a)}$
$h(t) \cdot \frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$
$h(t) \cdot \frac{ae^{at}-be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$

1.3 Laplace Transformationen

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n, $n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
t^p, $p > 0$	$\frac{\Gamma(p+1)}{s^{p+1}}$
$t^{n-\frac{1}{2}}$, $n = 1, 2, 3, \ldots$	$\frac{1 \cdot 2 \cdot \ldots \cdot (2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\sin(\omega t) + \omega t\cos(\omega t)$	$\frac{2\omega s^2}{(s^2+\omega^2)^2}$
$\sin(\omega t) - \omega t\cos(\omega t)$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
$\cos(\omega t) + \omega t\sin(\omega t)$	$\frac{s(s^2+3\omega^2)}{(s^2+\omega^2)^2}$
$\cos(\omega t) - \omega t\cos(\omega t)$	$\frac{s(s^2-\omega^2)}{(s^2+\omega^2)^2}$
$\sin(\omega t + \varphi)$	$\frac{s\sin(\varphi)+\omega\cos(\varphi)}{s^2+\omega^2}$
$\cos(\omega t + \varphi)$	$\frac{s\cos(\varphi)-\omega\sin(\varphi)}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$t\sinh(\omega t)$	$\frac{2\omega s}{(\omega^2-s^2)^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$t\cosh(\omega t)$	$\frac{\omega^2+s^2}{(a^2-s^2)^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$
$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
t^ne^{at}, $n = 1, 2, 3, \ldots$	$\frac{n!}{(s-a)^{n+1}}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{1}{2}}}$
$f(ct)$	$\frac{1}{c}F(\frac{s}{c})$
$f(t+T) = f(t)$	$\frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$

2 Fourier

2.1 Eigenschaften Periodischer Funktionen

$f(x)$ periodisch, wenn $f(x) = f(x+p)\ \forall x$. Periode $p \Rightarrow L = \frac{p}{2}$ bestimmen.
Eigenschaften für $f(x) = f(x+p)$, $h(x) = h(x+p)\ \forall x$

- $f(x)$ periodisch $\Rightarrow f'(x)$ auch
- Für $f(x)$ p_1 und für $g(x)$ p_2, für $f(x) + g(x)$ p mit p als kgV(p_1, p_2)
- $f(ax)$ ist $\frac{p}{a}$-periodisch
- $a \cdot f(x) + b \cdot h(x)$ ist p periodisch
- $f(x)$ und $h(x)$ sind $n \cdot p$-periodisch

2.1.1 Gerade/Ungerade Funktionen

- **Gerade (even):** $f(-x) = f(x)$
- **Ungerade (odd):** $f(-x) = -f(x)$

$f(x)$ ist gerade und $h(x)$ ist ungerade, dann gilt

- $f(x) \cdot h(x)$ ist ungerade
- $f(x) \cdot f(x)$ ist gerade
- $h(x) \cdot h(x)$ ist gerade

Periodische Fortsetzung

- Gegeben: $f(x)$ von 0 bis L
- Gesucht: $2L$-periodische Erweiterung

2.2 Fourier Reihen

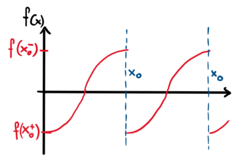
Ziel: $f(x)$ $2L$-periodische Summe von **sinus** und **cosinus** termen, $L = \frac{p}{2}$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(\frac{n\pi}{L}x) + b_n\sin(\frac{n\pi}{L}x)$$

- $a_0 = \frac{1}{2L}\int_{-L}^{+L} f(x)dx$, falls $f(x)$ **ungerade** $\Rightarrow a_0 = 0$
- $a_n = \frac{1}{L}\int_{-L}^{+L} f(x) \cdot \cos(\frac{n\pi}{L}x)dx$, falls $f(x)$ **ungerade** $\Rightarrow a_n = 0$
- $b_n = \frac{1}{L}\int_{-L}^{+L} f(x) \cdot \sin(\frac{n\pi}{L}x)$, falls $f(x)$ **gerade** $\Rightarrow b_n = 0$

2.2.1 Konvergenz bei Unstetigkeit (uncontinuousity)

Die Fourier-Reihe von $f(x)$ konvergiert in x_0 zu: $\frac{f(x_0^+)+f(x_0^{-1})}{2}$



2.3 Komplexe Fourier Reihe

$$\sin(t) = \frac{e^{it}-e^{-it}}{2i} \text{ und } \cos(t) = \frac{e^{it}+e^{-it}}{2}$$

die Fourier Reihe lässt sich folgendermassen umformen

$$f(x) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2}(e^{i\frac{n\pi}{L}x} + e^{-i\frac{n\pi}{L}x}) - i\frac{b_n}{2}(e^{i\frac{n\pi}{L}x} - e^{-i\frac{n\pi}{L}x})$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2}\right)e^{i\frac{n\pi}{L}x} + \left(\frac{a_n + ib_n}{2}\right)e^{-i\frac{n\pi}{L}x}$$

mit $c_0 = a_0$, $c_n = \left(\frac{a_n - ib_n}{2}\right)e^{i\frac{n\pi}{L}x}$ und $c_{-n} = \left(\frac{a_n + ib_n}{2}\right)e^{-i\frac{n\pi}{L}x}$ folgt

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}$$

- $c_n = \frac{1}{2L}\int_{-L}^{+L} f(x)e^{-i\frac{n\pi}{L}x}dx$

2.4 Fourier Integral

Bisher wurden **periodische** Funktionen thematisiert, nun **nicht periodische**.
Fourier Integral der nicht periodischen Funktion $f(x)$

$$f(x) = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]d\omega$$

- $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\cos(\omega v)dv$
  - $f(x)$ **gerade:** $A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos(\omega v)dv$, **ungerade:** $A(\omega) = 0$
- $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\sin(\omega v)dv$
  - $f(x)$ **ungerade:** $B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin(\omega v)dv$, **gerade:** $B(\omega) = 0$

Das Fourier Integral **existiert nur**, wenn: $\int_{-\infty}^{+\infty} |f(x)|dx < \infty$

2.5 Fourier Transformation

Fourier Transformation: $\mathcal{F}(f) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx$

Inverse Fourier Transformation: $\mathcal{F}^{-1}(\hat{f}) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\omega)e^{i\omega x}d\omega$

2.5.1 Eigenschaften

- $\mathcal{F}(af(x) + bg(x)) = a\hat{f}(\omega) + b\hat{g}(\omega)$
- $\frac{d}{d\omega}\mathcal{F}(f(x)) = -i\mathcal{F}(xf(x))$
- $\mathcal{F}(\frac{df(\omega)}{dx}) = i\omega\hat{f}(\omega)$
- $\mathcal{F}(f(x) * g(x)) = \sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$
- $\mathcal{F}(\frac{\partial f(x,y)}{\partial y}) = \frac{\partial}{\partial y}\hat{f}(\omega, y)$

3 PDE's

3.1 Wichtige Gleichungen

- 1D-Wellengleichung: $\frac{\partial^2}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
- 2D-Wellengleichung: $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$
- 1D-Wärmegleichung: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
- 2D-Poissongleichung: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$
- 2D-Laplacegleichung: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
- 3D-Laplacegleichung: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

