D-MAVT
D-MATL

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Analysis III

Serie 5

Topics: Periodic extension of a function, Fourier series, convergence of Fourier series.

Remark: In the following 'Fourier series' always means the real Fourier series of a function. Otherwise we will always specify 'complex Fourier series'.

1. Determine whether the following functions are even, odd, or neither. Justify your answer.

a) $f(x) = x^2 + 2$

b) $f(x) = x + 1$

c) $f(x) = \sinh(x^3 + x)$

d) $f(x) = \sin(\pi x) + \sin(x^2)$

e) $f(x) = \mathfrak{Re}(\mathrm{e}^{i\sin(x)})$

2. **a)** Consider the function

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

Extend f to an even function on the interval $[-\pi, \pi]$ and then finally to an even, 2π-periodic function on $\mathbb{R}$ and call this function f_e. Sketch the graph of f_e and find its Fourier series.

b) Consider the function

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} \leq x < \pi \\ 0, & x = \pi \end{cases}$$

Do the same for the odd, 2π-periodic extension[1] of f (call this f_o).

[1] We added the condition $f(\pi) = 0$, to avoid problems when we want to extend f to an odd function.

3. **a)** Sketch the graph of the $2L$-periodic extension of

$$f(x) = x, \quad x \in [-L, L)$$

in the interval $[-2L, 2L]$. In which points this extension is not continuous?

b) Compute its Fourier series.

c) Evaluate the Fourier series at an appropriate point x_0 and use the convergence result to calculate the following numerical series

$$\sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} = ?$$

4. Find the complex Fourier series of the same function $f(x)$ considered in Exercise **3.** Verify that the coefficients c_n of this series

$$\sum_{n=-\infty}^{+\infty} c_n e^{i\frac{n\pi}{L}x}$$

are related as written in the script to the real coefficients a_n, b_n found in the previous exercise.

Hand in on Moodle by: Wednesday 23 October 2024.