D-MAVT
D-MATL

Prof. A. Iozzi
ETH Zürich
Autumn 2024

Analysis III

Serie 2

<u>**Topics:**</u> Inverse Laplace transform, Laplace transform of derivatives, Heaviside function, t-shifting theorem and applications to ODEs.

1. Find the <u>inverse</u> Laplace transform $f = \mathscr{L}^{-1}(F)$ of the following functions:

a) $F(s) = \dfrac{3}{s^5}$

b) $F(s) = \dfrac{1}{s^2 + 25}$

c) $F(s) = \dfrac{7}{(s+3)(s-2)}$

d) $F(s) = \dfrac{s+1}{(s+2)(s^2+s+1)}$

e) $F(s) = \dfrac{s}{(s-1)^2(s^2+2s+5)}$

<u>*Hint:*</u> It may be useful to simplify the expression using partial fraction decomposition. Then to write the denominator as $(s+a)^2 + \omega^2$ for an opportune choice of a, ω. And finally, use the s-shifting property.

2. Find the Laplace transform of the following functions.

a) $f(t) = -8t^2 \mathrm{e}^{-4t}$

b) $f(t) = \cosh^2(t/2)$

c) $f(t) = t \sinh(3t)$

d) $f(t) = t^2 \cos(\pi t)$

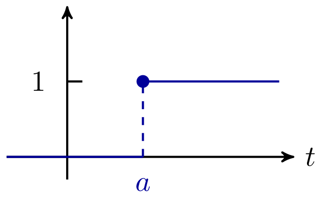
e) $f(t) = u(t-3)(t-3)^2$

f) $f(t) = u(t-5\pi)\cos(t)$

g) $f(t) = -u(t-3)t^2 + u(t-5)\cos(t)$

Recall: For a fixed $a \in \mathbb{R}$, the function

$$u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$$



is the Heaviside, or *unit step*, function (shifted by a).

3. Find the Laplace transform of the following functions

a) $f(t) = \begin{cases} \sin(t), & 0 \leq t \leq 4\pi \\ 0. & \text{otherwise} \end{cases}$

b) $f(t) = \begin{cases} 5t^2, & t \geq 100 \\ 0. & \text{otherwise} \end{cases}$

4. First application to ODEs

Solve the following initial value problems with constant coefficients.

a) $\begin{cases} 2y'' + 2y' - 4y = 0 \\ y(0) = 1 \\ y'(0) = 5 \end{cases}$

b) $\begin{cases} 2y'' + 3y' - 2y = te^{-2t} \\ y(0) = 0 \\ y'(0) = -2 \end{cases}$

c) $\begin{cases} 3y'' = -8 + u(t-4)(t-4) \\ y(0) = 3 \\ y'(0) = 1 \end{cases}$

[*Recall:* Under opportune hypothesis of regularity of the function y, the Laplace transform of its derivatives is:

$$\mathcal{L}\left(y^{(n)}\right)(s) = s^n Y(s) - \sum_{k=0}^{n-1} s^{n-k-1} y^{(k)}(0)\,.]$$

 Look at the next page!

5. ODEs with nonconstant coefficients

Solve the following initial value problem with nonconstant coefficients.

$$\begin{cases} ty'' - ty' + y = 2 \\ y(0) = 2 \\ y'(0) = -4 \end{cases}$$

Hint: In this case the coefficients are nonconstant but very easy to transform when multiplied by a function. That is:

$$\mathcal{L}\left(ty'\right) = -\frac{d}{ds}\mathcal{L}\left(y'\right) = -\frac{d}{ds}\left(sY(s) - y(0)\right) = -sY'(s) - Y(s).$$
$$\mathcal{L}\left(ty''\right) = -\frac{d}{ds}\mathcal{L}\left(y''\right) = -\frac{d}{ds}\left(s^2Y(s) - sy(0) - y'(0)\right) = -s^2Y'(s) - 2sY(s) + y(0).$$

6. **a)** Using the Laplace transform of the derivative, and the Laplace transform of $1/\sqrt{t}$ found in Exercise **1.c)** of Serie 1, prove that

$$\mathcal{L}\left(\sqrt{t}\right) = \frac{\sqrt{\pi}}{2s\sqrt{s}}.$$

b) More generally, prove that for every integer $n \geq 0$:

$$\mathcal{L}\left(t^n\sqrt{t}\right) = \frac{(2n+2)!\sqrt{\pi}}{(4s)^{n+1}(n+1)!\sqrt{s}}.$$

Hand in on Moodle by: Wednesday 2 October 2024.