3.2 Namensgebung

Allgemeine Form einer PDE 2er Ordnung:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

Form der Kurve:

$$Ax^2 + 2Bxy + Cy^2 = F(x, y, u, u_x, u_y)$$

- $AC - B^2 < 0 \Rightarrow$ **Hyperbolische PDE** (Wellengleichung)
- $AC - B^2 = 0 \Rightarrow$ **Parabolische PDE** (Wärmegleichung)
- $AC - B^2 > 0 \Rightarrow$ **Elliptische PDE** (Laplace/Poisson)
- Vorzeichen ändert sich mit $x, y \Rightarrow$ **Gemischte PDE**

3.3 Koeffizientenvergleich

Als Beispiel betrachten wir hier die Wärmegleichung:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(0,t) = u(L,t) &= 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= \sin(\frac{\pi}{L}x) \end{aligned}$$

1. $u(x,t) = \sum_{n=1}^{\infty}[B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)] \sin(\frac{n\pi}{L}x), \quad \lambda_n = \frac{n\pi}{L}c$
2. $u(x, 0) = B_n \sin(\frac{n\pi}{L}x) = 0 \Rightarrow B_n = 0$
3. $u_t(x,t) = \sum_{n=1}^{\infty} B_n^* \cdot \frac{n\pi}{L}c \cdot \cos(\frac{n\pi}{L}ct) \cdot \sin(\frac{n\pi}{L}x) =$
4. $u_t(x,0) = B_n^* \frac{n\pi}{L}c \cdot \sin(\frac{n\pi}{L}x) = \sin(\frac{\pi}{L}x) \Rightarrow n = 1$
5. $B_1^* = \frac{L}{c\pi}$ und $B_{n\geqslant 2}^* = 0$
6. $u(x,t) = \frac{L}{c\pi}\sin(\frac{\pi}{L}ct)\sin(\frac{\pi}{L}x)$

3.4 Wellengleichung

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(0,t) = u(L,t) &= 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

3.4.1 Herleitung

1. **Ansatz:** $u(x,t) = F(x)G(t)$
2. **Einsetzen in** $u_{tt} = c^2 u_{xx}$**:**
   a) $F(x)\ddot{G}(t) = c^2 F''(x)G(t)$
   b) $\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k, \quad k \in \mathbb{R}, k = \text{const.}$
3. **Separation:** $F'' = Fk$ und $\ddot{G} = c^2 Gk$
4. **Fallunterscheidung:**
   a) $k = 0$: $u(x,t) = 0 \Rightarrow$ **triviale Lösung**
   b) $k > 0$: $F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x} \Rightarrow$ **triviale Lösung**
   c) $k < 0$:
   $$F(x) = A\sin(\sqrt{-k}x) + B\cos(\sqrt{-k}x)$$
   $$G(t) = C\sin(\sqrt{\alpha}ct) + D\cos(\sqrt{\alpha}ct)$$
5. **Randbedingungen** $u(0,t) = u(L,t) = 0$
   a) $u(0,t) = F(0)G(t) \Rightarrow F(0) = 0 \Rightarrow B = 0$
   b) $u(L,t) = F(L)G(t) \Rightarrow F(L) = 0 \Rightarrow \sqrt{-k} = \frac{n\pi}{L}$
   c) $G(t) = C\sin(\lambda_n t) + D\cos(\lambda_n t) \Rightarrow \frac{n\pi}{L}c = \lambda_n$
6. **Zusammensetzen mit** $B_n = AD$, $B_n^* = AC$**:**
   $$u_n(x,t) = [B_n\cos(\lambda_n t) + B_n^* \sin(\lambda_n t)]\sin(\frac{n\pi}{L}x)$$
7. **Allgemeine Lösung:**
   $$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$
8. **Fourier für PDE:** (Auch Koeffizientenvergleich möglich)
   a) λ_n berechnen
   b) $B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx \Rightarrow f(x) = u(x,0)$
   c) $B_n^* = \frac{2}{L\lambda_n}\int_0^L g(x)\sin(\frac{n\pi}{L}x)dx \Rightarrow g(x) = u_t(x,0)$

3.4.2 Direkter Weg

1. **Allgemeine Lösung:**
   $$u(x,t) = \sum_{n=1}^{\infty}[B_n\cos(\lambda_n t) + B_n^*\sin(\lambda_n t)]\sin(\frac{n\pi}{L}x)$$
2. **Fourier:**
   - λ_n berechnen
   - $B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx \Rightarrow f(x) = u(x,0)$
   - $B_n^* = \frac{2}{L\lambda_n}\int_0^L g(x)\sin(\frac{n\pi}{L}x)dx \Rightarrow g(x) = u_t(x,0)$

3.5 Wellengleichung nach d'Alembert

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$$

3.6 Wärmegleichung

$$\begin{aligned} u_t &= c^2 u_{xx} \\ u(0,t) = u(L,t) &= 0 \\ u(x,0) &= f(x) \end{aligned}$$

3.6.1 Herleitung

1. **Ansatz:** $u(x,t) = F(x)G(t)$
2. **Einsetzen in** $u_t = c^2 u_{xx}$
   a) $F(x)\dot{G}(t) = c^2 F''(x)G(t)$
   b) $\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = k, \quad k \in \mathbb{R}, k = \text{const}$
3. **Separation:** $F'' = Fk$ und $\dot{G} = c^2 Gk$
4. **Fallunterscheidung:**
   a) $k = 0$: $u(x,t) = 0 \Rightarrow$ **triviale Lösung**
   b) $k > 0$: $F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x} \Rightarrow$ **triviale Lösung**
   c) $k < 0$: Set $k = -p^2$
   $$F(x) = A\sin(px) + B\cos(px)$$
   $$G(t) = Ce^{c^2 kt}$$
5. **Randbedingungen** $u(0,t) = u(L,t) = 0$**:**
   a) $u(0,t) = F(0)G(t) = 0 \Rightarrow F(0) = 0 \Rightarrow B = 0$
   b) $u(L,t) = F(L)G(t) = 0 \Rightarrow F(L) = 0 \Rightarrow p = \frac{n\pi}{L}$
   c) $G(t) = Ce^{-c^2(\frac{n\pi}{L})^2 t}$ mit $k = -\left(\frac{n\pi}{L}\right)^2 = -p^2$
6. **Zusammensetzen mit** $B_n = AC$**:**
   $$u_n(x,t) = F_n G_n = B_n \sin(\frac{n\pi}{L}x)e^{-\lambda_n^2 t}, \quad \lambda_n = \frac{cn\pi}{L}$$
7. **Allgemeine Lösung:**
   $$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L}x)e^{-\lambda_n^2 t}$$
8. **Fourier für PDE:**
   a) λ_n berechnen
   b) $B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)dx \Rightarrow f(x) = u(x,0)$

3.6.2 Direkter Weg

1. **Allgemeine Lösung:**
   $$u(x,t) = \sum_{n=1}^{\infty} B_n\sin(\frac{n\pi}{L}x)e^{-\lambda_n^2 t}$$
2. **Fourier:**
   - λ_n berechnen
   - $B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)dx \Rightarrow f(x) = u(x,0)$

