1 Wellengleichung

$$u_{tt} = c^2 u_{xx}$$

Separationsansatz: $u(x,t) = X(x)T(t) \Rightarrow \frac{X''}{X} = \frac{\ddot{T}}{Tc^2} = \overset{k}{\cancel{\alpha}}$

1.1 $\overset{K}{\cancel{\alpha}} = 0$

1.1.1 $\alpha = 0 \Rightarrow X'' = 0$

$$X(x) = Ax + B$$

$$X'(x) = A$$

1.1.2 $\alpha = 0 \Rightarrow \ddot{T} = 0$

$$T(t) = Ct + D$$

$$\dot{T}(t) = C$$

1.2 $\overset{K}{\cancel{\alpha}} > 0$

1.2.1 $\alpha > 0 \Rightarrow X'' - \alpha X = 0$

$$X(x) = Ae^{\sqrt{\alpha}x} + Be^{-\sqrt{\alpha}x}$$

$$X'(x) = \sqrt{\alpha}Ae^{\sqrt{\alpha}x} - \sqrt{\alpha}Be^{-\sqrt{\alpha}x}$$

1.2.2 $\alpha > 0 \Rightarrow \ddot{T} - \alpha c^2 T = 0$

$$T(t) = Ce^{c\sqrt{\alpha}t} + De^{-c\sqrt{\alpha}t}$$

$$\dot{T}(t) = c\sqrt{\alpha}Ce^{c\sqrt{\alpha}t} - c\sqrt{c\alpha}De^{-c\sqrt{\alpha}t}$$

1.3 $\overset{K}{\cancel{\alpha}} < 0$ def: $K = -\alpha$

1.3.1 $\alpha < 0 \Rightarrow X'' + \alpha X = 0$

$$X(x) = A\sin(\sqrt{\alpha}x) + B\cos(\sqrt{\alpha}x)$$

$$X'(x) = \sqrt{\alpha}A\cos(\sqrt{\alpha}x) - \sqrt{\alpha}B\sin(\sqrt{\alpha}x)$$

1.3.2 $\alpha < 0 \Rightarrow \ddot{T} + \alpha c^2 T = 0$

$$T(t) = C\sin(\sqrt{\alpha}ct) + D\cos(\sqrt{\alpha}ct)$$

$$\dot{T}(t) = \sqrt{\alpha}cC\cos(\sqrt{\alpha}ct) - \sqrt{\alpha}cD\sin(\sqrt{\alpha}ct)$$

2 Wärmeleitgleichung

$$u_t = c^2 u_{xx}$$

Separationsansatz: $u(x,t) = X(x)T(t) \Rightarrow \frac{X''}{X} = \frac{\dot{T}}{Tc^2} = \cancel{\alpha}K$

2.1 $\overset{K}{\cancel{\alpha}} = 0$

2.1.1 $\alpha = 0 \Rightarrow X'' = 0$

$$X(x) = Ax + B$$

$$X'(x) = A$$

2.1.2 $\alpha = 0 \Rightarrow \dot{T} = 0$

$$T(t) = C$$

$$\dot{T}(t) = 0$$

2.2 $\overset{K}{\cancel{\alpha}} > 0$

2.2.1 $\alpha > 0 \Rightarrow X'' - \alpha X = 0$

$$X(x) = Ae^{\sqrt{\alpha}x} + Be^{-\sqrt{\alpha}x}$$

$$X'(x) = \sqrt{\alpha}Ae^{\sqrt{\alpha}x} - \sqrt{\alpha}Be^{-\sqrt{\alpha}x}$$

2.2.2 $\alpha > 0 \Rightarrow \dot{T} - \alpha c^2 T = 0$

$$T(t) = Ce^{c^2\alpha t}$$

$$\dot{T}(t) = Cc^2\alpha e^{c^2\alpha t}$$

2.3 $\overset{K}{\cancel{\alpha}} < 0$ def. $K = -\alpha$

2.3.1 $\alpha < 0 \Rightarrow X'' + \alpha X = 0$

$$X(x) = A\sin(\sqrt{\alpha}x) + B\cos(\sqrt{\alpha}x)$$

$$X'(x) = \sqrt{\alpha}A\cos(\sqrt{\alpha}x) - \sqrt{\alpha}B\sin(\sqrt{\alpha}x)$$

2.3.2 $\alpha < 0 \Rightarrow \dot{T} + \alpha c^2 T = 0$

$$T(t) = Ce^{-c^2\alpha t}$$

$$\dot{T}(t) = -Cc^2\alpha e^{-c^2\alpha t}$$

3 Laplace-Gleichung

$$\nabla^2 u = \Delta u = u_{x_1x_1} + u_{x_2x_2} + \ldots$$

Ansatz: $u(x,t) = X(x)T(t) \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \alpha$

