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Analysis III

Solutions Serie 1

1. Find the Laplace transform $F(s) := \mathscr{L}(f)(s)$ of the following functions:

a) $f(t) = 2t^3 + 8t - 2$

Solution:

From the lecture we know that the polynomial $g_n(t) = t^n$ $(n \in \mathbb{N})$, has Laplace transform, defined for each $s > 0$: $G_n(s) = \frac{n!}{s^{n+1}}$. Using the linearity of $\mathscr{L}$ we have

$$F(s) = \mathscr{L}(2t^3 + 8t - 2)(s) = 2\mathscr{L}(t^3)(s) + 8\mathscr{L}(t)(s) - 2\mathscr{L}(1)(s) = 2\frac{3!}{s^4} + 8\frac{1}{s^2} - 2\frac{1}{s} = \\ = \frac{-2s^3 + 8s^2 + 12}{s^4}.$$

b) $f(t) = 3t^2 - 2t + 4$

Solution:

From the lecture we know that the polynomial $g_n(t) = t^n$ $(n \in \mathbb{N})$, has Laplace transform, defined for each $s > 0$: $G_n(s) = \frac{n!}{s^{n+1}}$. Using the linearity of $\mathscr{L}$ we have

$$F(s) = \mathscr{L}(3t^2 - 2t + 4)(s) = 3\mathscr{L}(t^2)(s) - 2\mathscr{L}(t)(s) + 4\mathscr{L}(1)(s) = 3\frac{2!}{s^3} - 2\frac{1}{s^2} + 4\frac{1}{s} = \\ = \frac{4s^2 - 2s + 6}{s^3}.$$

c) $f(t) = \dfrac{1}{\sqrt{t}}$, using that

$$\Gamma\left(\frac{1}{2}\right) \left(= \int_0^{+\infty} t^{-1/2} e^{-t} dt\right) = \sqrt{\pi}$$

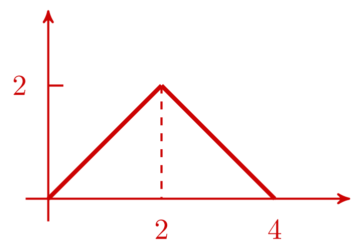
Solution:

For each $s > 0$, we make the change of variables $u = st$, so that $dt = 1/s\, du$, and

$$F(s) = \int_0^{+\infty} t^{-1/2} e^{-st} dt = \frac{1}{s} \int_0^{+\infty} s^{1/2} u^{-1/2} e^{-u} du = \frac{1}{\sqrt{s}} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{s}}.$$

2. Find the Laplace transform of the following functions:

a)



Solution:

We have

$$f(t) = \begin{cases} t, & 0 \le t \le 2 \\ 4-t, & 2 \le t \le 4 \\ 0, & \text{otherwise.} \end{cases}$$

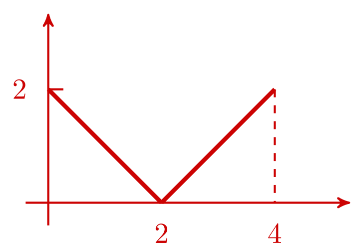
Again, integrating by parts

$$\mathscr{L}(f)(s) = \int_0^2 e^{-st} t\,dt + \int_2^4 e^{-st}(4-t)\,dt = -\frac{t}{s}e^{-st}\Big|_0^2 + \frac{1}{s}\int_0^2 e^{-st}\,dt - \frac{4-t}{s}e^{-st}\Big|_2^4 - \frac{1}{s}\int_2^4 e^{-st}\,dt =$$

$$= -\frac{2}{s}e^{-2s} - \frac{1}{s^2}e^{-st}\Big|_0^2 + \frac{2}{s}e^{-2s} + \frac{1}{s^2}e^{-st}\Big|_2^4 = -\frac{1}{s^2}e^{-2s} + \frac{1}{s^2} + \frac{1}{s^2}e^{-4s} - \frac{1}{s^2}e^{-2s} =$$

$$= \frac{1}{s^2}\left(1 - 2e^{-2s} + e^{-4s}\right) = \frac{(1-e^{-2s})^2}{s^2}.$$

b)



Solution 1: (direct computation)

We have

$$f(t) = \begin{cases} 2-t, & 0 \le t \le 2 \\ t-2, & 2 \le t \le 4 \\ 0, & \text{otherwise.} \end{cases}$$

Again, integrating by parts

$$\mathscr{L}(f)(s) = \int_0^2 e^{-st}(2-t)\,dt + \int_2^4 e^{-st}(t-2)\,dt = \frac{2s + e^{-2s} - 1}{s^2} + \frac{e^{-4s}(-2s + e^{2s} - 1)}{s^2}$$

$$= \frac{2}{s} - \frac{(1-e^{-2s})^2}{s^2} - 2\frac{e^{-4s}}{s}.$$

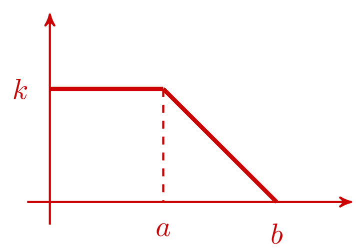
Look at the next page!