3.7 Laplace-Gleichung - Dirichlet Problem auf einem Rechteck

$$\Delta u = \nabla^2 u = u_{xx} + u_{yy}$$

Stationäre Temperaturverteilung ($\dot{u} = 0$) auf einer Platte (Rechteck)
Randbedingungen:

- An 3 Seiten: Temperatur $= 0$
- An 4ten Seite: Temperatur $= f(x)$ (oder $g(y)$)

$$\begin{aligned} \Delta u &= 0 \\ u(0,y) = u(a,y) &= 0 \\ &= u(x,0) \\ u(x,b) &= f(x) \end{aligned}$$

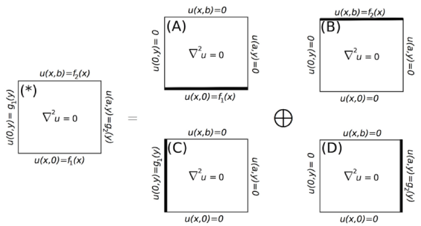
3.7.1 Herleitung mit Separationsansatz

1. **Ansatz:** $u(x,y) = F(x)G(y)$
2. **Einsetzen in** $u_{xx} + u_{yy} = 0$
   a) $F''G + F\ddot{G} = 0 \Rightarrow F''G = -F\ddot{G}$
   b) $\frac{F''}{F} = -\frac{\ddot{G}}{G} = -k, \quad k \in \mathbb{R},\ k = \text{const.}$
3. **Separation:** $F'' = -Fk$ und $\ddot{G} = Gk$
4. **Fallunterscheidung:**
   a) $k = 0$: $u(x,y) = 0 \Rightarrow$ **triviale Lösung**
   b) $k < 0$: $u(x,y) = 0 \Rightarrow$ **triviale Lösung**
   c) $k > 0$:
   $$F(x) = A\sin(\sqrt{\gamma}x) + B\cos(\sqrt{\gamma}x)$$
   $$G(y) = C\sinh(\tfrac{n\pi}{a}y)$$
5. **Randbedingungen:**
   a) $u(0,y) = F(0)G(y) = 0 \Rightarrow F(0) = 0 \Rightarrow B = 0$
   b) $u(a,y) = F(a)G(y) = 0 \Rightarrow F(a) = 0 \Rightarrow \sqrt{\gamma} = \frac{n\pi}{a}$
6. **Zusammensetzen mit** $A_n = AC$:
   $$u_n(x,y) = F_nG_n = A_n\sinh(\tfrac{n\pi}{a}y)\sin(\tfrac{n\pi}{a}x)$$
7. **Allgemeine Lösung:**
   $$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n\sinh(\tfrac{n\pi}{a}y)\sin(\tfrac{n\pi}{a}x)}$$
8. **Fourier für PDE:** Mit der Randbedingung $u(x,b)$ folgt
   $$u(x,b) = \sum_{n=1}^{\infty} A_n\sinh(\tfrac{n\pi}{a}b)\sin(\tfrac{n\pi}{b}x) = f(x)$$
   - $A_n\sinh(\frac{n\pi}{a}b)$ ist der Fourier Koeffizient
   - $A_n = \frac{2}{a\sinh(\frac{n\pi}{a}b)}\int_0^a f(x)\sin(\frac{n\pi}{a}x)dx$

3.7.2 Direkter Weg

1. **Allgemeine Lösung:**
   $$u(x,t) = \sum_{n=1}^{\infty} A_n\sinh(\tfrac{n\pi}{a}y)\sin(\tfrac{n\pi}{a}x)$$
2. **Fourier:**
   a) $A_n = \frac{2}{a\sinh(\frac{n\pi}{a}b)}\int_0^a f(x)\sin(\frac{n\pi}{a}x)dx$

3.8 Superposition für ein Dirichlet Problem



- **Lösung für** A: $u_1(x,y) = \sum_{n=1}^{\infty} A_n\sinh(\frac{n\pi x}{a})\sin(\frac{n\pi(b-y)}{a})$
  $$A_n = \frac{2}{a\sinh(\frac{n\pi b}{a})}\int_0^a f_1(x)\sin(\tfrac{n\pi x}{a})dx$$
- **Lösung für** B: $u_2(x,y) = \sum_{n=1}^{\infty} B_n\sinh(\frac{n\pi x}{a})\sin(\frac{n\pi y}{a})$
  $$B_n = \frac{2}{a\sinh(\frac{n\pi b}{a})}\int_0^a f_2(x)\sin(\tfrac{n\pi x}{a})dx$$
- **Lösung für** C: $u_3(x,y) = \sum_{n=1}^{\infty} C_n\sinh(\frac{n\pi(a-x)}{b})\sin(\frac{n\pi y}{b})$
  $$C_n = \frac{2}{b\sinh(\frac{n\pi a}{b})}\int_0^b g_1(y)\sin(\tfrac{n\pi y}{a})dy$$
- **Lösung für** D: $u_1(x,y) = \sum_{n=1}^{\infty} D_n\sinh(\frac{n\pi x}{b})\sin(\frac{n\pi y}{b})$
  $$D_n = \frac{2}{b\sinh(\frac{n\pi a}{b})}\int_0^b g_2(y)\sin(\tfrac{n\pi y}{a})dy$$
- **Lösung für** $A + B + C + D$: $u = u_1 + u_2 + u_3 + u_4$

3.9 Wärmegleichung für einen unendlichen Stab

$$u_t = c^2u_{xx}$$
$$u(x,0) = f(x)$$

1. **Ansatz:** $u(x,t) = F(x)G(x)$
2. **Einsetzen in** $u_t - c^2u_{xx} = 0$
   a) $F\dot{G} - c^2F''G = 0 \Rightarrow F\dot{G} = c^2F''G$
   b) $\frac{\dot{G}}{c^2G} = \frac{F''}{F} = k, \quad k \in \mathbb{R},\ k = \text{const.}$
3. **Separation:** $F'' = Fk$ und $\dot{G} = c^2Gk$
4. **Fallunterscheidung:**
   a) $k = 0$: $u(x,t) = e^{-c^2kt}(Ae^{\sqrt{-k}x} + Be^{-\sqrt{-k}x})$
   u wird grösser wenn t steigt $\Rightarrow$ physikalisch unmöglich
   b) $k > 0$: $k := p^2$
   $$F_p(x) = A(p)\cos(px) + B(p)\sin(px)$$
   $$G_p(t) = e^{-c^2p^2t}$$
5. **Allgemeine Lösung:**
   $$\boxed{u(x,t) = \int_0^{\infty}[A(p)\cos(px) + B(p)\sin(px)]e^{-c^2p^2t}dx}$$
6. **Randbedingungen:**
   a) $u(x,0) = f(x) = \int_0^{\infty}[A(p)\cos(px) + B(p)\sin(px)]$
7. **Fourier:**
   - $A(p) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\cos(pv)dv$
   - $B(v) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\sin(pv)dv$
8. **Endresultat:**
   $$\boxed{u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{+\infty} f(v)\exp\left[-\left(\frac{x-v}{2c\sqrt{t}}\right)^2\right]dv}$$

