D-MAVT
D-MATL

Prof. A. Iozzi
ETH Zürich
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Analysis III

Serie 12

Topics: Laplace equation on a rectangle. Inhomogeneous Wave equation. Heat equation on an infinite bar.

1. Find the general solution $u = u(x,y)$ of the following Laplace equation on a rectangle, with nonzero boundary conditions on two edges of the rectangle:

$$\begin{cases} \Delta u = 0, & 0 \leq x \leq a, 0 \leq y \leq b \\ u(x,0) = u(0,y) = 0, & 0 \leq x \leq a, 0 \leq y \leq b \\ u(x,b) = f(x), & 0 \leq x \leq a \\ u(a,y) = g(y), & 0 \leq y \leq b \end{cases}$$

where $f(x)$ and $g(y)$ are two arbitrary functions with the only (obvious) condition that they are compatible with the other boundary conditions, which means:

$$f(0) = f(a) = g(b) = g(0) = 0.$$

Choose your preferred method to solve it, in particular you are allowed to use any formula previously learned from the Lecture notes or from any other exercise.

[*Hint:* You can combine the results from §4.5.1. of the Lecture notes and Exercise **4.** of Serie 11.]

2. Find the solution of the following wave equation (**with inhomogeneous boundary conditions**) on the interval $[0, \pi]$:

$$u = u(x,t) \text{ such that } \begin{cases} u_{tt} = c^2 u_{xx}, & t \geq 0,\ x \in [0,\pi] \\ u(0,t) = 3, & t \geq 0 \\ u(\pi,t) = 5, & t \geq 0 \\ u(x,0) = x^2 + \frac{1}{\pi}(2-\pi^2)x + 3, & x \in [0,\pi] \\ u_t(x,0) = 0. & x \in [0,\pi] \end{cases} \tag{1}$$

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w'' = 0$, $w(0) = 3$, and $w(\pi) = 5$.

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

c) (i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (1).

[*Hint:* You can use the following formulas:

$$\int x\sin(nx)\,dx = \frac{\sin(nx) - nx\cos(nx)}{n^2} \quad (+\text{ constant})$$

$$\int x^2\sin(nx)\,dx = \frac{(2-n^2x^2)\cos(nx) + 2nx\sin(nx)}{n^3} \quad (+\text{ constant})$$]

3. Find the solution of the heat equation on an infinite bar

$$\begin{cases} u_t = c^2 u_{xx}, & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = f(x) = \begin{cases} \sinh(x), & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases} & x \in \mathbb{R} \end{cases}$$

in Fourier integral form - formula (4.25) of the Lecture Notes.

4. Solve the following heat equation on an infinite bar:

$$\begin{cases} u_t = \frac{1}{2}u_{xx}, & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = x\mathrm{e}^{-\frac{1}{2}x^2}, & x \in \mathbb{R} \end{cases}$$

via the Fourier transform with respect to x.
Hint: Use that, for $a > 0$,

$$\mathcal{F}\left(x\mathrm{e}^{-ax^2}\right)(\omega) = \frac{-i\omega}{(2a)^{3/2}}\mathrm{e}^{-\frac{\omega^2}{4a}}.$$

Hand in on Moodle by: Wednesday 11 December 2024.