Analysis 3 Summary

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1 Laplace Transforms

1.1 Definition

The Laplace transform is given by:

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) dt \quad (1)$$

1.2 Properties of the Laplace transform

- **Linearity**

$$L\{af(t) + bg(t)\} = aF(s) + bG(s)$$

- **t-Shifting**

$$L\{f(t-a)\} = e^{-as}F(s)$$

- **s-Shifting**

$$L\{e^{at} \cdot f(t)\} = F(s-a) = L\{f(t)\}(s-a)$$

- **Time Scaling**

$$L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$$

- **Differentiation**

$$L\{t \cdot f(t)\} = -\frac{d}{ds}[F(s)]$$

- **Integration**

$$\int_0^\infty f(t)dt = \frac{1}{s}F(s)$$

1.2.1 example

Determine Laplace transform of $3\sin(2t) + 6t^4$

$$\mathcal{L}\{3\sin(2t) + 6t^4\} = 3\mathcal{L}\{\sin(2t)\} + 6\mathcal{L}\{t^4\}$$

We separate it using **linearity**

$$\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}$$

$$\mathcal{L}\{t^4\} = \frac{4!}{s^5} = \frac{24}{s^5}$$

We substitute it back into the original expression:

$$\mathcal{L}\{3\sin(2t) + 6t^4\} = 3 \cdot \frac{2}{s^2+4} + 6 \cdot \frac{24}{s^5}$$

1.3 Inverse Laplace Transform

There is no direct formula to compute the inverse of the Laplace transformation, however we can express any transformed function in a simpler way such that we can recognize simpler LTs such as the LT of t^n and e^{at} and also take advantage of the *Frequency Shifting property*

1.3.1 example

Determine the inverse LT of $F(s) = \frac{7}{s^2+s-6}$

To express such function in a simpler way we can use **partial fraction decomposition**

$$\rightarrow F(s) = \frac{-7}{5*(s+3)} + \frac{7}{5*(s-2)}$$

The inverse LT of the two fractions can now be simply determined by the formula

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

1.4 Heaviside Function

The Heaviside function, denoted as $H(t)$, is defined as:

$$H(t-a) = \begin{cases} 0, & \text{for } t < a, \\ 1, & \text{for } t \geq a. \end{cases}$$

The Laplace transform of the Heaviside function is given by:

$$\mathcal{L}\{H(t-a)\} = \int_a^\infty e^{-st}\frac{e^{-as}}{s}\,dt.$$

1.5 Dirac Delta Function

The Dirac delta function, denoted as $\delta(t)$, is defined as:

$$\delta(t) = \begin{cases} \frac{1}{\epsilon}, & \text{if } t \in (a, a+\epsilon) \\ 0, & \text{otherwise} \end{cases}$$

In case $\epsilon \to \infty$, we have:

$$\delta(t) = \lim_{\epsilon\to\infty} \begin{cases} \infty, & \text{if } t = a \\ 0, & \text{if } t \neq a \end{cases}$$

and

$$\int_0^\infty \delta(t)dt = 1$$

1.5.1 properties

When multiplying the dirac delta function with any other function we obtain:

$$\int_0^\infty f(t)\delta(t)\,dt = f(a)$$

This is called **Shifting property**

1.6 Solving DE with LT

- **Step 1:** Use the *differentiation property* to turn y, y', y'', ..., $y^{(n)}$ into algebraic terms.
  Reminder:
  - $\mathcal{L}\{y\} = Y(s)$
  - $\mathcal{L}\{y'\} = sY(s) - y(0)$
  - $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$
- **Step 2:** If your ODE is non-homogeneus, apply the LT to the non homogeneous term
- **Step 3:** Solve your newly found algebraic equation
- **Step 4:** Find the *Inverse Laplace Transform*

Honorable Mention:
if the initial conditions are not given with argument 0, rather with any other number (e.g. $y(a) = \ldots$ where $a \neq 0$), you need to perform a **step 1.5** where you do the substitution $\eta = t - a \Rightarrow t = \eta + a$. You then define a function $u(\eta) = y(\eta + a)$ and continue the usual steps to find $u(\eta)$

1.6.1 example

Solve the following system of ODEs using the LT:

$$\begin{cases} 2y'' + 2y' - 4y = 0 \\ y(0) = 1 \\ y'(0) = 5 \end{cases}$$

We apply the LT to y and its derivatives and substitute them in the equation

$$\begin{cases} 2s^2Y(s) + 2sY(s) - 4Y(s) = 0 \\ Y(0) = 1 \\ sY(0) = 6 \end{cases}$$

We can then solve this equation alrebraically

1.7 Important Laplace transforms

$u(t)$	$U(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(\omega t)$	$\omega/(s^2+\omega^2)$
$\cos(\omega t)$	$s/(s^2+\omega^2)$
$t\sin(\omega t)$	$2\omega s/(s^2+\omega^2)^2$

2 Fourier Analysis

2.1 Periodicity of Functions

2.1.1 Definition

A function $f(x)$ is periodic if:

1. f is defined for sufficiently many $x \in \mathbb{R}$, and
2. there exists $p \in \mathbb{R}$, $p > 0$, such that $f(x) = f(x+p)$ holds true. p is then referred to as the period.

2.1.2 Properties

Periodic functions satisfy the following Properties:

1. If $f(x)$ is p-periodic, then $f'(x)$ is also periodic.
2. If $f(x)$ is p_1-periodic and $g(x)$ is p_2-periodic, then $f(x) + g(x)$ is p-periodic, where p is the least common multiple (LCM) of p_1 and p_2.
3. $f(ax)$ is p/a-periodic.
4. $a \cdot f(x) + b \cdot h(x)$ is p-periodic.
5. If $f(x)$ and $h(x)$ are both n-periodic, then $f(x) \cdot h(x)$ is also n-periodic.