3.9.1 Mit Fourier-Transformation

1. **Fourier Transformation von** $u_t = c^2u_{xx}$ **und R.B.** $u(x,0) = f(x)$
   a) $\mathcal{F}(u_t) = \mathcal{F}(c^2u_{xx})$
   b) $\mathcal{F}(u_{xx}) = -\omega^2\hat{u}$
   c) $$\begin{aligned}\mathcal{F}(u_t(x,t)) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} u_te^{i\omega x}dx = \frac{1}{\sqrt{2\pi}}\frac{d}{dt}\int_{-\infty}^{+\infty} u(x,t)e^{i\omega x}\\ &= \frac{d}{dt}\mathcal{F}(u(x,t)) = \frac{d}{dt}\hat{u}\end{aligned}$$
   d) $\frac{d}{dt}\hat{u} = -c^2\omega^2\hat{u}$
   e) $\hat{u}(\omega,0) = \hat{f}(\omega)$
2. **Neue DGL für** $\hat{u}(\omega,t)$ **lösen**
3. $\hat{u}(\omega,t) = \ldots\ \mathcal{F}^{-1}(\hat{u}(\omega,t)) = u(x,t)$

3.10 Dirichlet Problem auf einem symetrischen Gebiet

Laplacegleichung auf einer Kreisscheibe

Polarkoordinaten $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta)\end{cases} \iff \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan(\frac{y}{x})\end{cases}$

$$\begin{aligned}\Delta u = u_{xx} + u_{yy} &= 0 \quad &, x^2 + y^2 < \mathbb{R}^2\\ u &= f \quad &, x^2 + y^2 = \mathbb{R}^2\end{aligned}$$

In Polarkoordinaten:
$u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r = 0,\ r \in [0,R),\ \theta \in [0,2\pi]$
$u(R,\theta) = f(\theta),\ \theta \in [0,2\pi)$

3.10.1 Lösung mit Separation der Variablen

1. **Ansatz:** $u(r,\theta) = F(r)G(\theta)$
   a) $u_r = rF'(r)G(\theta)$
   b) $u_{rr} = r^2F''(r)G(\theta)$
   c) $u_{\theta\theta} = F(r)G''(\theta)$
2. **Einsetzen in** $u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r = 0$:
   a) $F''G + \frac{1}{r^2}FG'' + \frac{1}{r}F'G = 0$
   b) $\frac{r^2F''+rF'}{F} = -\frac{G''}{G} = k \quad k \in \mathbb{R},\ k = \text{const.}$
3. **Separation:** $r^2F'' + rF' - Fk = 0$ und $G'' + Gk = 0$
4. **Fallunterscheidung:**
   a) $k = 0$: $F(x) = 0$ und $G(\theta) = \text{const.}$
   b) $k < 0$: $u(x,t) = 0 \Rightarrow$ **triviale Lösung**
   c) $k > 0$:
   $$F_n(r) = P_nr^n + Q_nr^{-n} \text{ mit } Q_n = 0, \text{ weil Scheibe beschränkt}$$
   $$G_n(\theta) = A_n^*\cos(n\theta) + B_n^*\sin(n\theta)$$
5. **Zusammensetzen mit** $A_n = P_nA_n^*$, $B_n = P_nB_n^*$:
   $$u_n(r,\theta) = F_nG_n = r^n(A_n\cos(n\theta) + B_n\sin n\theta)$$
6. **Allgemeine Lösung:**
   $$\boxed{u(x,t) = \sum_{n=0}^{\infty} r^n(A_n\cos(n\theta) + B_n\sin(n\theta))}$$
7. **Randbedingung** $u(R,\theta) = f(\theta)$**:**
   a) $f(\theta) = A_0 + \sum_{n=1}^{\infty} R^n(A_n\cos(n\theta) + B_n\sin(n\theta))$
8. **Fourier:**
   - $A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)d\varphi$
   - $A_n = \frac{1}{R^n\pi}\int_0^{2\pi} f(\varphi)\cos(n\varphi)d\varphi$
   - $B_n = \frac{1}{R^n\pi}\int_0^{2\pi} f(\varphi)\sin(n\varphi)d\varphi$
9. **Endresultat:**
   $$\boxed{u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} = \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2}f(\varphi)d\varphi}$$

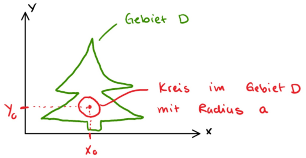
3.10.2 Direkter Weg

1. Poisson Integral Kernel: $\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2} \quad (= K(r,\theta,R,\varphi))$
2. $u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} = \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2}f(\varphi)d\varphi$

3.11 Mean Value Principle und Maximum Principle

Harmonische Funktionen erfüllen die Poissongleichung $\Delta u = 0$. Auch wenn die Lösung u des Problems unbekannt ist, kann man mit dem **Maximumsprinzip** wichtige Aussagen treffen.

3.11.1 Mittelwerte - Maxima - Minima



$$
\begin{aligned}
u(x_0, y_0) = &\quad u(r = 0, \theta) \\
= &\quad \frac{1}{2\pi} \int_0^{2\pi} K(0, \theta, R = a, \varphi) \cdot u(a, \varphi) d\varphi \\
= &\quad \frac{1}{2\pi} \int_0^{2\pi} u(a, \varphi) d\varphi \quad \text{(Mittelwert von } u(a, \varphi))
\end{aligned}
$$

- u ist eine harmonische Funktion auf dem Gebiet D
- (x_0, y_0) ist ein beliebiger Punkt auf D
- $a \in \mathbb{R}$ so dass ein Kreis mit Radius a vollständig in D

$\Rightarrow$ Der Funktionswert in jedem beliebigen Punkt P = Mittelwert der Funktionswerte auf jedem beliebigen Kreis, wenn P der Mittelpunkt des Kreises ist
$\Rightarrow$ Minima und Maxima von u sind auf dem Rand ∂D oder $u = \text{const.}$

3.11.2 Beispiel Inhomogene Wellengleichung

$$
\text{Inhom.:} \begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 3 \\ u(\pi, t) = 5 \\ u(x,0) = x^2 + \frac{x}{\pi}(2 - \pi^2) + 3 \\ u_t(x,0) = 0 \end{cases} \qquad \text{Bekannt:} \begin{cases} v_{tt} = c^2 v_{xx} \\ v(0,t) = 0 \\ v(\pi, t) = 0 \\ v(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}
$$

