Analysis III

## Serie 6

Topics: Complex Fourier series, Fourier integral and Fourier transform.

- **1.** Let g be the 2L-periodic extensions to  $\mathbb{R}$  of  $x^2$  from [-L, L).
  - a) Sketch a graph of this function.
  - **b**) Prove that the Fourier series of g is

$$f(x) = \frac{L^2}{3} + \sum_{n=1}^{+\infty} (-1)^n \frac{4L^2}{\pi^2 n^2} \cos\left(\frac{n\pi}{L}x\right)$$

c) Because g is well-behaved and continuous everywhere, its Fourier series f converges to it in every point. In particular

$$L^2 = g(L) = f(L).$$

Deduce from this equality the value of the Riemann Zeta function  $\zeta(s)$  evaluated at s = 2

$$\zeta(2) := \sum_{n=1}^{+\infty} \frac{1}{n^2}$$

- **2.** For a > 0, consider the function  $\cosh(ax)$  on the interval  $[-\pi, \pi)$  and extend it on all  $\mathbb{R}$  to a function of period  $2\pi$ .
  - a) Compute its complex Fourier series.
  - b) Use this result to find the value of the following series:

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2 + a^2} = ?$$

Please turn!

- **3.** The function  $f(x) = \left| \cos\left(\frac{x}{2}\right) \right|$  is periodic of period  $2\pi$ .
  - a) Compute its Fourier series.
  - b) Use this result to find the value of the following series:

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{4n^2 - 1} = ?$$

4. Compute the Fourier integral of the function  $f(x) = e^{-\pi |x|}$  and use it to compute the values of the following integral:

$$\int_{\mathbb{R}} \frac{\cos(\omega x)}{\omega^2 + \pi^2} \, d\omega$$

(for each  $x \in \mathbb{R}$ ).

5. Find the Fourier transform  $\widehat{f} = \mathcal{F}(f)$  of the following functions:

a) 
$$f(x) = \begin{cases} e^{2ix}, & -1 \le x \le 1\\ 0, & \text{otherwise} \end{cases}$$
  
b)  $f(x) = \begin{cases} x, & 0 \le x \le 1\\ -x, & -1 \le x \le 0\\ 0, & \text{otherwise.} \end{cases}$ 

Hand in on Moodle by: Wednesday 30 October 2024.