Tricks: When determining the periodicity of some funny function, you can compute its limit for $x \to \infty$
If it isn't bounded, it isn't periodic.

2.2 Even/Odd functions

$$f(x) = \begin{cases} \text{even}, & \text{if } f(x) = f(-x) \\ \text{odd}, & \text{iff}(-x) = -f(x) \end{cases}$$

2.2.1 multiplication

$\times$	**Even**	**Odd**
Even	Even	Odd
Odd	Odd	Even

2.3 Fourier Series

2.3.1 goal

The goal with Fourier series is to express any function as sum (linear combination) of sine and cosine functions.

2.3.2 Definition

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right)\right]$$

$$a_0 = \frac{1}{T}\int_a^b f(x)\,dx$$

$$a_n = \frac{2}{T}\int_a^b f(x)\cos\left(\frac{2\pi nx}{T}\right)\,dx$$

$$b_n = \frac{2}{T}\int_a^b f(x)\sin\left(\frac{2\pi nx}{T}\right)\,dx$$

2.3.3 Computation

To compute the Fourier Series it is often convenient to make use of the *orthogonality of trigonometric functions*

$$\int_0^{2\pi}\cos(mx)\cos(nx)\,dx = \begin{cases} \pi & \text{if } m = n \neq 0, \\ 0 & \text{if } m \neq n. \end{cases}$$

$$\int_0^{2\pi}\sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } m = n \neq 0, \\ 0 & \text{if } m \neq n. \end{cases}$$

$$\int_0^{2\pi}\cos(mx)\sin(nx)\,dx = 0$$

2.3.4 Fourier Series for even/odd functions

$$f(x) = a_0 + \sum_{n=1}^{\infty} 2a_n \cos(nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$$

2.3.5 Complex Fourier Series

A way of expressing the Fourier Series using complex numbers

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$
$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n x}{T}}$$

$$c_n = \frac{1}{T}\int_a^b f(x)e^{-i\frac{2\pi n x}{T}}\,dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} |c_n|\, e^{i\left(\frac{2\pi n x}{T} + \arg(c_n)\right)}$$

2.4 Minimum Square Error

$$E_N(f) = \int_{-\pi}^{\pi} |f(x) - P_N(x)|^2 dx$$
$$P_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}[a_n\cos(nx) + b_n\sin(nx)]$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

2.5 Fourier Integral

$$f(x) = \int_0^{\infty}[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]d\omega,$$
$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)dv,$$
$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)dv,$$

where for even (Gerade) functions:

$$B(\omega) = 0,\ A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos(\omega v)dv,$$

and for odd (Ungerade) functions:

$$A(\omega) = 0,\ B(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\sin(\omega v)dv.$$

Existence:
The integral exists if $\int_{-\infty}^{\infty} |f(x)|dx < \infty$

2.6 Fourier Transformation

2.6.1 Definition

If function f is absolutely integrable, then the Fourier Transformation of f:

$$\hat{f} = F(f)(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

2.6.2 Properties

1. **Linearity:** The Fourier transform of a linear combination of functions is the same linear combination of their Fourier transforms. Mathematically, this can be written as: $F(af + bg) = aF(f) + bF(g)$, where a and b are constants, and f and g are functions.
2. **Time and Frequency Scaling:** The Fourier transform has the property that scaling a function in the time domain corresponds to an inverse scaling in the frequency domain. This can be written as: $F(f(at)) = \frac{1}{|a|}F(\omega/a)$.
3. **Time and Frequency Shifting:** The Fourier transform of a function that is shifted in time is a phase-shifted version of the Fourier transform of the original function. This can be written as: $F(f(t-a)) = e^{-i\omega a}F(\omega)$.

3 Partial Differential Equations (PDEs)

3.1 Definition

A partial differential equation (PDE) is an equation in which a function u and some partial derivatives of u are involved.

- **Linear:** They are linear if both u and the partial derivatives appear with degree 1.
- **Homogeneous:** They are homogeneous if they are linear and if each term contains either u or a partial derivative.
- **Order:** The order of a PDE is the maximum order among all the involved derivatives.

3.2 Important PDEs

- One-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

(linear, 2nd order, homogeneous, hyperbolic)

- One-dimensional heat equation:

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

(linear, 2nd order, homogeneous, parabolic)

- Two-dimensional Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(linear, 2nd order, homogeneous, elliptic)

- Two-dimensional Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

(linear, 2nd order, inhomogeneous, elliptic)

- Two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(linear, 2nd order, homogeneous, hyperbolic)

- Two-dimensional heat equation:

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(linear, 2nd order, homogeneous, parabolic)

- Three-dimensional Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

(linear, 2nd order, homogeneous, elliptic)

3.2.1 Second Order Linear PDEs

A linear second-order PDE can be expressed in the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

And is classified as

- Hyperbolic if $AC - B^2 < 0$
- Parabolic if $AC - B^2 = 0$
- Elliptic if $AC - B^2 > 0$

