Analysis III

Solutions Serie 10

1. Let $u(x,t)$ be the solution of the following problem (1-dimensional wave equation on the line).

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R} \\ u_t(x,0) = 0, & x \in \mathbb{R} \end{cases}$$

where

$$f(x) = \begin{cases} e^{\frac{x^2}{x^2-1}}, & |x| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

a) Sketch a graph of $f(x)$, which is the solution at the initial time.

Solution:

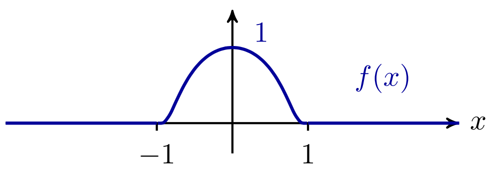
Here are some hints on how to plot this function in the interval $(-1,1)$. First of all we can observe that the function is always positive and never zero (because it's the exponential of something). Moreover we have

$$\lim_{|x|\to 1^-} \frac{x^2}{x^2-1} = -\infty \qquad \Longrightarrow \qquad \lim_{|x|\to 1^-} e^{\frac{x^2}{x^2-1}} = \lim_{t\to-\infty} e^t = 0,$$

so that the function joins continuously with the definition of $f(x)$ outside the interval.
The function is clearly even, so the graph is symmetric with respect the y-axis. One last observation that can help plotting the function quite accurately is either that it is decreasing from 0 to 1 (via derivative), or observing that actually

$$e^{\frac{x^2}{x^2-1}} = e^{\frac{x^2-1+1}{x^2-1}} = e^{\left(1+\frac{1}{x^2-1}\right)} = e \cdot e^{\frac{1}{x^2-1}}$$

and the latter is easier to understand. To summarize:



 Please turn!

b) Sketch a graph of the solution $u(x,t)$ at the time $t = 2$.

Solution:

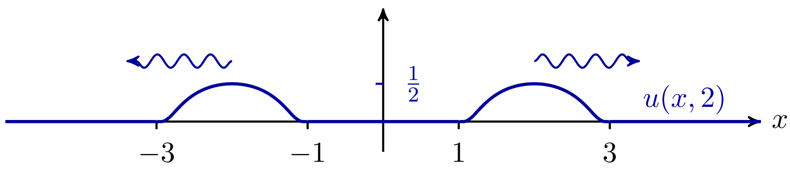
D'Alembert solution of the wave equation is

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

which in this case simplifies to

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right).$$

This means that the initial 'wave' $f(x)$ is split in half and generates two waves, of its same shape (but each of half-height) going in opposite directions.



Interpretation: The wave equation has been discovered by d'Alembert in 1746, as the equation of a vibrating string. As the name suggests, this equation describes the behaviour of many waves: water waves, sound waves, light waves, seismic waves, and so on.
It is perhaps more enlightning to think about it in these terms, for example the Exercise you just solved describes quite well the shape of the waves you create if you let a stone fall in a lake (you should look at the surface of the lake from a side, so you see just a 1-dimensional profile). You can notice a little discrepancy with reality, because in reality usually there is not just one perfect block of two waves going in opposite directions, but also some other turbolences. This is in part due to the fact that actually the initial speed/impulse $g(x)$ is not zero. Adding this contribute to the solution would lead to an even more accurate picture.

c) Prove that, for each fixed $x \in \mathbb{R}$:

$$\lim_{t\to+\infty} u(x,t) = 0.$$

Solution:

If $g(x) = 0$ then

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right).$$

Now for each fixed $x \in \mathbb{R}$ if we let t going to $+\infty$, the points $x \pm t$ will go, respectively, to $\pm\infty$. Therefore

$$\lim_{t\to+\infty} u(x,t) = \frac{1}{2}\left(\lim_{t\to+\infty} f(x+t) + \lim_{t\to+\infty} f(x-t)\right) = \frac{1}{2}\left(\lim_{s\to+\infty} f(s) + \lim_{s\to-\infty} f(s)\right) = 0$$

whenever we start from a function with $\lim\limits_{|x|\to+\infty} f(x) = 0$.

Interpretation: In terms of what said in the previous remark/intepretation, this is telling you that the waves propagate in time, but eventually, in each point they will disappear.

2. Find the solution $u = u(x,t)$ of the 1-dimensional wave equation on the interval $[0, L]$ with the following boundary and initial conditions:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & \\ u(0,t) = 0 = u(L,t), & t \geq 0 \\ u(x,0) = 0, & 0 \leq x \leq L \\ u_t(x,0) = \sin\left(\frac{\pi}{L}x\right), & 0 \leq x \leq L \end{cases}$$

in the following two 'different' ways:

a) Use the formula in the solution via Fourier series.

Solution:

The formula for the solution via Fourier series in this case in which the initial function $f = 0$ looks like:

$$u(x,t) = \sum_{n=1}^{+\infty} B_n^* \sin\left(\frac{cn\pi}{L}t\right) \sin\left(\frac{n\pi}{L}x\right) .$$

To find the coefficients B_n^* we impose the initial condition:

$$u_t(x,0) = \sum_{n=1}^{+\infty} B_n^* \frac{cn\pi}{L} \sin\left(\frac{n\pi}{L}x\right) = \sin\left(\frac{\pi}{L}x\right) \quad \Longrightarrow \quad \begin{cases} B_1^* = \frac{L}{c\pi}, \\ B_{n\geq 2}^* = 0 \,. \end{cases}$$

$$\Longrightarrow \quad \boxed{u(x,t) = \frac{L}{c\pi} \sin\left(\frac{c\pi}{L}t\right) \sin\left(\frac{\pi}{L}x\right)}$$

b) Consider the 2-L periodic extension $u^*(x,t)$ of the solution $u(x,t)$. Then use d'Alembert's formula which in this case ($f = 0$) becomes:

$$u^*(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g^*(s)\, ds$$

where $g^*(s)$ is a priori the $2L$-periodic extension[1] of the initial datum $g(s) = \sin\left(\frac{\pi}{L}s\right)$ to all $s \in \mathbb{R}$.