3.1 $\alpha = 0$

3.1.1 $\alpha = 0 \Rightarrow X'' = 0$

$$X(x) = Ax + B$$

$$X'(x) = A$$

3.1.2 $\alpha = 0 \Rightarrow Y'' = 0$

$$Y(y) = Cy + D$$

$$Y'(y) = C$$

3.2 $\alpha > 0$

3.2.1 $\alpha > 0 \Rightarrow X'' - \alpha X = 0$

$$X(x) = Ae^{\sqrt{\alpha}x} + Be^{-\sqrt{\alpha}x}$$

$$X'(x) = \sqrt{\alpha}Ae^{\sqrt{\alpha}x} - \sqrt{\alpha}Be^{-\sqrt{\alpha}x}$$

3.2.2 $\alpha > 0 \Rightarrow Y'' + \alpha Y = 0$

$$Y(y) = A\sin(\sqrt{\alpha}y) + B\cos(\sqrt{\alpha}y)$$

$$Y'(y) = \sqrt{\alpha}A\cos(\sqrt{\alpha}y) - \sqrt{\alpha}B\sin(\sqrt{\alpha}y)$$

3.3 $\alpha < 0$

3.3.1 $\alpha < 0 \Rightarrow X'' + \alpha X = 0$

$$X(x) = A\sin(\sqrt{\alpha}x) + B\cos(\sqrt{\alpha}x)$$

$$X'(x) = \sqrt{\alpha}A\cos(\sqrt{\alpha}x) - \sqrt{\alpha}B\sin(\sqrt{\alpha}x)$$

3.3.2 $\alpha < 0 \Rightarrow Y'' - \alpha Y = 0$

$$Y(y) = Ae^{\sqrt{\alpha}y} + Be^{-\sqrt{\alpha}y}$$

$$Y'(y) = \sqrt{\alpha}Ae^{\sqrt{\alpha}y} - \sqrt{\alpha}Be^{-\sqrt{\alpha}y}$$

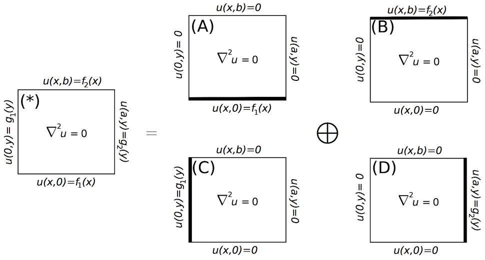
3.3.3 Anmerkung:

$E_1 e^{\sqrt{\alpha}y} - E_1 e^{-\sqrt{\alpha}y} = E_2 \sinh(\sqrt{\alpha}y)$
$E_1 e^{\sqrt{\alpha}y} + E_1 e^{-\sqrt{\alpha}y} = E_2 \cosh(\sqrt{\alpha}y)$

3.4 Allgemeine Lösung der PDE

$u(x,y) = [C\cosh(kx) + D\sinh(kx)][A\cos(ky) + B\sin(ky)]$

3.5 Superposition eines Dirichlet Problem



Lösung für A:

$$u_1(x,y) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{a})\sinh(\frac{n\pi(b-y)}{a})$$

$$A_n = \frac{2}{a\sinh(\frac{n\pi b}{a})}\int_0^a f_1(x)\sin(\frac{n\pi x}{a})\mathrm{d}x$$

Lösung für B:

$$u_2(x,y) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{a})\sinh(\frac{n\pi y}{a})$$

$$B_n = \frac{2}{a\sinh(\frac{n\pi b}{a})}\int_0^a f_2(x)\sin(\frac{n\pi x}{a})\mathrm{d}x$$

Lösung für C:

$$u_3(x,y) = \sum_{n=1}^{\infty} C_n \sinh(\frac{n\pi(a-x)}{a})\sin(\frac{n\pi y}{b})$$

$$C_n = \frac{2}{b\sinh(\frac{n\pi a}{b})}\int_0^b g_1(y)\sin(\frac{n\pi y}{b})\mathrm{d}y$$

Lösung für D:

$$u_4(x,y) = \sum_{n=1}^{\infty} D_n \sinh(\frac{n\pi x}{b})\sin(\frac{n\pi y}{b})$$

$$D_n = \frac{2}{b\sinh(\frac{n\pi a}{b})}\int_0^b g_2(y)\sin(\frac{n\pi y}{b})\mathrm{d}y$$

Lösung für A+B+C+D=(*):

$$u = u_1 + u_2 + u_3 + u_4$$