3.3 Solving the wave equation

Consider a one-dimensional wave equation of the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions for $x \in [0, L]$:

$$u(0, t) = u(L, t) = 0$$

and initial conditions:

$$u(x, 0) = f(x)\frac{\partial u}{\partial t}(x, 0) = g(x)$$

We can apply *separation of variables*:

$$u(t, x) = G(t)F(x)$$

And compute the derivatives of $u(t, x)$:

$$u_{xx} = F(x)''G(t)$$
$$u_{tt} = F(x)G''(t)$$

We can now insert them in the original equation:

$$F(x)G''(t) = c^2F''(x)G(t)$$

By reformulating and adding a *separation constant k* we get:

$$\frac{G''}{c^2G} = \frac{F''}{F} = k$$

We can therefore extrapolate two ODEs:

$$\begin{cases} F'' = kF \\ G'' = c^2kG \end{cases}$$

From the initial conditions previously defined we get:

$$\begin{cases} F(0) = 0 \\ F(L) = 0 \end{cases}$$

From here we need to solve our ODEs system in various instances of k:

- $k = 0$

$$\begin{cases} F'' = 0 \\ G'' = 0 \end{cases}$$

By integrating twice we get

$$F(x) = ax + b$$

By inserting the initial conditions we get

$$a = b = 0$$

- $k > 0$

We ensure that $k > 0$ by defining a new constant a such that $k = a^2$

We can now rewrite our system of ODEs:

$$F'' - a^2 F = 0$$

And solve it as we already know:

$$F(x) = Ae^{\sqrt{a^2}x} + Be^{-\sqrt{a^2}x}$$

By inserting the initial condition we discover that the equation holds only if $A = 0$ and therefore $F(x) = 0$ and $u(x,t) = 0$

- $k < 0$

Similar to the previous case, we define a new constant a such that $k = -a^2$

We can now rewrite our system of ODEs:

$$F'' + a^2 F = 0$$

And solve it as we already know:

$$F(x) = A\cos\left(\sqrt{a^2}x\right) + B\sin\left(\sqrt{a^2}x\right)$$

Inserting the initial conditions yields:

$$F(0) = A = 0$$

$$F(L) = B\sin\left(\sqrt{a^2}\right)L$$

This holds either for $B = 0$ or $B \neq 0$. This last option finally gives us the following non-trivial solution, for which:

$$\sin\left(\sqrt{a^2}x\right) \overset{!}{=} 0$$

$$\Rightarrow a^2 L \overset{!}{=} n\pi$$

$$\Rightarrow a^2 = \left(\frac{n\pi}{L}\right)^2$$

We have therefore found the following solution:

$$F(x) = B\sin\left(\frac{n\pi}{L}x\right)$$

By inserting $k = a^2 = (\frac{n\pi}{L})^2$ in the second equation of our original system of ODEs we obtain:

$$G'' = c^2\left(\frac{n\pi}{L}\right)^2 G$$

By solving it as we already know we obtain

$$G_n = C_n\cos\left(\frac{cn\pi}{L}t\right) + D_n\sin\left(\frac{cn\pi}{L}t\right)$$

We now finally have solutions for $F(x)$ and $G(t)$. We can simply combine them to find a solution for $u(x,t)$:

$$u_n(x,t) = \left(B_n\cos(\lambda_n t) + \widetilde{B}_n\sin(\lambda_n t)\right)\sin\left(\frac{n\pi x}{L}\right)$$

where $\lambda_n = \frac{cn\pi}{L}$

Now, by recalling the *superposition property* for differential equations, which states that the addition of multiple solutions of a differential equation is again a solution of the same differential equation, we can write a new solution by summing over all $n \in \mathbb{N}^+$

The general solution can be expressed as:

$$u(x,t) = \sum_{n=1}^{\infty}\left(B_n\cos(nt) + \widetilde{B}_n\sin(nt)\right)\sin\left(\frac{n\pi x}{L}\right)$$

The coefficients are given by:

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\widetilde{B}_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

3.4 example

Solve the following 1-Dimentional wave equation using the Fourier Series:

$$\begin{cases} u_{tt} = c^2 u_{xx}, x \in [0,L],\ t \geq 0 \\ u(0,t) = u(L,t) = 0, t \geq 0 \\ u(x,0) = 0, 0 \leq x \leq L \\ u_t(x,0) = 0, 0 \leq x \leq L \end{cases}$$

We apply the usual formulas:

$$u(x,t) = \sum_{n=1}^{\infty}\left[B_n\cos(\lambda_n t) + B_n^*\sin(\lambda_n t)\right]$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\widetilde{B}_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\lambda_n = \frac{n\pi c}{L}$$

From the given initial condition we figure that:

$$f(x) = 0, B_n = 0 \text{and} g(x) = x$$

Inserting in the formulas above yields:

$$B_n^* = \frac{(-1)^{n+1}\cdot 2L^2}{c\pi^2 n^2}$$

3.4.1 D'Alembert's solution to the 1-D wave eq.

The solution by separation of variables only hold for the three boundry conditions. If we don't have boundary conditions (i.e. the rope is not fixed) we need to introduce a new solution called *D'Alembert's solution*:

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

3.4.2 example

Solve the following wave equation using D'Alembert's formula:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = e^{-x^2}\sin^2(x) + x \\ u_t(x,0) = xe^2 - x^2 \end{cases}$$

Using the formula defined in the previous subsection, we insert $u(x,0)$ as $f(x)$ and obtain:

$$\begin{aligned} u(x,t) =& \frac{1}{2}\big(e^{-(x+ct)^2}\sin^2(x+ct) + x + ct \\ &+ e^{-(x-ct)^2}\sin^2(x-ct) + x - ct \\ &+ \frac{1}{2c}\int_{x-ct}^{x+ct} se^{-s^2}\,ds\big) \end{aligned}$$

From here we can solve the integral and simplify further if needed

3.5 Fourier Series solution to the 1-D heat eq.

Consider a one-dimentional heat equation of the form:

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$$

Where c is the *thermal diffusivity* of the material, defined as

$$\alpha^2 = \frac{K}{\sigma\rho}$$

We assume the following boundry conditions:

$$\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$$

and the following initial condition:

$$u(x,0) = f(x)$$

In a similar fashion to how we solved the wave equation, we first apply *separation of variables*, then solve the system of ODEs and then formulate the solution using the Fourier series. This yields:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t}\sin\left(\frac{n\pi x}{L}\right)$$

The coefficient is given by:

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

3.5.1 example

Find the solution to the following time-dependant heat equation using separation of variables:

$$\begin{cases} u_t = t^3 u_{xx} \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = \sin\left(\frac{3\pi x}{L}\right) + 2\sin\left(\frac{\pi x}{L}\right) \end{cases}$$

We proceed in a fashion analogous to how we solved the Wave equation and therefore start by applying the usual Ansatz $u(x,t) = F(x)G(t)$ and obtain:

$$F(x)G'(t) = t^3 F''(x)G(t)$$

By rewriting and equating it to the separation constant k we obtain:

$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{t^3 G(t)} = k$$

Which yields the following system of ODEs:

$$\begin{cases} F'' = kF \\ G' = t^3 G \end{cases}$$

Now, by considering the usual boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ we understand that $F(0) = F(L) = 0$ since the only other scenario in which the boundary conditions are met is the trivial $G(t) = 0$ (boooooring)

From here we can solve our first ODE in various instances of k:

- $k > 0$

$$F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$$

Unfortunately the only scenario in which this solution satisfies the boundary conditions is if both C_1 and C_2 are equal to 0 which is very uninteresting.

- $k = 0$

$$F(x) = C_1 x + C_2$$

Unfortunately this is also not compatible with the boundry conditions outside of the trivial $C_1 = C_2 = 0$ case.

- $k < 0$

As we did when solving the wave equation, we ensure that k is negative by imposing $k = -p^2$ where p is an arbitrary real number. We therefore obtain the solution:

$$F(x) = A\cos(px) + B\sin(px)$$

By imposing the boundry conditions we obtain $p_L = n\pi$ for $n \in \mathbb{Z}_{\geq 1}$

We therefore have the following non-trivial soluions:

$$F_n(x) = B_n\sin\left(\frac{n\pi}{L}x\right)$$

We can now find possible solutions for G_n by inserting $k_n = -\frac{n^2\pi^2}{L^2}$ in the second equation of our system of ODEs, therefore obtaining the equation:

$$G' = -t^3\frac{n^2\pi^2}{L^2}G$$

Which has the following non-trivial solutions:

$$G_n(t) = C_n e^{-\frac{n^2\pi^2}{L^2}t^4}$$

We now finally have solutions for $F(x)$ and $G(t)$. We can simply combine them to find a solution for $u(x,t)$:

$$u_n(x,t) = A_n e^{-\frac{n^2\pi^2}{L^2}t^4} \sin\left(\frac{n\pi}{L}x\right)$$

By applying the superposition principle we obtain:

$$\sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right)$$

By imposing the initial condition $u(x,0) = \sin\left(\frac{3\pi x}{L}x\right) + 2\sin\left(\frac{\pi x}{L}x\right)$ we obtain:

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = \sin\left(\frac{3\pi x}{L}x\right) + 2\sin\left(\frac{\pi x}{L}x\right)$$

And conclute that:

$$A_n = \begin{cases} 2, & \text{if } n = 1, \\ 1, & \text{if } n = 3, \\ 0, & \text{otherwise} \end{cases}$$

3.5.2 Generalization to multiple dimentions

We can scale the heat equation to multiple dimentions using the *Laplace-Operator* ∇

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$

Reminder: The Laplace operator, denoted by ∇^2 or Δ is defined as the divergence of the gradient of a scalar field:

$$\nabla^2 u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

3.6 Time-Independant heat eq.

We often observe steady-state situations, where the temperature distribution reaches an equilibrium and is independent of time (i.e. $u \neq u(t)$). In this case $u_t = 0$, and the time-independent heat equation thus simplifies to the Laplace equation:

$$\Delta u = 0$$

3.6.1 Laplace equation

The laplace equation is, as already mentioned, defined as simplifies to the Laplace equation:

$$\Delta u = 0$$

For this equation we have the following boundary conditions:

- **Dirichlet Boundary Conditions**

$$u|_{\partial R} = g$$

- **Neumann Boundary Conditions**

  Also known as natural BCs. Here, a boundary condition prescribes how derivatives of the distribution look like along the border of the region. This case will (probably) be discussed at the very end of this course.

3.6.2 example

Determine the temperature distribution $u(x,y)$ on a thin, rectangular plate with thermal diffusivity c and and the dimensions $b \times c$

In this problem, ∂R is defined as

$$\partial R = \{(x,y) : 0 \leq x \leq a, 0 \leq y \leq b\}$$

Since we are looking for a time-independant function in two dimentions we can make use of the *Laplace equation*:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

We can now impose the *Dicherlet boundary conditions* by fixing the temperature distribution along the border of ∂R:

$$\begin{cases} u(0,y) = u(a,y) = u(x,0) = 0 \\ u(x,b) = f(x) \end{cases}$$

This problem can now be solved through the usual procedure of separation of variables, which yields:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right)$$

$$A_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)} \int_0^b f(x) \sin\left(\frac{n\pi}{a}x\right)\,dx$$

