D-MAVT
D-MATL

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Autumn 2024

Analysis III

Serie 9

Topics: PDE: Wave equation. Fourier series solution of the one-dimensional wave equation.

1. For $k \in \mathbb{R}$, find the Fourier series solution $u = u(x,t)$ of the 1-dimensional wave equation on the interval $[0,1]$ with the following boundary and initial conditions:

$$\begin{cases} u_{tt} = u_{xx}, & \\ u(0,t) = 0 = u(1,t), & t \geq 0 \\ u(x,0) = kx(1-x^2), & 0 \leq x \leq 1 \\ u_t(x,0) = 0, & 0 \leq x \leq 1 \end{cases}$$

Use the method of separation of variables from scratch and describe each step of it.

2. Find the solution $u = u(x,t)$ of the 1-dimensional wave equation on the interval $[0,L]$ with the following boundary and initial conditions:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & \\ u(0,t) = 0 = u(L,t), & t \geq 0 \\ u(x,0) = 0, & 0 \leq x \leq L \\ u_t(x,0) = \sin\left(\frac{\pi}{L}x\right), & 0 \leq x \leq L \end{cases}$$

Find the solution via Fourier series, you don't need to detail the steps. So use directly the formula given in §4.3 of the Lecture Notes.

3. Find all possible solutions of the following PDEs with variables separated, i.e. solutions of the form $u(x,t) = F(x)G(t)$:

a) $xu_x + u_t = 0$

b) $u_x + u_t + xu = 0$

c) $t^3 u_x + \cos(x)u - 2u_{xt} = 0$

4. Wave Equation with inhomogeneous boundary conditions

Find the solution of the following wave equation (**with inhomogeneous boundary conditions**) on the interval $[0, \pi]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t \geq 0,\ x \in [0, \pi] \\ u(0,t) = 3\pi^2, & t \geq 0 \\ u(\pi, t) = 7\pi, & t \geq 0 \\ u(x,0) = 2\sin(5x) + \sin(4x) + (7 - 3\pi)x + 3\pi^2, & x \in [0, \pi] \\ u_t(x,0) = 0. & x \in [0, \pi] \end{cases} \tag{1}$$

You must proceed as follows.

a) Find the unique function $w = w(x)$ with $w''(x) = 0$, $w(0) = 3\pi^2$, and $w(\pi) = 7\pi$.

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

c) (i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (1).

Hand in on Moodle by: Wednesday 20 November 2024.