D-MAVT
D-MATL

Analysis III

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Solutions Serie 11

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1. Find, via Fourier series, the solution of the 1-dimensional heat equation with the following initial condition:

$$\begin{cases} u_t = 4\, u_{xx}, & x \in [0,1],\ t \geq 0 \\ u(0,t) = u(1,t) = 0, & t \geq 0 \\ u(x,0) = f(x), & x \in [0,1] \end{cases}$$

where

$$f(x) = \sin(\pi x) + \sin(5\pi x) + \sin(10\pi x).$$

Use the method of separation of variables from scratch, showing all the steps.

Solution:

With variables separated $u(x,t) = F(x)G(t)$ the differential equation becomes:

$$F(x)\dot{G}(t) = 4F''(x)G(t),$$

which is convenient to rewrite as

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{4G(t)}$$

because it becomes clear that we are comparing a function of x with a function of t, and the only way that this equality might be true is that both these functions are equal and constant:

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{4G(t)} = k, \qquad k \in \mathbb{R}.$$

The boundary conditions are

$$u(0,t) = F(0)G(t) = 0 \quad \text{and} \quad u(1,t) = F(1)G(t) = 0 \quad \forall t \in [0,+\infty)$$

which in order to be true, excluding the trivial solution $G(t) \equiv 0$, become:

$$F(0) = F(1) = 0.$$

In other words the initial PDE with boundary conditions becomes the system of coupled equations

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(1) = 0, \end{cases} \qquad \text{and} \quad \dot{G}(t) = 4kG(t).$$

We first solve the system for $F(x)$, distinguishing the cases of k positive, zero, or negative. For $k > 0$ the general solution of the ODE is

$$F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x},$$

which is, however, not compatible with the boundary conditions, in the sense that the only solution of this form satisfying the boundary conditions is the trivial solution: $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_1 + C_2 \quad \Leftrightarrow \quad C_2 = -C_1 \quad \Longrightarrow F(x) = C_1 \left(e^{\sqrt{k}x} - e^{-\sqrt{k}x} \right)$$

but then imposing the other condition:

$$0 = F(1) = C_1 \left(e^{\sqrt{k}} - e^{-\sqrt{k}} \right) \quad \Leftrightarrow \quad \begin{array}{l} \text{either } C_1 = 0 \\ \text{or } e^{2\sqrt{k}} = 1 \end{array}$$

which implies $C_1 = 0$ (and consequently $C_2 = -C_1 = 0$) because $2\sqrt{k} \neq 0$ and therefore its exponential is not 1.

For $k = 0$ the general solution is $F(x) = C_1 x + C_2$ which is also not compatible with boundary conditions unless $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_2 \quad \Longrightarrow F(x) = C_1 x$$

and then

$$0 = F(1) = C_1.$$

It remains the case $k < 0$, in which its convenient to write it in the form $k = -p^2$ for positive real number p, and general solutions of $F'' = -p^2 F$ are:

$$F(x) = A\cos(px) + B\sin(px).$$

$F(0) = 0$ if and only if $A = 0$. $F(1) = 0$ if and only if $B\sin(p) = 0$, so if we want nontrivial solutions $B \neq 0$, we need to have

$$p = n\pi$$

for some integer $n \geq 1$. Conclusion: we have a nontrivial solution for each $n \geq 1$, $k = k_n = -n^2\pi^2$:

$$F_n(x) = B_n \sin(n\pi x)$$

The corresponding equation for $G(t)$ is

$$\dot{G} = -4n^2\pi^2 G$$

which has general solution

$$G_n(t) = C_n e^{-4n^2\pi^2 t}$$

The conclusion is that for every $n \geq 1$ we have a solution

$$u_n(x,t) = F_n(x)G_n(t) = B_n \sin(n\pi x) e^{-4n^2\pi^2 t}$$

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and by the superposition principle:

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x) e^{-4n^2\pi^2 t}$$

where the coefficients B_n are determined by the initial condition

$$f(x) = u(x,0) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x).$$

This case is particularly easy because $f(x)$ is already expressed as a linear combination of these functions and there is no need to compute any integral to get

$$B_n = \begin{cases} 1, & n = 1, 5, 10 \\ 0, & \text{otherwise.} \end{cases}$$

Finally, the solution will be

$$u(x,t) = \sin(\pi x) e^{-4\pi^2 t} + \sin(5\pi x) e^{-100\pi^2 t} + \sin(10\pi x) e^{-400\pi^2 t}$$

2. An aluminium bar of length $L = 1(m)$ has thermal diffusivity of (around)[1]

$$c^2 = 0.0001 \left(\frac{\text{m}^2}{\text{sec}}\right) = 10^{-4} \left(\frac{\text{m}^2}{\text{sec}}\right).$$

It has initial temperature given by $u(x,0) = f(x) = 100\sin(\pi x)$ (°C), and its ends are kept at a constant 0°C temperature. Find the first time t^* for which the whole bar will have temperature ≤ 30°C.
In mathematical terms, solve

$$\begin{cases} u_t = 10^{-4} u_{xx}, \\ u(0,t) = u(1,t) = 0, & t \geq 0 \\ u(x,0) = 100\sin(\pi x), & 0 \leq x \leq 1. \end{cases}$$

and find the smallest t^* for which

$$\max_{x\in[0,1]} u(x,t^*) \leq 30.$$

You can use the formula from the lecture notes.

