D-MAVT
D-MATL

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Autumn 2024

Analysis III

Serie 8

Topics: Introduction to PDEs, classification of 2nd order PDEs. Definition of wave equation and heat equation. Fourier transform.

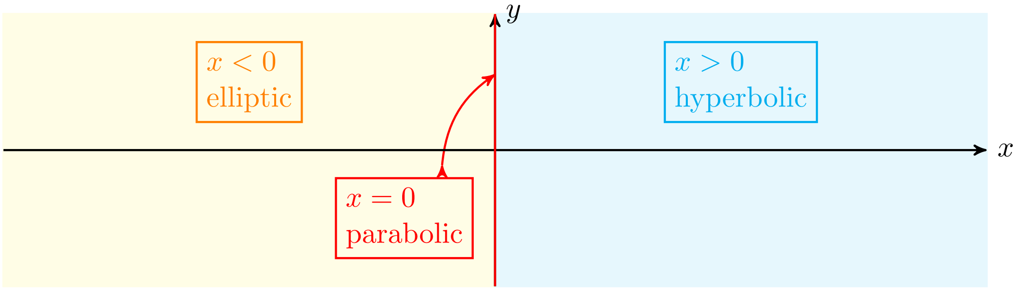
Theory reminder on the classification of PDEs: A 2nd order PDE is an equation of the form:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

where the coefficients A, B, C may also be functions of x, y. We say that the PDE is, respectively, hyperbolic, parabolic or elliptic, if the function $AC - B^2$ is, respectively, always smaller, equal, or greater than zero. When the sign changes in different regions of the plane (x, y), the equation is called *of mixed type*.
For example the Euler-Tricomi equation

$$u_{xx} - xu_{yy} = 0$$

has $AC - B^2 = 1 \cdot (-x) - (0)^2 = -x$, and therefore is of mixed type: hyperbolic in the half plane $x > 0$, elliptic in the other half plane $x < 0$, and parabolic on the line $x = 0$.



Please turn!

1. Consider the following PDEs - in what follows, $u = u(x, y)$ is a function of two variables.

$$u_{xx} + 2u_{xy} + u_{yy} + 3u_x + xu = 0, \tag{1}$$
$$u_{xx} + 2u_{xy} + 2u_{yy} + u_y = 0, \tag{2}$$
$$u_{xx} + 8u_{xy} + 2u_{yy} + e^x u_x = 0, \tag{3}$$
$$yu_{xx} + 2xu_{xy} + u_{yy} - u_y = 0, \tag{4}$$
$$(x + 1)u_{xx} + 2yu_{xy} + x^2 u_{yy} = 0. \tag{5}$$

Which of this is hyperbolic? Parabolic? Elliptic? Of mixed type?
In the last case, try to understand in which region of the plane (x, y) they are hyperbolic, parabolic or elliptic[1].

2. Consider the following functions.

a) $u(x, t) = e^{-100t} \cos(2x)$

b) $u(x, t) = \sin(2x) \cos(8t)$

c) $u(x, t) = e^{-36t} \sin(3x)$

Which PDE between the heat equation, $u_t = c^2 u_{xx}$, and the wave equation, $u_{tt} = c^2 u_{xx}$, does each of these solve? Write down also which is the constant c in each case.

3. Find the general solution $u = u(x, y)$ for the following PDEs:

a) $u_y + 2yu = 0$

b) $u_{yy} = 4xu_y$.

4. Compute the Fourier transform of the following function:

$$f(x) = \begin{cases} \sqrt{2\pi}(1 + x), & 0 \le x \le \pi, \\ 0, & \text{otherwise.} \end{cases}$$

5. Let $a > 0$ be a positive number. A Gaussian function with parameter a is of the form e^{-ax^2}. The Fourier transform of the Gaussian function is given by:

$$\mathcal{F}(\mathrm{e}^{-ax^2})(\omega) = \frac{1}{\sqrt{2a}} \mathrm{e}^{-\frac{\omega^2}{4a}}$$

Use this equality to calculate the following integrals:

[1] You can plot the curve $\{AC - B^2 = 0\}$ on - say - Wolfram|Alpha to understand its shape.

a) $\displaystyle\int_{\mathbb{R}} \mathrm{e}^{-ax^2} dx$

b) $\displaystyle\int_{\mathbb{R}} x\mathrm{e}^{-ax^2} dx$

c) $\displaystyle\int_{\mathbb{R}} x^2\mathrm{e}^{-ax^2} dx$

Hint: Use the properties of the Fourier transform, from §3.6 of the Lecture Notes.

Hand in by: Wednesday 13 November 2024.