D-MAVT
D-MATL

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Analysis III

Serie 10

Topics: Wave equation: solutions via Fourier series and d'Alembert's formula.

1. Let $u(x,t)$ be the solution of the following problem (1-dimensional wave equation on the line).

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R} \\ u_t(x,0) = 0, & x \in \mathbb{R} \end{cases}$$

where

$$f(x) = \begin{cases} e^{\frac{x^2}{x^2-1}}, & |x| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

a) Sketch a graph of $f(x)$, which is the solution at the initial time.

b) Sketch a graph of the solution $u(x,t)$ at the time $t = 2$.

c) Prove that, for each fixed $x \in \mathbb{R}$:

$$\lim_{t \to +\infty} u(x,t) = 0.$$

2. Find the solution $u = u(x,t)$ of the 1-dimensional wave equation on the interval $[0, L]$ with the following boundary and initial conditions:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & \\ u(0,t) = 0 = u(L,t), & t \geq 0 \\ u(x,0) = 0, & 0 \leq x \leq L \\ u_t(x,0) = \sin\left(\frac{\pi}{L}x\right), & 0 \leq x \leq L \end{cases}$$

in the following two 'different' ways:

a) Use the formula in the solution via Fourier series. (You already computed this solution in ex. 2 of Serie 9.)

b) Consider the 2-L periodic extension $u^*(x,t)$ of the solution $u(x,t)$. Then use d'Alembert's formula which in this case ($f=0$) becomes:

$$u^*(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds$$

where $g^*(s)$ is a priori the $2L$-periodic extension[1] of the initial datum $g(s) = \sin\left(\frac{\pi}{L}s\right)$ to all $s \in \mathbb{R}$.

Finally verify that the solutions obtained in the two ways **a)** and **b)** are indeed the same for all $0 \le x \le L$.

3. Let $c > 0$. Consider the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \ge 0, \\ u(x,0) = \frac{1}{c}\left((x^2-2)\sin(x) + 2x\cos(x)\right), & x \in \mathbb{R}, \\ u_t(x,0) = x^2\cos(x), & x \in \mathbb{R}. \end{cases}$$

Find the solution $u(x,t)$. You may use D'Alembert formula.
[Simplify the expression as much as possible: no unsolved integrals].

4. Let $c > 0$. Consider the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \ge 0 \\ u(x,0) = e^{-x^2}\sin^2(x) + x, & x \in \mathbb{R} \\ u_t(x,0) = xe^{-x^2}. & x \in \mathbb{R} \end{cases}$$

a) Find the solution $u(x,t)$. You may use D'Alembert formula.
[Simplify the expression as much as possible: no unsolved integrals].

b) For a fixed $a \in \mathbb{R}$, determine the asymptotic limit

$$\lim_{t\to+\infty} u(a,t).$$

Hand in by: Wednesday 27 November 2024.

[1] in this case the function is already $2L$-periodic ...