3.7 1D heat eq. on an infinite rod

The heat equation for an infinetely long rod is defined as:

$$\begin{cases} u_t &= c^2 u_{xx} \\ u(x,0) &= f(x) \end{cases}$$

We can solve this equation either with the Fourier integral or with the Fourier transform

3.7.1 Fourier Integral solution

Using separation of variable leads to:

$$u(x,t) = \int_{-\infty}^{\infty} [A(p)\cos(px) + B(p)\sin(px)]e^{-cp^2t}\,dp$$

Where

$$\begin{cases} A(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\cos(pv)\,dv \\ B(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\sin(pv)\,dv \end{cases}$$

Inserting the initial conditions yields:

$$u(x,t) = \int_{-\infty}^{\infty} [A(p)\cos(px) + B(p)\sin(px)]\,dp \stackrel{!}{=} f(x)$$

3.7.2 Fourier Transform solution

The solution to the 1D heat equation on an infinite rod using the Fourier transform is given by:

$$u(\omega,t) = \mathcal{F}[f(x)] \cdot e^{-c^2\omega^2 t}$$

3.8 Laplace Equation on a Region with Radial Symmetry

3.8.1 Fouier Series Solution

Consider the following Dicherlet problem on a disk D with radius R:

$$\begin{cases} \Delta u = 0, & \text{on } \mathbb{R}, \\ \dfrac{\partial u}{\partial n} = g, & \text{on } \partial\mathbb{R}, \end{cases}$$

Given the circular geometry, it's easier to work in polar coordinates:

$$\Delta u = u_{xx} + u_{yy} = u_{rr} + \frac{u_{\theta\theta}}{r^2} + \frac{u_r}{r} = \frac{1}{3}$$

$$\begin{cases} u \in D = \{(r,\theta) : 0 \leq r < R,\ 0 \leq \theta < 2\pi\}, \\ u \in \partial D = \{(R,\theta) : 0 \leq \theta < 2\pi\}. \end{cases}$$

We can now apply the usual method of separation of variables:

$$u(r,\theta) = F(r)G(\theta)$$

And obtain the following system of ODEs:

$$\begin{cases} r^2F'' + rF' = kF \\ G'' = -kG \end{cases}$$

And as usual obtain the following general solution:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n\cos(n\theta) + B_n\sin(n\theta))$$

Where

$$\begin{cases} A_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(\phi)\,d\phi, \\ A_n = \dfrac{1}{\pi R^n}\displaystyle\int_0^{2\pi} f(\phi)\cos(n\phi)\,d\phi, \\ B_n = \dfrac{1}{\pi R^n}\displaystyle\int_0^{2\pi} f(\phi)\sin(n\phi)\,d\phi. \end{cases}$$

3.8.2 Poisson Integral Form

By reformulating the formulas above for A_0, A_n and B_n we find the *Poisson Integral Form*:

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} K(r,\theta,R,\phi)f(\phi)\,d\phi$$

The *Poisson integral kernel* is given by:

$$K(r,\theta,R,\phi) = \frac{1-r^2}{1-2rR\cos(\theta-\phi)+R^2}$$

The Poisson integral form allows us to find the following important properties:

- **Mean Value Property**

  If we compute the value of u at the center of the disk, the Poisson integral kernel reduces to

$$K(r,\theta,R,\phi) = 1$$

  and therefore

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} u(R,\phi)d\phi$$

- **Maxiumum Principle**

  A function which satisfies the Laplace equation $\Delta u = 0$ on a region R is called harmonic on R. For any point (x_0, y_0) inside R, we can form a circle around this point with radius a, such that the resulting disk D_a is fully contained within R. Since u is harmonic on R, it must also satisfy the Laplace equation on D_a. Defining a local reference frame in polar coordinates with its origin at (x_0, y_0), we can express the solution u (within D_a) in this local reference frame through (13).

  The value $u(x_0, y_0)$ corresponds to $u(0,\theta)$ in the local reference frame. According to (15), we thus have

$$u(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} u(a,\phi)\,d\phi =$$

$$\frac{1}{2\pi}\int_0^{2\pi} u(x_0 + a\cos(\phi), y_0 + a\sin(\phi))\,d\phi$$

  This leads us to the following extended interpretation of the mean value property: The value of u at a point inside R is equal to the mean value of all values of u along the border of any circle within R around this point.

  As a consequence, the maximum of a harmonic function must be on the border of the region R on which it is harmonic, unless it is constant.

3.9 Well-posed and Ill-posed Problem

A *PDE is well-posed* if it satisfies all of the following conditions:

- **Existence:** The problem has a solution.
- **Uniqueness:** The problem has a solution.
- **Stability:** The solution depends continuously on the boundary and initial conditions

If at least one of these properties does not hold, the *PDE is Ill-posed*.

3.10 Neumann Problem

The *Neumann Problem* is a boundry value problem. It is defined as:

$$\begin{cases} \Delta u = 0, & \text{on } R, \\ \frac{\partial u}{\partial n} = g, & \text{on } \partial R, \end{cases}$$

The Neumann problem is not well-posed

4 ODE Review (Analysis II)

4.1 Introduction

- **Key Concepts: Ordinary Differential Equations (ODEs)**
  - *Ordinary Differential Equation (ODE):* $F(x, y(x), y'(x), ..., y^{(n)}(x)) = 0$
    $\Rightarrow$ "Entire function depends only on one variable"
  - *General Solution:* Set of all solutions to an ODE (Family of curves with n (= order) free parameters)
  - *Particular Solution:* Values assigned to free parameters through an Initial Value Problem (IVP).
  - *Singular Solution:* Cannot be found by substitution into the general solution.
  - *Order:* Highest derivative of the sought function.
  - *Linear:* Sought function and all its derivatives appear only linearly.
    (not allowed: $sin(y), y^2, e^y$, allowed: y', x^2, e^x)
  - *Homogeneous:* Each term contains the sought function or one of its derivatives. Otherwise, the ODE is *inhomogeneous* and has a *forcing term*.
  - *Regular:* Exactly one curve passes through every point.

