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Analysis III

Autumn 2024

Serie 3

Topics: Laplace transform of Dirac delta function, convolution product with applications to ODEs and integral equations.

1. Find the inverse Laplace transform of the following functions.

a) $F(s) = \dfrac{e^{-2s}}{s^2 + 4}$

b) $F(s) = \dfrac{e^{-s}}{(s+1)^3}$

c) $F(s) = \dfrac{1}{s(s^2+1)}$

d) $F(s) = \dfrac{1}{(s^2+1)^2}$

e) $F(s) = \dfrac{s}{(s^2-16)^2}$

2. Compute the following convolutions.

a) $e^{at} * e^{bt}, \qquad a, b \in \mathbb{R}$

b) $\sin(t) * \cos(t)$

c) $t^m * t^n, \qquad m, n \in \mathbb{N}$

Hint: Exercise **a)** requires a different discussion for the cases $a \neq b$ and $a = b$.

3. Find the solution $f(t)$ of the following initial value problem:

$$\begin{cases} f''(t) - a^2 f(t) = a, \quad t > 0, \\ f(0) = 2, \quad f'(0) = a, \end{cases}$$

where $a > 0$ is a positive constant.

4. **a)** Use the Laplace transform to find the solution of the following initial value problem:

$$\begin{cases} y' = g(t) \\ y(0) = c \end{cases}$$

b) Find the solution $y : [0, \infty) \to \mathbb{R}$ of the following integral equation:

$$y(t) + \int_0^t y(\tau)\cosh(t-\tau)\,d\tau = t + \mathrm{e}^t.$$

5. Find the solution $f = f(t)$ of the following initial value problem:

$$\begin{cases} f''(t) + \omega^2 f(t) = \omega\,\delta(t-a), \quad t > 0 \\ f(0) = 1, \quad f'(0) = \omega, \end{cases}$$

where $\omega, a > 0$ are positive constants.

Theory reminder on the convolution product:
The convolution of two functions $f, g : [0, +\infty) \to \mathbb{R}$ is defined as the new function $f * g : [0, +\infty) \to \mathbb{R}$:

$$(f * g)(t) := \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau.$$

The useful property of the convolution product with respect to the Laplace transform is:

$$\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g). \tag{1}$$

Therefore the inverse Laplace transform of a product is the convolution of the Laplace transforms:

$$\mathcal{L}^{-1}(FG) = \mathcal{L}^{-1}(F) * \mathcal{L}^{-1}(G). \tag{2}$$

Hand in on Moodle by: Wednesday 9 October 2024.