1. $v(x,t) = u(x,t) - w(x)$
   a) $v(0,t) = u(0,t) - w(0) = 3 - w(0) = 0 \Rightarrow w(0) = 3$
   b) $v(\pi,t) = u(\pi,t) - w(\pi) = 5 - w(\pi) = 0 \Rightarrow w(\pi) = 5$
   c) $v_{tt} = c^2 v_{xx} \rightarrow u_{tt} = c^2(u_{xx} - w_{xx}) \Rightarrow w_{xx} = w'' = 0$
   d) $w(x)$ ist von der Form $w(x) = Ax + B$
2. $w(x) = Ax + B$
   a) Mit den RB aus 1. $\Rightarrow w(x) = \frac{2}{\pi} + 3$
   b) $f(x) = v(x,0) = u(x,0) - w(x) = x^2 - \pi x$
   c) $g(x) = v_t(x,0) = u_t(x,0) - w_t(x) = 0$
3. Aufgabenstellung neu formulieren
   a) Neue Aufgabe: $\begin{cases} v_{tt} = c^2 v_{xx} \\ v(0,t) = 0 \\ v(\pi,t) = 0 \\ v(x,0) = x^2 - \pi x \\ u_t(x,0) = 0 \end{cases}$
   b) $v(x,t)$ berechnen
   c) $u(x,t) = v(x,t) + w(x)$

4 Anhang

4.1 Laplace

1. Beide Seiten der Gleichung $\mathcal{L}(\ldots) \Rightarrow \mathcal{L}(y(t)) = Y(s)$ etc.
2. Die erhaltene Gleichung nach $Y(s)$ auflösen
3. Erneut auf beiden Seiten $\mathcal{L}^{-1}(\ldots) \Rightarrow \mathcal{L}^{-1}(Y(s)) = y(t)$ etc.

4.1.1 Beispiel

$$y(t) + \frac{1}{\sqrt{2}} \int_0^t y(\tau) \sin(\sqrt{2}(t - \tau)) d\tau = t$$

1. $Y(s) + \frac{1}{\sqrt{2}} \mathcal{L}(\int_0^t y(\tau) \sin(\sqrt{2}(t-\tau)) d\tau) = \frac{1}{s^2}$
   a) $\int_0^t y(\tau) \sin(\sqrt{2}(t-\tau)) d\tau = f * g \Rightarrow \mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$
2. $Y(s) + \frac{Y(s)\sqrt{2}}{\sqrt{2}(s^2+2)} = \frac{1}{s^2} \Rightarrow \frac{1}{s^2} = Y(s)(1 + \frac{1}{s^2+2}) \Rightarrow Y(s) = \frac{s^2+2}{s^2(s^2+3)}$
3. $\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{s^2+2}{s^2(s^2+3)}\right) \Rightarrow \frac{s^2+2}{s^2(s^2+3)} = \frac{2}{3s^2} + \frac{1}{3(s^2+3)}$
4. $y(t) = \frac{2}{3}t + \frac{\sin(\sqrt{3}t)}{s\sqrt{3}}$

4.2 Fourier

1. Ist die Funktion periodisch (2.) oder nicht (5.)?
2. Periode p und daraus L bestimmen ($p = 2\pi = 2L \Rightarrow L = \frac{p}{2} = \pi$)
3. Ist die funktion gerade (even) oder ungerade (odd)?
   a) **Even:** $f(x) = f(-x) \Rightarrow B_n = 0$
   b) **Odd:** $-f(x) = f(-x) \Rightarrow A_0, A_n = 0$
4. A_0, A_n, B_n bestimmen und einsetzen
5. Das Fourier Integral aufstellen
6. Ist die funktion gerade (even) oder ungerade (odd)?
   a) **Even:** $f(x) = f(-x) \Rightarrow B(\omega) = 0,\ A(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \cos(\omega v) dv$
   b) **Odd:** $-f(x) = f(-x) \Rightarrow A(\omega) = 0,\ B(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin(\omega v) dv$
7. Existenz des Fourier Integral überprüfen

$$\int_{-\infty}^{+\infty} |f(x)| dx < \infty$$

4.2.1 Beispiel

$f(x) = x(\pi - x)$ for $x \in [0, \pi]$, $f_{\text{odd}}(x)$ is the 2π periodic extension

1. The function is periodic
2. $p = 2\pi \Rightarrow L = \pi$
3. The function is odd $A_0, A_n = 0$
4. $B_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \sin(\frac{n\pi}{L} \cdot x) = \frac{2}{\pi} \int_0^\pi x(\pi - x) \sin(nx) dx = \ldots$

$$f_{\text{odd}}(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^3} \sin(nx) = \frac{8}{\pi} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^3} \sin((2j+1)x)$$

4.3 Trigonometry

4.3.1 Sinus und Cosinus

α	0 / $0°$	$\pi/6$ / $30°$	$\pi/4$ / $45°$	$\pi/3$ / $60°$	$\pi/2$ / $90°$	π / $180°$	**T**	**0-Stellen**
$\sin\alpha$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0	$2\cdot\pi$	$k\cdot\pi$
$\cos\alpha$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1	$2\cdot\pi$	$\frac{\pi}{2}+k\cdot\pi$
$\tan\alpha$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	–	0	π	$k\cdot\pi$
$\cot\alpha$	–	$\sqrt{3}$	1	$\sqrt{3}/3$	0	–	π	$\frac{\pi}{2}+k\cdot\pi$

$cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}) \quad cosh(x) = \frac{1}{2}(e^x + e^{-x})$
$sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) \quad sinh(x) = \frac{1}{2}(e^x - e^{-x})$
$e^{2ix} = cos(2x) + isin(2x) \quad e^{-2ix} = cos(2x) - isin(2x)$
$sec(x) = \frac{2cos(x)}{cos(2x)+1}$

	$\int_0^{\frac{\pi}{4}}$	$\int_0^{\frac{\pi}{2}}$	$\int_0^{\pi}$	$\int_0^{2\pi}$	$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$	$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$	$\int_{-\pi}^{\pi}$
sin	$\frac{\sqrt{2}-1}{\sqrt{2}}$	1	2	0	0	0	0
sin^2	$\frac{\pi-2}{8}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{\pi-2}{4}$	$\frac{\pi}{2}$	π
sin^3	$\frac{8-5\sqrt{2}}{12}$	$\frac{2}{3}$	$\frac{4}{3}$	0	0	0	0
sin^4	$\frac{3\pi-8}{32}$	$\frac{3\pi}{16}$	$\frac{3\pi}{8}$	$\frac{3\pi}{4}$	$\frac{3\pi-8}{16}$	$\frac{3\pi}{8}$	$\frac{3\pi}{4}$
cos	$\frac{1}{\sqrt{2}}$	1	0	0	$\sqrt{2}$	2	0
cos^2	$\frac{2+\pi}{8}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{2+\pi}{4}$	$\frac{\pi}{2}$	π
cos^3	$\frac{5}{6\sqrt{2}}$	$\frac{2}{3}$	0	0	$\frac{5}{3\sqrt{2}}$	$\frac{4}{3}$	0
cos^4	$\frac{8+3\pi}{32}$	$\frac{3\pi}{16}$	$\frac{3\pi}{8}$	$\frac{3\pi}{4}$	$\frac{8+3\pi}{16}$	$\frac{3\pi}{8}$	$\frac{3\pi}{4}$
$sin \cdot cos$	$\frac{1}{4}$	$\frac{1}{2}$	0	0	0	0	0
$sin^2 \cdot cos$	$\frac{1}{6\sqrt{2}}$	$\frac{1}{3}$	0	0	$\frac{1}{3\sqrt{2}}$	$\frac{2}{3}$	0
$sin \cdot cos^2$	$\frac{4-\sqrt{2}}{12}$	$\frac{1}{3}$	$\frac{2}{3}$	0	0	0	0