4.2 General Solution of a First-Order ODE

- **First-Order ODE:** $y' = f(x, y)$
  The general solution is a one-parameter family of regular curves.
- **Theorem:** Let $f(x, y)$ be continuous and have continuous partial derivatives with respect to y. For every $(x_0, y_0) \in \mathbb{D}(f)$, the Initial Value Problem (IVP) $y' = f(x, y), y(x_0) = y_0$ has **exactly one** solution.

4.3 Separable Differential Equations

- **Separable DE** (can be transformed to:) $y' = \frac{g(x)}{h(y)}$

$$h(y) \cdot \frac{dy}{dx} = g(x)$$

$$\int h(y)\, dy = \int g(x)\, dx$$

- Common Substitutions:
  - $\frac{y(x)}{x} := u$ (replace y' and the right side)
  - $ax + by(x) + c = u(x)$
  - $y' = u$ (when only derivatives are present)

4.4 Linear Differential Equations

- **Linear:** The sought function and all its derivatives appear only linearly. Form: $y' + f(x) \cdot y = g(x)$
  (not allowed: $\sin(y), y^2, e^y$, allowed: y', x^2, e^x)
- **Linear DE:** (I) e.g., $y' + p(x) \cdot y = a(x)$
  with $a(x)$: **forcing term**
  (H) e.g., $y' + p(x) \cdot y = 0$, separable
- **Theorem:** Homogeneous linear DEs are separable.

4.4.1 Solving Inhomogeneous DEs:

- General solution of linear DE:

$$y = y_h + y_p$$

  - y_h using 4.3
- Variant 1: **Ansatz**
  - Choose an ansatz for y_p from a table
  - Substitute required **derivatives & ansatz** into DE
  - Coefficient comparison
  - $y = y_h + y_p$
  - If the ansatz doesn't work:
    Multiply y_p by x!

We consider a differential equation of the form $y' + c \cdot y = g(x)$

Forcing Term	Ansatz
Constant	$y_p = A$
Linear Function	$y_p = Ax + B$
Quadratic Function	$y_p = Ax^2 + Bx + C$
Polynomial of Degree n	$y_p = Ax^n + Bx^{n-1} + ... + Z$
$a \cdot \sin(\omega x - \varphi)$	
$a \cdot \cos(\omega x - \varphi)$	$y_p = A \cdot \sin(\omega x - \varphi)$
$a \cdot \sin(\omega x - \varphi)$	$+B \cdot \cos(\omega x - \varphi)$
$+b \cdot \cos(\omega x - \varphi)$	
$a \cdot e^{bx}$	$y_p = A \cdot e^{bx}$
$a \cdot x \cdot e^{bx}$	No suitable ansatz
$\frac{1}{x^n}$	$y_p = A \cdot \ln(x)$

- Variant 2: **Variation of Parameters** (Lagrange)
  - Use the computed y_h as the ansatz for y_p and assume that the arbitrary constant C can depend on $x \Rightarrow C(x)$
  - Substitute $y_p\prime$ and y_p into the DE
  - Solve for $C\prime(x)$
    $\Rightarrow$ all non-derivative terms involving $C(x)$ must vanish!
  - Integrate to obtain $C(x)$
  - Substitute $C(x)$ back into C in $y_h \Rightarrow y = y_h + y_p$
- Variant 3: **Direct Formula**
  - DE of the form $y' + p(x)y = q(x)$
  - Apply the formula
    $y(x) = e^{-P(x)} \int_0^x e^{P(\xi)} q(\xi) d\xi + y(0)$
    Where $P(x) = \int_c^x p(s)ds$

4.5 Higher-Order Differential Equations, General

- DE of nth order: $F(x, y, y', ..., y^{(n)}) = 0$
- **Existence Theorem:** The Initial Value Problem $y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(n-1)}(x_0) = y_{n-1}$ for the DE $y^{(n)} = f(x, y, ..., y^{(n-1)})$ has a unique solution (if f is continuous and $y^{\cdot\cdot}$ is continuously differentiable).

4.6 Linear Differential Equations with Constant Coefficients of nth Order

- **Linear DE of nth Order:**
  (I) e.g., $y^{(n)}(x) + ... + y' \cdot p_1(x) + y \cdot p_0(x) = q(x)$
  with $q(x)$: **forcing term**
  (H) e.g., $y^{(n)}(x) + ... + y' \cdot p_1(x) + y \cdot p_0(x) = 0$
- Analogous to n=1: $y = y_h + y_p$
- **Theorem:** Linear combinations of solutions of (H) are also solutions of (H).
- **Theorem:** If y_1, ..., y_n are n linearly independent solutions of (H), then $y(x) = C_1 \cdot y_1(x) + C_2 \cdot y_2(x) + ... + C_n \cdot y_n(x)$