Solution 2: (using the Heaviside function)

Let us call $g(t)$ this function and $f(t)$ the function in the previous point (**a)**). We can observe that $g(t) = 2 - f(t) - 2u(t-4)$, so it follows by linearity and t-shifting property that:

$$\mathscr{L}(g)(s) = \mathscr{L}(2)(s) - \mathscr{L}(f)(s) - \mathscr{L}(2u(t-4)) = \frac{2}{s} - \frac{(1-e^{-2s})^2}{s^2} - 2\frac{e^{-4s}}{s}.$$

c)



Solution:

We have

$$f(t) = \begin{cases} k, & 0 \le t \le a \\ -\frac{k}{b-a}t + \frac{kb}{b-a}, & a \le t \le b \\ 0, & \text{otherwise.} \end{cases}$$

Then, integrating by parts

$$\begin{aligned}
\mathscr{L}(f)(s) &= k\int_0^a e^{-st}\,dt + \int_a^b e^{-st}\left(-\frac{k}{b-a}t + \frac{kb}{b-a}\right)dt \\
&= -\frac{k}{s}e^{-st}\Big|_0^a - \frac{1}{s}e^{-st}\left(-\frac{k}{b-a}t + \frac{kb}{b-a}\right)\Big|_a^b - \frac{1}{s}\frac{k}{b-a}\int_a^b e^{-st}\,dt \\
&= -\frac{k}{s}e^{-sa} + \frac{k}{s} + \frac{1}{s}e^{-sa}k + \frac{1}{s^2}\frac{k}{b-a}e^{-st}\Big|_a^b = \frac{k}{s} + \frac{1}{s^2}\frac{k}{b-a}(e^{-sb} - e^{-sa}).
\end{aligned}$$

3. Compute $\mathscr{L}(\sin(\omega t))(s)$. Remember that:

$$\sin(\omega t) = \sum_{k=0}^{+\infty}(-1)^k\frac{(\omega t)^{2k+1}}{(2k+1)!}$$

Solution:

We have

$$\mathscr{L}(\sin(\omega t))(s) = \mathscr{L}\left(\sum_{k=0}^{+\infty}(-1)^k\frac{(\omega t)^{2k+1}}{(2k+1)!}\right)(s) = \sum_{k=0}^{+\infty}(-1)^k\frac{\omega^{2k+1}}{(2k+1)!}\mathscr{L}(t^{2k+1})(s) =$$

$$= \sum_{k=0}^{+\infty}(-1)^k\frac{\omega^{2k+1}}{\cancel{(2k+1)!}}\cdot\frac{\cancel{(2k+1)!}}{s^{2k+2}} = \frac{\omega}{s^2}\sum_{k=0}^{+\infty}\left(-\frac{\omega^2}{s^2}\right)^k = \frac{\omega}{s^2}\cdot\frac{1}{1+\frac{\omega^2}{s^2}} = \frac{\omega}{s^2+\omega^2}.$$

4. Exercise **1.c)** uses the fact that $\Gamma(1/2) = \sqrt{\pi}$; this exercise proves it.
Let us call $I := \Gamma(1/2)$ this value:

$$I = \Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2}\mathrm{e}^{-t}\,dt.$$

(i) Use an opportune change of variables to prove that:

$$I = 2\int_0^{+\infty} e^{-x^2}\,dx.$$

Solution:
We change variables $t = x^2$, for which $dt = 2xdx$, and:

$$I = \Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2}e^{-t}dt \overset{t=x^2}{=} \int_0^{+\infty} x^{-1}e^{-x^2}2xdx = 2\int_0^{+\infty} e^{-x^2}dx.$$

(ii) Justify why

$$2\int_0^{+\infty} e^{-x^2}dx = \int_{-\infty}^{+\infty} e^{-x^2}dx.$$

Solution:
The function in the integral is symmetric with respect to $x = 0$. Thus the integral over all real numbers is twice the integral from 0 to $+\infty$.

(iii) Compute the square of this integral (fill the dots by changing coordinates to polar coordinates on $\mathbb{R}^2$):

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}dy\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}dxdy =$$
$$= \cdots = \pi$$

Solution:

We fill the dots using polar coordinates[1], for which $dxdy = rdrd\theta$:

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2} dy\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dxdy =$$
$$= \int_0^{+\infty}\int_0^{2\pi} e^{-r^2} rdrd\theta = 2\pi \int_0^{+\infty} e^{-r^2} rdr = 2\pi \cdot \left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_0^{+\infty} = \pi.$$

(iv) From (iii) we can deduce that the desired value is one of the two square roots of π: $I = \pm\sqrt{\pi}$. Why can we exclude the negative value?
Why can we exclude the negative value?

Solution:

Because the function in the integral is ≥ 0, therefore also its integral must be.

[1] Remember: $x = r\cos(\theta)$, $y = r\sin(\theta)$.