D-MAVT
D-MATL

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Analysis III

Serie 4

Topics: Periodic functions and their properties. Fourier series of a periodic function.

Theory reminder on periodic functions:
A function $f : \mathbb{R} \to \mathbb{R}$ is periodic of period $P > 0$ if $f(x+P) = f(x)$ for all $x \in \mathbb{R}$. The fundamental period is the smallest positive number P for which f is periodic of period P. For example for positive α, the functions $\sin(\alpha x), \cos(\alpha x)$ are periodic with fundamental period $\frac{2\pi}{\alpha}$.

Theory reminder on convergence of Fourier series:
Let f be any $2L$-periodic function. From the script we know that if f is well behaved (for example everywhere continuous except a discrete set of points and with left and right derivatives at every point) then, calling by F its Fourier series, we have for every point x_0

$$F(x_0) = \frac{1}{2}\left(f^+(x_0) + f^-(x_0)\right), \qquad \text{where} \quad f^\pm(x_0) = \lim_{x \to x_0^\pm} f(x) = \lim_{\varepsilon \to 0^+} f(x_0 \pm \varepsilon)$$

In particular if f is continuous in x_0 then $F(x_0) = f(x_0)$ because left and right limit of f coincide.

1. This exercise relates the periodicity of a function with its derivative(s) and its properties of boundedness.
Let $f : \mathbb{R} \to \mathbb{R}$ be any function.

a) Prove that if f is periodic and continuous, then it is bounded.

b) Prove that if f is differentiable, and periodic of period P, then also f' is periodic with the same period.

c) From **a)** and **b)** deduce that if f is periodic and smooth (smooth, then it is bounded and all its derivative are bounded as well.

d) Use **a)** and **b)** to give a very simple proof that $\sin(x^2)$ is not periodic.

2. Suppose that f and g are periodic functions of fundamental periods P and Q, respectively. What can you say about their sum $f+g$? More precisely, given minimal sufficient conditions on P and Q the sum $f+g$ is periodic, and if so, which is the period.

3. Determine which of the following functions is periodic and which is not. For the periodic ones, determine their fundamental period. For the non-periodic ones, explain/prove why they are not periodic.

a) $\cos(\frac{2\pi x}{L})$, where $L>0$ is a constant.

b) $\sin(2x)+x^3$

c) $\cos(4x)+2\cos(2x)$

d) $\cos(15x)+3\sin(6x)$

[*Hint:* *Using Exercises* ***1.*** *and* ***2.*** *helps.*]

4. Let now f and g be, respectively, the $2L$-periodic extensions to $\mathbb{R}$ of x and x^2 from $[-L,L)$. Sketch a graph of these functions.

a) Are f and g well behaved in the sense specified above?

b) What are the points of discontinuity of f and g?

c) What are the mean values of the left and right limit of f in its points of discontinuity?

$$\frac{1}{2}\left(f^+(x_0)+f^-(x_0)\right) = \ ?$$

d) The Fourier serie of f is

$$F(x)=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{2L}{\pi n}\sin\left(\frac{n\pi}{L}x\right)$$

Does the Fourier series F of f converge to these values in these points?

5. Compute the Fourier serie of $\cos^3(x)$ in $[-\pi,\pi]$ to show the following trigonometric identity

$$\cos^3(x)=\frac{3}{4}\cos(x)+\frac{1}{4}\cos(3x).$$

[*Hint:* *You can use that the coefficients* $b_n=0$ *for all* n. *You will see the justification in chapter 3.2. And you can use the trigonometric formula* $\cos(x)\cos(nx)=\frac{1}{2}\left(\cos((n+1)x)+\cos((n-1)x)\right)$.]

Hand in on Moodle by: Wednesday 16 October 2024.