4.6.1 Homogeneous

$$a_n y^{(n)} + ... + a_2 y'' + a_1 y' + a_0 y = 0$$

- Solve homogeneous DE of nth order:
  1. Substitute $y = e^{\lambda x} \rightarrow$ characteristic polynomial
  2. Determine roots λ_i of the characteristic polynomial
     - $\lambda_1 \neq \lambda_2 \neq ...$, real
       $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + C_3 e^{\lambda_3 x} + ...$
     - $\lambda_1 = \lambda_2 = ...$, real
       $y = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_2 x} + C_3 x^2 e^{\lambda_3 x} + ...$
     - $\lambda_{1,2} = \lambda_{3,4} = ... = a \pm ib$
       $y = e^{ax}(C_1 \cos(bx) + C_2 \sin(bx)) +$
       $x e^{ax}(C_3 \cos(bx) + C_4 \sin(bx)) +$
       $x^2 e^{ax}(C_5 \cos(bx) + C_6 \sin(bx)) + ...$

4.6.2 Inhomogeneous

$$a_n y^{(n)} + ... + a_2 y'' + a_1 y' + a_0 y = q(x)$$

- Solve inhomogeneous DE of nth order:
  1. Determine homogeneous solution y_h
  2. Particular solution y_p
     - Use ansatz or variation of parameters (Lagrange)

4.7 Variation of Parameters (Lagrange) for $n = 2$:

- **Lagrange for** $n = 2$:
  1. DE in the form $y'' + p_1 \cdot y' + p_0 \cdot y = q(x)$
  2. Homogeneous solution: $y_h = C_1 \cdot y_1(x) + C_2 \cdot y_2(x)$
  3. Assume: $C_1' \cdot y_1 + C_2' \cdot y_2 = 0$
  4. Determine C_1, C_2 $\begin{cases} C_1(x) = -\int \frac{q(x) y_2(x)}{W(x)}\, dx \\ C_2(x) = \int \frac{q(x) y_1(x)}{W(x)}\, dx \end{cases}$
     $W(x) = y_1 y_2' - y_1' y_2$, $\quad q(x)$: forcing term
  5. General solution: $y = C_1(x) y_1(x) + C_2(x) y_2(x)$

5 Basics

5.1 Trigonometric Relations 97-100

5.1.1 Trigonometric Values 97

Ra	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
Deg	0°	30°	45°	60°	90°	180°
$\sin(\alpha)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos(\alpha)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\tan(\alpha)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0

5.1.2 Identities

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2(x) = \frac{1}{\cos^2(x)}$

Potenzen^n bei Integration

- $\int_0^{\pi/2} \sin^n \mathrm{d}x = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} \mathrm{d}x - \left(\frac{1}{n} \cos x \sin^{n-1} x\right)$
- $\int_0^{\pi/2} \cos^n \mathrm{d}x = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} \mathrm{d}x - \left(\frac{1}{n} \sin x \cos^{n-1} x\right)$

  $n \geq 2,\ r, s \in \mathbb{Z}$ (für beide, sin und cos)
- Green part is omitted with definite integration!
- $\int_0^{\pi/2}$: $n = 1 : 1, n = 2 : \frac{\pi}{4}, n = 3 : \frac{2}{3}, n = 4 : \frac{3\pi}{16}$
- Achtung: $\int_0^{\pi/2} \sin^2(2x)\mathrm{d}x = \int_0^{\pi/2} \sin^2(x)\mathrm{d}x$

5.2 Hyperbolic functions

5.2.1 General

- $\cosh^2(x) - \sinh^2(x) = 1$
- $\cosh(x) + \sinh(x) = e^x$
- $\coth(x) = \frac{\cosh(x)}{\sinh(x)}$
- $\tanh(a \pm b) = \frac{1}{\coth(a \pm b)} = \frac{\tanh(a) \pm \tanh(b)}{1 \pm \tanh(a)\tanh(b)}$

5.3 Integrals (Reminder: $+C$!) 72

Basic

- $\int k dx = kx + C$
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
- $\int \frac{1}{x^n} = \frac{-1}{(n-1)x^{n-1}} + C, n \neq 1$
- $\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$
- $\int a^x dx = \frac{a^x}{ln(a)} + C$
- $\int e^x dx = e^x + C$
- $\int \log_a(x) dx = x \log_a(x) - x \log_a(e) + C$

Logarithm

- $\int \ln(ax) dx = x \ln(ax) - x$
- $\int x \ln(ax) dx = \frac{x^2}{4}(2\ln(ax) - 1) + C$
- $\int \frac{\ln(ax)}{x} dx = \frac{1}{2}(\ln(ax))^2 + \text{C}$

Exponential

- $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$
- $\int x e^x dx = (x-1)e^x + C$
- $\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax} + C$

Rational Functions

- $\int \frac{1}{\sqrt{x}} = 2\sqrt{x} + C$
- $\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1},\ n \neq -1 + C$
- $\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} + C$
- $\int \frac{ax+b}{cx+d} dx = \frac{ax}{c} - \frac{ad-bc}{c^2} \ln|cx+d| + C$
- $\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a} + C$
- $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax+b| + C$
- $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{ax^2+bx+c} dx = \frac{2}{\sqrt{4ac-b^2}} \arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) + C$
- $\int \frac{1}{(x-a)(x-b)} dx = \frac{1}{a-b} \ln\left|\frac{x-a}{x-b}\right| + C$
- $\int \frac{x}{a^2+x^2} dx = \frac{1}{2} \ln\left|a^2+x^2\right| + C$
- $\int \frac{x^2}{a^2+x^2} dx = x - a \arctan\left(\frac{x}{a}\right) + C$
- $\int \frac{x^3}{a^2+x^2} dx = \frac{1}{2}x^2 - \frac{1}{2}a^2 \ln\left|a^2+x^2\right| + C$
- $\int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln|a+x| + C$
- $\int \frac{x}{ax^2+bx+c} = \frac{1}{2a} \ln\left|ax^2+bx+c\right| - \frac{b}{a\sqrt{4ac-b^2}} \arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) + C$

