Serie 11

Topics: Heat equation: solutions via Fourier series. Laplace equation on a rectangle.

1. Find, via Fourier series, the solution of the 1-dimensional heat equation with the following initial condition:

$$\begin{cases} u_t = 4\, u_{xx}, & x \in [0,1],\ t \geq 0 \\ u(0,t) = u(1,t) = 0, & t \geq 0 \\ u(x,0) = f(x), & x \in [0,1] \end{cases}$$

where

$$f(x) = \sin(\pi x) + \sin(5\pi x) + \sin(10\pi x).$$

Use the method of separation of variables from scratch, showing all the steps.

2. An aluminium bar of length $L = 1(m)$ has thermal diffusivity of (around)[1]

$$c^2 = 0.0001 \left(\frac{\mathrm{m}^2}{\mathrm{sec}}\right) = 10^{-4} \left(\frac{\mathrm{m}^2}{\mathrm{sec}}\right).$$

It has initial temperature given by $u(x,0) = f(x) = 100 \sin(\pi x)$ (°C), and its ends are kept at a constant 0°C temperature. Find the first time t^* for which the whole bar will have temperature ≤ 30°C.
In mathematical terms, solve

$$\begin{cases} u_t = 10^{-4} u_{xx}, & x \in [0,1],\ t \geq 0 \\ u(0,t) = u(1,t) = 0, & t \geq 0 \\ u(x,0) = 100 \sin(\pi x), & x \in [0,1]. \end{cases}$$

and find the smallest t^* for which

$$\max_{x \in [0,1]} u(x,t^*) \leq 30.$$

You can use the formula from the lecture notes.

[1] we are approximating the standard value which would be $c^2 \approx 0.000097\mathrm{m}^2/\mathrm{sec}$ to make computations easier.

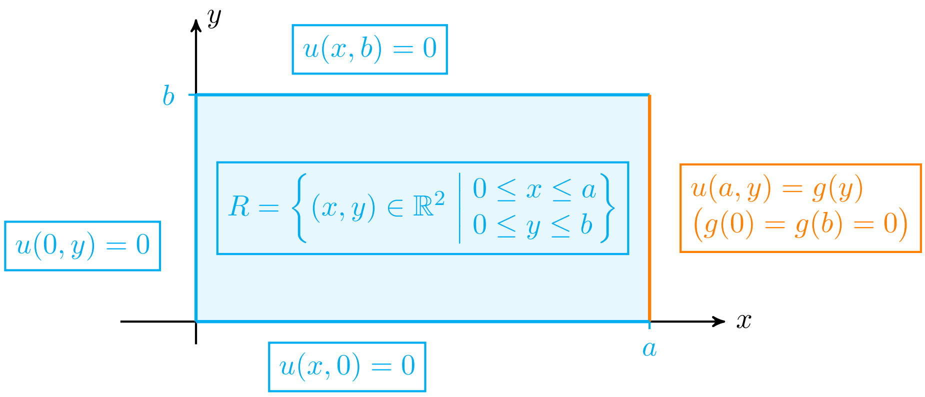
Please turn!

3. Consider the following time-dependent version of the heat equation on the interval $[0, L]$, in which the constant varies linearly with time. We also impose boundary conditions and we look for solutions:

$$u = u(x,t) \quad \text{s.t.} \quad \begin{cases} u_t = 2tc^2 u_{xx}, & x \in [0, L], t \in [0, +\infty) \\ u(0,t) = 0, & t \in [0, +\infty) \\ u(L,t) = 0, & t \in [0, +\infty) \end{cases}$$

Find all possible solutions of the specific form $u(x,t) = F(x)G(t)$.

4. Adapt the method used to solve the previous Laplace equation in the case in which the only nontrivial initial boundary condition is on the right vertical segment of the rectangle



$$\begin{cases} \Delta u = 0, & (x,y) \in R \\ u(x,0) = u(x,b) = 0, & 0 \le x \le a \\ u(0,y) = 0, & 0 \le y \le b \\ u(a,y) = g(y), & 0 \le y \le b \end{cases}$$

where $g(y)$ is any function with prescribed boundary conditions

$$g(0) = g(b) = 0.$$

Hand in by: Wednesday 4 December 2024.