4.3.2 Additionstheoreme und Identitäten

- $\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$
- $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$
- $\tan(a \pm b) = \frac{\tan(a) \pm \tan(b)}{a \mp \tan(a)\tan(b)}, \quad \tan^2(\frac{a}{2}) = \frac{1-\cos(a)}{\sin(a)} = \frac{\sin(a)}{1+\cos(a)}$
- $\sin(a)\sin(b) = \frac{1}{2}(\cos(a+b) - \cos(a-b))$
- $\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$
- $\sin(2a) = 2\sin(a)\cos(a)$
- $\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$
- $\sin(3a) = 3\sin(a) - 4\sin^3 a, \quad \cos(3a) = 4\cos^3(a) - 3\cos(a)$
- $\tan(3a) = \frac{3\tan(a) - \tan^3(a)}{1 - 3\tan^2(a)}$
- $\sin^2(\frac{a}{2}) = \frac{1-\cos(a)}{2}, \quad \cos^2(\frac{a}{2}) = \frac{1+\cos(a)}{2}$

4.3.3 Weiteres

- $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ und $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$
- $\sin(n\pi) = 0$ und $\cos(n\pi) = (-1)^n$
- $\sin(\frac{n\pi}{2}) = \begin{cases} 1\,, n = 2k+1 \text{ (ungerade)} \\ -1\,, n = 2k+3 \text{ (ungerade)} \\ 0\,, n = 2k \text{ (bzw.} n: \text{gerade)} \end{cases}$
- $\sin(\frac{3n\pi}{2})\sin(\frac{n\pi}{2}) = \begin{cases} -1\,, n \text{ gerade} \\ 0\,, n \text{ ungerade} \end{cases}$

4.4 Hyperbolische Funktionen

- $\cosh(x) = \frac{e^{x}+e^{-x}}{2}$ und $\sinh(x) = \frac{e^{x}-e^{-x}}{2}$
- $\text{arcosh}(x) = \ln(x+\sqrt{x^2-1})$ und $\text{arsinh}(x) = \ln(x+\sqrt{x^2+1})$
- $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$ und $\text{artanh}(x) = \frac{1}{2}\ln(\frac{1+x}{1-x})$

4.5 Wichtige Integrale

- $\int_{-L}^{+L} \cos(\frac{n\pi}{L}x)\cos(\frac{m\pi}{L}x)dx = \begin{cases} 0\,, n \neq m \\ L\,, n = m \neq 0 \\ 2L\,, n = m = 0 \end{cases}$
- $\int_{-L}^{+L} \sin(\frac{n\pi}{L}x)\sin(\frac{m\pi}{L}x)dx = \begin{cases} 0\,, n \neq m \\ L\,, n = m \neq 0 \\ 0\,, n = m = 0 \end{cases}$
- $\int_{-L}^{+L} \sin(\frac{n\pi}{L}x)\cos(\frac{m\pi}{L}x)dx = 0$
- $\int x\cos(nx)dx = \frac{\cos(nx)+nx\sin(nx)}{n^2} + C$
- $\int x^2\cos(nx)dx = \frac{(n^2x^2-2)\sin(nx)+2nx\cos(nx)}{n^3} + C$
- $\int x\sin(nx)dx = \frac{\sin(nx)-nx\cos(nx)}{n^2} + C$
- $\int x^2\sin(nx)dx = \frac{(2-n^2x^2)\cos(nx)+2nx\sin(nx)}{n^3} + C$
- $\int \frac{1}{1+x^2}dx = \arctan(x) + C$
- $\int_{-L}^{+L} e^{-\pi x}\cos(\omega x)dx = \frac{\pi}{\pi^2+\omega^2}$
- $\int_{-\infty}^{+\infty} f(x)u(t-a)dx = \int_{b}^{+\infty} f(x)$

5 D'alembert with ∞

$$\begin{cases} u_{tt} = u_{xx} \Rightarrow c = 1 \\ u(x,0) = f(x) = \begin{cases} \cos(x)\,, |x| \leqslant 2\pi \\ 0\,, |x| > 2\pi \end{cases} \\ u_t(x,0) = g(x) = \begin{cases} 0\,, |x| \leqslant 2\pi \\ e^{-x}\ |x| > 2\pi \end{cases} \end{cases}$$

5.1 Find $(0,\pi)$

- $u(x,t) = \frac{1}{2}(f(x+ct)+f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$ mit $c = 1$
- $u(0,\pi) = \frac{1}{2}(f(\pi)+f(-\pi)) + \frac{1}{2}\int_{-\pi}^{\pi} g(\tau)d\tau$ mit $x = 0 < 2\pi,\ t = \pi$
- $u(0,\pi) = \frac{1}{2}(\cos(\pi)+\cos(-\pi)) = -1$

5.2 Find $\lim_{a\to\infty} u(a,a)$

- $\lim_{a\to\infty} u(a,a) = \lim_{a\to\infty} \frac{1}{2}(f(2a)-f(0)) + \frac{1}{2}(\int_{0}^{2\pi} 0d\tau + \int_{2\pi}^{\infty} e^{-x})$

6 Special Case Wave Equation

Find the solution of following wave equation with homogeneous Neumann conditions (= the derivative u_x on the boundary is zero):

$$u = u(x,t) \text{ s.t. } \begin{cases} u_{tt} = c^2u_{xx} & x \in [0,\pi], t \geqslant 0 \\ u_x(0,t) = u_x(\pi,t) = 0 & t \geqslant 0 \\ u(x,0) = 1+\cos(4x) & x \in [0,\pi] \\ u_t(x,0) = 3 & x \in [0,\pi] \end{cases}$$

Use the method of separation of variables, showing all the steps.