Solution:

[1] in this case the function is already $2L$-periodic ...

Please turn!

As already pointed out in the text of the exercise in this case the function is already $2L$-periodic, therefore we can use the formula with $g^*(s) = g(s)$ itself:

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin\left(\frac{\pi}{L}s\right)\,ds = -\frac{L}{2c\pi}\left(\cos\left(\frac{\pi}{L}s\right)\right)\Big|_{s=x-ct}^{s=x+ct} =$$
$$= \boxed{\frac{L}{2c\pi}\left(\cos\left(\frac{\pi}{L}(x-ct)\right) - \cos\left(\frac{\pi}{L}(x+ct)\right)\right)}$$

Finally verify that the solutions obtained in the two ways **a)** and **b)** are indeed the same for all $0 \le x \le L$.

Solution:

It's enough to remember the formula:

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$

and apply it in this case with $\alpha = \frac{\pi}{L}x$, $\beta = \frac{c\pi}{L}t$.

3. Let $c > 0$. Consider the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \ge 0, \\ u(x,0) = \frac{1}{c}\left((x^2-2)\sin(x) + 2x\cos(x)\right), & x \in \mathbb{R}, \\ u_t(x,0) = x^2\cos(x), & x \in \mathbb{R}. \end{cases}$$

Find the solution $u(x,t)$. You may use D'Alembert formula.

Solution:

D'Alembert's formula for the solution of the wave equation is:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

With our given initial conditions we get

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\left(\frac{1}{c}\left(((x+ct)^2-2)\sin(x+ct)+2(x+ct)\cos(x+ct)\right)\right) \\
&\quad + \frac{1}{2}\left(\frac{1}{c}\left(((x-ct)^2-2)\sin(x-ct)+2(x-ct)\cos(x-ct)\right)\right) \\
&\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} s^2\cos(s)ds \\
&= \frac{1}{2c}\Big(((x+ct)^2-2)\sin(x+ct)+2(x+ct)\cos(x+ct)\Big) \\
&\quad + \frac{1}{2c}\Big(((x-ct)^2-2)\sin(x-ct)+2(x-ct)\cos(x-ct)\Big) \\
&\quad + \frac{1}{2c}\left((s^2-2)\sin(s)+2s\cos(s)\right)\Bigg|_{x-ct}^{x+ct} \\
&= \frac{1}{2c}\Big(((x+ct)^2-2)\sin(x+ct)+2(x+ct)\cos(x+ct)\Big) \\
&\quad + \frac{1}{2c}\Big(((x-ct)^2-2)\sin(x-ct)+2(x-ct)\cos(x-ct)\Big) \\
&\quad + \frac{1}{2c}\Big(((x+ct)^2-2)\sin(x+ct)+2(x+ct))\cos(x+ct)\Big) \\
&\quad - \frac{1}{2c}\Big(((x-ct)^2-2)\sin(x-ct)+2(x-ct)\cos(x-ct)\Big) \\
&= \frac{1}{c}\Big(((x+ct)^2-2)\sin(x+ct)+2(x+ct)\cos(x+ct)\Big).
\end{aligned}$$

Hence, the solution is

$$\boxed{u(x,t) = \frac{1}{c}\Big(((x+ct)^2-2)\sin(x+ct)+2(x+ct)\cos(x+ct)\Big).}$$

4. Let $c > 0$. Consider the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = e^{-x^2}\sin^2(x) + x, & x \in \mathbb{R} \\ u_t(x,0) = xe^{-x^2}. & x \in \mathbb{R} \end{cases}$$

a) Find the solution $u(x,t)$. You may use D'Alembert formula. [Simplify the expression as much as possible: no unsolved integrals].

Solution:

D'Alembert's formula for the solution of the wave equation is:

$$u(x,t) = \frac{1}{2}\left(f(x+ct)+f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

Please turn!

With our given initial conditions we get

$$u(x,t) = \frac{1}{2}\left(e^{-(x+ct)^2}\sin^2(x+ct) + x + ct + e^{-(x-ct)^2}\sin^2(x-ct) + x - ct\right) +$$

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} se^{-s^2}ds =$$

$$= \frac{1}{2}\left(e^{-(x+ct)^2}\sin^2(x+ct) + e^{-(x-ct)^2}\sin^2(x-ct) + 2x\right) +$$

$$+ \frac{1}{2c}\left(-\frac{1}{2}e^{-s^2}\right)\Bigg|_{x-ct}^{x+ct} =$$

$$= \boxed{\frac{1}{2}\left(e^{-(x+ct)^2}\sin^2(x+ct) + e^{-(x-ct)^2}\sin^2(x-ct) + 2x\right) - \frac{1}{4c}\left(e^{-(x+ct)^2} - e^{-(x-ct)^2}\right).}$$

b) For a fixed $a \in \mathbb{R}$, determine the asymptotic limit

$$\lim_{t\to+\infty} u(a,t).$$

Solution:

Let's observe first that, for a fixed $a \in \mathbb{R}$,

$$\lim_{t\to+\infty} e^{-(a\pm ct)^2} = 0,$$

while clearly the terms $\sin^2(a \pm ct)$ are bounded (by 1).
Therefore, between the 5 addends we have in the solution, only the third will contibute to the limit -with limit a- and we get

$$\boxed{\lim_{t\to+\infty} u(a,t) = a.}$$