D-MAVT
D-MATL

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Analysis III

Serie 1

Topics: Definition of Laplace transform. First properties: linearity, s-shifting. Inverse Laplace transform.

1. Find the Laplace transform $F(s) := \mathscr{L}(f)(s)$ of the following functions:

a) $f(t) = 2t^3 + 8t - 2$

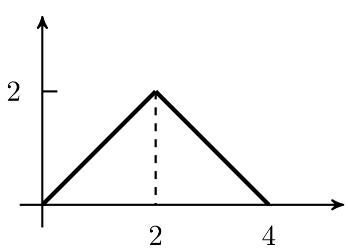
b) $f(t) = 3t^2 - 2t + 4$

c) $f(t) = \dfrac{1}{\sqrt{t}}$, using that

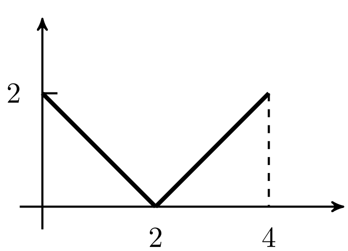
$$\Gamma\left(\frac{1}{2}\right)\left(= \int_0^{+\infty} t^{-1/2} e^{-t} dt\right) = \sqrt{\pi}$$

2. Find the Laplace transform of the following functions:

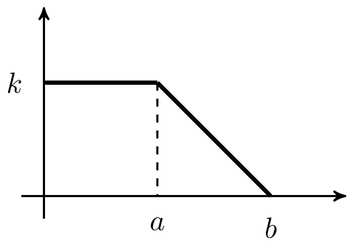
a)



b)



c)



3. **Theory reminder for Ex 4):** The Laplace transform of a finite linear combination of functions is the linear combination of their Laplace transforms:

$$\mathscr{L}\left(a_1 f_1 + \cdots + a_m f_m\right) = a_1 \mathscr{L}(f_1) + \cdots + a_m \mathscr{L}(f_m), \quad a_1, \ldots, a_m \in \mathbb{R}.$$

The same thing is true for infinite linear combination of functions under opportune conditions of convergence (which are all satisfied in the following case). For example we can compute the Laplace transform of the exponential from its power series expansion:

$$\mathscr{L}(e^{at})(s) = \mathscr{L}\left(\sum_{k=0}^{+\infty} \frac{(at)^k}{k!}\right)(s) = \sum_{k=0}^{+\infty} \frac{a^k}{k!} \mathscr{L}(t^k)(s) =$$

$$= \sum_{k=0}^{+\infty} \frac{a^k}{\cancel{k!}} \cdot \frac{\cancel{k!}}{s^{k+1}} = \frac{1}{s} \sum_{k=0}^{+\infty} \left(\frac{a}{s}\right)^k = \frac{1}{s} \cdot \frac{1}{1 - \frac{a}{s}} = \frac{1}{s-a}.$$

Compute $\mathscr{L}(\sin(\omega t))(s)$ using this method. Remember that:

$$\sin(\omega t) = \sum_{k=0}^{+\infty} (-1)^k \frac{(\omega t)^{2k+1}}{(2k+1)!}$$

4. Exercise **1.c)** uses the fact that $\Gamma(1/2) = \sqrt{\pi}$; this exercise proves it. Let us call $I := \Gamma(1/2)$ this value:

$$I = \Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2} e^{-t}\, dt.$$

(i) Use an opportune change of variables to prove that:

$$I = 2 \int_0^{+\infty} e^{-x^2}\, dx.$$

 Look at the next page!

(ii) Justify why

$$2\int_0^{+\infty} e^{-x^2}dx = \int_{-\infty}^{+\infty} e^{-x^2}dx.$$

(iii) Compute the square of this integral (fill the dots by changing coordinates to polar coordinates on $\mathbb{R}^2$):

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}dy\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}dxdy =$$
$$= \cdots = \pi$$

(iv) From (iii) we can deduce that the desired value is one of the two square roots of π: $I = \pm\sqrt{\pi}$. Why can we exclude the negative value?

Hand-in on Moodle by: Wednesday 25 September 2024.