1. $\begin{cases} F''(x) = kF(x) \\ \ddot{G}(t) = kc^2G(t) \end{cases}$
2. Nontrivial solutions $\Rightarrow$ the homogeneous Neumann conditions imply
$$\begin{cases} u_x(0,t) = F'(0)G(t) = 0 \Rightarrow F'(0) = 0 & t \geqslant 0 \\ u_x(\pi,t) = F'(\pi)G(t) = 0 \Rightarrow F'(\pi) = 0 & t \geqslant 0 \end{cases}$$
3. We have obtained a well-defined initial value problem for $F = F(x)$:
$$\begin{cases} F''(x) = kF(x) \\ F'(0) = F'(\pi) = 0 \end{cases}$$
4. $k > 0$:
    - $F(x) = Ae^{\sqrt{k}x} + Be^{\sqrt{k}x}$
    - $F'(x) = \sqrt{k}(Ae^{\sqrt{k}x} - Be^{\sqrt{k}x})$
    - $F'(0) = \sqrt{k}(A-B)$
    - $F'(\pi) = \sqrt{k}(Ae^{\sqrt{k}\pi} - Be^{-\sqrt{k}\pi})$
    - **No nontrivial solutions!**
5. $k = 0$:
    - $F(x) = Ax + B$
    - $F'(x) = B$
    - $F'(0) = B$
    - $F'(\pi) = B$
    - $F_0(x) = B$
6. $k < 0$:
    - $F(x) = A\cos(\sqrt{-k}x) + B\sin(\sqrt{-k}x)$
    - $F'(x) = \sqrt{k}(-A\sin(\sqrt{-k}x) + B\cos(\sqrt{-k}x))$
    - $F'(0) = \sqrt{-k}B$
    - $F'(\pi) = \sqrt{-k}(-A\sin(\sqrt{-k}\pi) + B\cos(\sqrt{-k}\pi))$
    - for $k = -n^2 \Rightarrow F(x) = B_n\cos(nx)$
7. Solve $G(t)$ for the values $k = 0$ and $k = -n^2$
    - $k = 0 \Rightarrow G_0(t) = Ct + D$
    - $k = -n^2 \Rightarrow G_n(t) = C_n\cos(cnt) + D_n\sin(cnt)$
8. With superposition we get
$$u(x,t) = F_0(x)G_0(t) + \sum_{n=1}^{\infty} F_n(x)G_n(t)$$
9. Renaming the constants we get
$$u(x,t) = A_0t + B_0 + \sum_{n=1}^{\infty}(A_n\cos(cnt) + B_n\sin(cnt))\cos(nx)$$
10. The time derivative is
$$u_t(x,t) = A_0 + \sum_{n=1}^{\infty} cn(-A_n\sin(cnt) + B_n\cos(cnt))\cos(nx)$$
11. Now we have to find these constants by imposing the initial conditions
$$\begin{cases} u(x,0) = 1+\cos(4x) = B_0 + \sum_{n=1}^{\infty} A_n\cos nx & B_0 = 1,\ A_4 = 1 \\ u_t(x,0) = 3 = A_0 + \sum_{n=1}^{\infty} cnB_n\cos(nx) & A_0 = 3 \end{cases}$$
12. The final solution is
$$u(x,t) = 3t + 1 + \cos(4ct)\cos(4x)$$

7 Laplace Equation and Maximum Principle

Consider the solution of the following Laplace problem on a disk D_R centered in the origin, of radius $R > 0$:

$$u = u(x,y) \text{ s.t } = \begin{cases} \nabla^2 u = 0 & \text{in } D_R \\ u = \frac{e^R}{2R^2}(x^2-y^2) & \text{in } \partial D_R \end{cases}$$

Find the unique $R > 0$ such that the maximum of u on the disk is π:

$$\max_{(x,y)\in D_R} u(x,y) = \pi$$

1. Find the maximum of $x^2 - y^2$ in the region $x^2 + y^2 = R^2$
2. $x^2 - y^2 = R^2(\cos^2(\varphi) - \sin^2(\varphi)) = R^2\cos(2\varphi)$
3. The maximum is $R^2 \Rightarrow R^2 = x^2 - y^2$
4. In $u_{\max} = \pi \Rightarrow \frac{e^R}{2R^2}R^2 = \pi$
5. $e^R = 2\pi \Rightarrow R = \ln(2\pi)$

8 Heat Equation via Fourier Transform

Remember that the solution of the heat equation

$$\begin{cases} u_t = c^2c_{xx} & x \in \mathbb{R}, t \geqslant 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \end{cases}$$

has Fourier transform

$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-c^2\omega^2 t}$$

For some particular cases of f, this $\hat{u}$ can be recognised as the Fourier transform of some function, and the original solution $u = u(x,t)$ can be found. Find the solution $u = u(x,t)$ of the following:

$$\begin{cases} u_t = c^2u_{xx} & x \in \mathbb{R}, t \geqslant 0 \\ u(x,0) = e^{-\frac{1}{2}x^2} & x \in \mathbb{R} \end{cases}$$

1. $\widehat{e^{-ax^2}} = \frac{1}{\sqrt{2a}}e^{-\frac{\omega^2}{4a}}$ in this case $a = \frac{1}{2}$
2. $\hat{u}(x,t) = e^{-\frac{\omega^2}{2}}e^{-c^2\omega^2 t} = e^{-\omega^2(\frac{1}{2}+c^2t)}$ (still a gaussian)
3. For some $a > 0$ we want
$$\frac{1}{2} + c^2t = \frac{1}{4a} \Rightarrow a = \frac{1}{2+4c^2t}$$
4. $u(x,t) = \mathcal{F}^{-1}(\hat{u}(x,t)) = \mathcal{F}^{-1}\left(e^{-\omega^2(\frac{1}{2}+c^2t)}\right)$
$$= \sqrt{2a}\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2a}}e^{-\frac{\omega^2}{4a}}\right) = \sqrt{2a}e^{-ax^2} = \sqrt{\frac{1}{1+2c^2t}}e^{-\frac{x^2}{2+4c^2t}}$$

9 PDE with Fourier Transform

Find the solution $u = u(x,t)$ of the following equation using the Fourier transform:

$$\begin{cases} u_x + u_t + u = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \end{cases}$$

1. Take the Fourier transform with respect to the x variable of the PDE and the initial condition and transform them into an ODE.
2. Solve the ODE
3. Take the inverse Fourier transform of the solution of the ODE to find the solution of the PDE

1. **Transform into ODE**
    a) $\mathcal{F}(u_x + u_t + u) = \mathcal{F}(u_x) + \mathcal{F}(u_t) + \mathcal{F}(u) = 0$
    b) $\mathcal{F}(u_x) = i\omega\hat{u}(\omega,t),\quad \mathcal{F}(u_t) = \frac{\partial}{\partial t}\hat{u}(\omega,t),\quad \mathcal{F}(u) = \hat{u}(\omega,t)$
    c) $\frac{\partial}{\partial t}\hat{u}(\omega,t) = \hat{u}(\omega,t)(-i\omega - 1)$
2. **Solve the ODE:**
    a) $\hat{u}(\omega,t) = A(\omega)e^{-t(i\omega+1)}$
    b) $\hat{u}(\omega,0) = \hat{f}(\omega) \Rightarrow A(\omega) = \hat{f}(\omega)$

c) $\hat{u}(\omega,t) = \hat{f}(\omega)e^{-t(i\omega+1)} = e^{-t}\hat{\omega}e^{-i\omega t}$

3. **Inverse Fourier Transform**

a) $\mathcal{F}^{-1}(\hat{u}(\omega,t)) = u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega,t)e^{i\omega x}d\omega$

b) $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t}\hat{f}(\omega)e^{i\omega x}e^{-i\omega t}d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t}\hat{f}(\omega)e^{i\omega(x-t)}d\omega$

c) $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mathcal{F}^{-1}(e^{-t}f))e^{i\omega(x-t)}d\omega = \mathcal{F}^{-1}(\mathcal{F}(e^{-t}f))(x-t)$

d) $= e^{-t}f(x-t)$

4. $\boxed{u(x,t) = e^{-t}f(x-t)}$

10 Fourier Series 1

Compute the real Fourier series of the function $f(x) = \sin(\frac{5\pi x}{L}) + \cos(\frac{4\pi x}{L}) + |x|$ on the interval $[-L, L]$. Where $|x|$ is the absolute value of x.