Roots

- $\int \sqrt{x-a} dx = \frac{2}{3}(x-a)^{\frac{3}{2}} + C$
- $\int \sqrt{ax+b} dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b} + C$
- $\int \sqrt{x^2+a} dx = \frac{1}{2}x\sqrt{x^2+a} + \frac{a}{2} \ln|x+\sqrt{x^2+a}| + C$
- $\int \sqrt{a^2-x^2} dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2} \arcsin\left(\frac{x}{a}\right) + C$
- $\int x\sqrt{x-a} dx = \frac{2}{3}a(x-a)^{\frac{3}{2}} + \frac{2}{5}(x-a)^{\frac{5}{2}} + C$
- $\int x\sqrt{x^2 \pm a^2} dx = \frac{1}{3}(x^2 \pm a^2)^{\frac{3}{2}} + C$
- $\int (ax+b)^{\frac{3}{2}} dx = \frac{2}{5a}(ax+b)^{\frac{5}{2}} + C$
- $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$
- $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{\sqrt{x \pm a}} dx = 2\sqrt{x \pm a} + C$
- $\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} + C$

Trigonometric Basic

- $\int \sin(x) dx = -\cos(x) + C$
- $\int \cos(x) dx = \sin(x) + C$
- $\int \tan(x) dx = -\ln|\cos(x)|$

Trigonometric Ar

- $\int \text{arsinh}(x)\, dx = x\,\text{arsinh}(x) - \sqrt{x^2+1} + C$
- $\int \text{arcosh}(x)\, dx = x\,\text{arcosh}(x) - \sqrt{x^2-1} + C$
- $\int \text{artanh}\ dx = x\,\text{artanh}(x) + \frac{1}{2}\ln\left(1-x^2\right) + C$

Trigonometric ^{2}

- $\int \sin^2(x) dx = \frac{1}{2}(x - \sin(x)\cos(x) + C$
- $\int \cos^2(x) dx = \frac{1}{2}(x + \sin(x)\cos(x) + C$
- $\int \tan^2(x) dx = \tan(x) - x + C$

Trigonometric $\frac{1}{\dots}$

- $\int \frac{1}{\sin(x)} dx = \ln\left|\frac{1-\cos(x)}{\sin(x)}\right| + C$
- $\int \frac{1}{\cos(x)} dx = \ln\left|\frac{1+\sin(x)}{\cos(x)}\right| + C$
- $\int \frac{1}{\sin^2(x)} dx = -\cot(x) + C$
- $\int \frac{1}{\cos^2(x)} dx = \tan(x) + C$
- $\int \frac{1}{1+\sin(x)} dx = \frac{-\cos(x)}{1+\sin(x)} + C$
- $\int \frac{1}{1+\cos(x)} dx = \frac{\sin(x)}{1+\cos(x)} + C$
- $\int \frac{1}{1-\sin(x)} dx = \frac{\cos(x)}{1-\sin(x)} + C$
- $\int \frac{1}{1-\cos(x)} dx = \frac{-\sin(x)}{1-\cos(x)} + C$

Trigonometric with ...(ax) and $x\cdot$ ...(ax)

- $\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + C$
- $\int \cos(ax) dx = \frac{1}{a}\sin(ax) + C$
- $\int \tan(ax) dx = -\frac{1}{a}\ln(\cos(ax)) + C$
- $\int x\sin(ax) dx = -\frac{1}{a}x\cos(ax) + \frac{1}{a^2}\sin(ax) + C$
- $\int x\cos(ax) dx = \frac{1}{a}x\sin(ax) + \frac{1}{a^2}\cos(ax) + C$

Hyperbolic

- $\int \sinh(x) dx = \cosh(x) + C$
- $\int \cosh(x) dx = \sinh(x) + C$
- $\int \tanh(x) dx = \ln(\cosh(x)) + C$
- $\int (x) dx = x(x) - \sqrt{x^2+1} + C$
- $\int (x) dx = x(x) - \sqrt{x^2-1} + C$
- $\int (x) dx = x(x) + \frac{1}{2}\ln\left(1-x^2\right) + C$

Other Trigonometric Integrals

- $\int x\sin(ax) dx = -\frac{1}{a}x\cos(ax) + \frac{1}{a^2}\sin(ax) + C$
- $\int x\cos(ax) dx = \frac{1}{a}x\sin(ax) + \frac{1}{a^2}\cos(ax) + C$
- $\int e^{bx}\sin(ax) dx = \frac{1}{a^2+b^2}e^{bx}\left(b\sin(ax) - a\cos(ax)\right) + C$
- $\int e^{bx}\cos(ax) dx = \frac{1}{a^2+b^2}e^{bx}\left(a\sin(ax) + b\cos(ax)\right) + C$

6 REMARKS - PLEASE READ

- While every effort has been made to ensure the accuracy of the content, I do not take responsibility for any errors or omissions.
- I strongly encourage you to verify the information presented and report any mistakes to improve the quality of this cheatsheet. Contact information can be found below.
- Sections 4 and 5 were taken from the Anaylisis I/II cheatsheet by Leon Anspruch, translated to English for the sake of consistency and adapted.
- Some of the theory, namely sections 3.3 and 3.8.2 alongside some other smaller things where taken from the theory sheets made by Pascal Strauch

Contact Information for Reporting Mistakes:
If you find any mistakes or have suggestions for improvement, please contact me at ebesana@ethz.ch without hesitation.