1. The first two terms have already the form of a Fourier series, so we don't need to compute their coefficients.
2. The function $|x|$ is **even** $\Rightarrow b_n = 0$
3. $|x| = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x)$
4. $a_0 = \frac{1}{L}\int_0^L |x|dx = \frac{1}{2}L$
5. $a_n = \frac{2}{L}\int_0^l x\cos(\frac{n\pi}{L}x)dx = \frac{2L}{n^2\pi^2}((-1)^n - 1)$
6. $a_n = \begin{cases} 0 & n = 2j \\ -\frac{4L}{n^2\pi^2} & n = 2j+1 \end{cases}$
7. $f(x) = \sin(\frac{5\pi x}{L}) + \cos(\frac{4\pi x}{L}) + \frac{L}{2} - \sum_{j=0}^{\infty} \frac{4L}{(2j+1)^2\pi^2}\cos(\frac{(2j+1)\pi}{L}x)$

11 Fourier Series 2

Solve the following integral equation using the Fourier transform

$$f(x) = \sqrt{2\pi}g(x) + \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

where

$$g(y) = \begin{cases} e^{-2y}, & y \geqslant 0 \\ 0, & y < 0 \end{cases}$$

1. Fourier on both sides and bring $\hat{f}(\omega)$ omn one side
   a) $\mathcal{F}(f(x)) = \hat{f}(\omega)$ and $\mathcal{F}(g(x)) = \hat{g}(\omega)$
   b) $\mathcal{F}\left(\int_{-\infty}^{\infty} f(x-y)g(y)dy\right) = \mathcal{F}(f*g) = \sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$
   c) $\hat{f}(\omega) - \sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega) = \sqrt{2\pi}\hat{g}(\omega) = \hat{f}(\omega)(1-\sqrt{2\pi}\hat{g}(\omega))$
2. Solve for $\hat{f}(\omega)$
$$\hat{f}(\omega) = \frac{\sqrt{2\pi}\hat{g}(\omega)}{1-\sqrt{2\pi}\hat{g}(\omega)}$$
3. Compute $\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}(2+i\omega)}$ and calculate $\hat{f}(\omega)$
$$\hat{f}(\omega) = \frac{\sqrt{2\pi}\frac{1}{\sqrt{2\pi}(2+i\omega)}}{1-\sqrt{2\pi}\frac{1}{\sqrt{2\pi}(2+i\omega)}} = \frac{\frac{1}{2+i\omega}}{\frac{2+i\omega-1}{2+i\omega}} = \frac{1}{1+i\omega}$$
4. Compute $f(x)$
$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega \text{ with } \hat{f}(\omega) = \frac{1}{1+\omega}$$

12 Fourier Series 3

The Fourier series of the $2L$-periodic extension of the function

$$f(x) = x \quad x \in [-L, L)$$

is given by

$$f(x) = x = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2L}{\pi n}\sin(\frac{n\pi}{L}x)$$

Compute the value of the following numerical series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

1. We chose $x_0 = \frac{L}{2} \Rightarrow \sin(\frac{n\pi}{2}) = \begin{cases} 0 & n = 2k \\ (-1)^k & n = 2k+1 \end{cases}$
2. For the whole equation we get
$$x_0 = \frac{L}{2} = \sum_{k=0}^{\infty} (-1)^{2k+2}\frac{2L}{\pi(2k+1)}(-1)^k = \frac{2L}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$
3. Solve for $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} \Rightarrow \frac{\pi}{4}$

13 Heat equation with $c(t)$

Consider the following time-dependent version of the Heat equation on the interval $[0, L]$. We also impose boundary conditions and we look for a solution $u = u(x,t)$ such that

$$\begin{cases} u_t = (1+t)u_{xx} & x \in [0,L], t \in [0,\infty) \\ u(0,t) = u(L,t) = 0 & t \in [0\infty) \\ u(x,0) = f(x) & x \in [0,L] \end{cases}$$

where f is a given function. Assume that the Fourier series of the $2L$ periodic odd extension of f is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{\pi^2}{(8+n)^2}\sin(\frac{n\pi}{L}x)$$

Find the solution $u(x,t)$ using separation of variable.

1. **Ansatz:** $u(x,t) = F(x)G(t)$
2. **Einsetzen in** $u_t = (1+t)u_{xx}$
   a) $F(x)\dot{G}(t) = (1+t)F''(x)G(t)$
   b) $\frac{\dot{G}}{c^2G} = \frac{F''}{F} = k, \quad k \in \mathbb{R}, k = \text{const}$
3. **Separation:** $F'' = Fk$ und $\dot{G} = (1+t)Gk$
4. **Fallunterscheidung:**
   a) $k = 0$: $u(x,t) = 0 \Rightarrow$ **triviale Lösung**
   b) $k > 0$: $F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x} \Rightarrow$ **triviale Lösung**
   c) $k < 0$: Set $k = -p^2$
$$F(x) = A\sin(px) + B\cos(px)$$
$$G(t) = Ce^{-p^2\int(1+t)dt} = Ce^{-p^2(t+\frac{t^2}{2})}$$
5. **Randbedingungen** $u(0,t) = u(L,t) = 0$:
   a) $u(0,t) = F(0)G(t) = 0 \Rightarrow F(0) = 0 \Rightarrow B = 0$
   b) $u(L,t) = F(L)G(t) = 0 \Rightarrow F(L) = 0 \Rightarrow p = \frac{n\pi}{L}$
   c) $G(t) = Ce^{-(\frac{n\pi}{L})^2(t+\frac{t^2}{2})} = G_n$
6. **Zusammensetzen mit** $B_n = AC$:
$$u_n(x,t) = F_nG_n = B_n\sin(\frac{n\pi}{L}x)e^{-(\frac{n\pi}{L})^2(t+\frac{t^2}{2})}$$
7. **Insert the boundary conditions:**
$$u(x,0) = \sum_{n=1}^{\infty} B_n\sin(\frac{n\pi}{L}x) = \sum_{n=1}^{\infty}\frac{\pi^2}{(8+n)^2}\sin(\frac{n\pi}{L}x)$$
8. **Compare the coefficient:** $\Rightarrow B_n = \frac{\pi^2}{(8+n)^2}$
9. **The solution is:** $u(x,t) = \sum_{n=1}^{\infty}\frac{\pi^2}{((8+n)^2)}\sin(\frac{n\pi}{L}x)e^{-(\frac{n\pi}{L})^2(t+\frac{t^2}{2})}$

14 MC-Questions

- Find the Laplace transfrom of $f(x) = \begin{cases} k, & a \leqslant t \leqslant b \\ 0 & \text{otherwise} \end{cases}$
$$\Rightarrow \mathcal{L}\{f(x)\} = k\int_a^b e^{-st}dt = \frac{k}{s}(e^{-sa} - e^{-sb})$$
- Find the Laplace transfrom of $f(x) = \begin{cases} t, & 0 \leqslant t \leqslant 1 \\ t-2 & 1 \leqslant t \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$
$$\Rightarrow \mathcal{L}\{f(x)\} = \int_0^1 te^{-st}dt + \int_1^2 (t-2)e^{-st}dt = \frac{(1-e^{-s})^2}{s^2}$$
- TO BE DONE!