Solution:

The parameters are length $L = 1$, thermal diffusivity $c^2 = 10^{-4}$ and consequently

$$\lambda_n^2 = \frac{c^2 n^2 \pi^2}{L^2} = 10^{-4} n^2 \pi^2.$$

[1] we are approximating the standard value which would be $c^2 \approx 0.000097\text{m}^2/\text{sec}$ to make computations easier.

The solution is

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x) e^{-\lambda_n^2 t}$$

and

$$f(x) = u(x,0) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x)$$

so that the only nontrivial coefficient will be $B_1 = 100$. The solution is explicitely given by

$$u(x,t) = 100 \sin(\pi x) e^{-10^{-4}\pi^2 t}.$$

For each fixed time $t \geq 0$, it is a multiple of $\sin(\pi x)$, therefore its maximum will be reached in $x = 1/2$ with value

$$M_t := \max_{x\in[0,1]} u(x,t) = u\left(\frac{1}{2}, t\right) = 100 \sin\left(\frac{\pi}{2}\right) e^{-10^{-4}\pi^2 t} = 100 e^{-10^{-4}\pi^2 t}.$$

This is a decreasing function of t, so that the required value t^* for which the bar will have temperature $\leq 30°\text{C}$ is given by imposing

$$M_{t^*} = 30 \quad \Leftrightarrow \quad 100 e^{-10^{-4}\pi^2 t^*} = 30 \quad \Leftrightarrow \quad t^* = \frac{10^4}{\pi^2} \ln\left(\frac{10}{3}\right)$$
$$\left(\approx 1219.88 \text{ sec} = 20 \text{ min } 19.88 \text{ sec}\right)$$

3. Consider the following time-dependent version of the heat equation on the interval $[0, L]$, in which the constant varies linearly with time. We also impose boundary conditions and we look for solutions:

$$u = u(x,t) \quad \text{s.t.} \quad \begin{cases} u_t = 2tc^2 u_{xx}, & x \in [0,L], t \in [0,+\infty) \\ u(0,t) = 0, & t \in [0,+\infty) \\ u(L,t) = 0, & t \in [0,+\infty) \end{cases}$$

Find all possible solutions of the specific form $u(x,t) = F(x)G(t)$.

Solution:

The differential equation becomes:

$$F(x)\dot{G}(t) = 2tc^2 F''(x) G(t),$$

which is convenient to rewrite as

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{2tc^2 G(t)}$$

because it becomes clear that we are comparing a function of x with a function of t, and the only way that this equality might be true is that both these functions are equal and constant:

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{2tc^2 G(t)} = k, \qquad k \in \mathbb{R}.$$

The boundary conditions are

$$u(0,t) = F(0)G(t) = 0 \quad \text{and} \quad u(L,t) = F(L)G(t) = 0 \quad \forall t \in [0, +\infty)$$

which in order to be true, excluding the trivial solution $G(t) \equiv 0$, become:

$$F(0) = F(L) = 0.$$

In other words the initial PDE with boundary conditions becomes the system of coupled equations

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(L) = 0, \end{cases} \qquad \text{and} \quad \dot{G}(t) = 2tkc^2 G(t).$$

We first solve the system for $F(x)$, distinguishing the cases of k positive, zero, or negative. For $k > 0$ the general solution of the ODE is

$$F(x) = C_1 \mathrm{e}^{\sqrt{k}x} + C_2 \mathrm{e}^{-\sqrt{k}x},$$

which is, however, not compatible with the boundary conditions, in the sense that the only solution of this form satisfying the boundary conditions is the trivial solution: $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_1 + C_2 \quad \Leftrightarrow \quad C_2 = -C_1 \quad \Longrightarrow F(x) = C_1 \left(\mathrm{e}^{\sqrt{k}x} - \mathrm{e}^{-\sqrt{k}x} \right)$$

but then imposing the other condition:

$$0 = F(L) = C_1 \left(\mathrm{e}^{\sqrt{k}L} - \mathrm{e}^{-\sqrt{k}L} \right) \quad \Leftrightarrow \quad \begin{array}{l} \text{either } C_1 = 0 \\ \text{or } \mathrm{e}^{2\sqrt{k}L} = 1 \end{array}$$

which implies $C_1 = 0$ (and consequently $C_2 = -C_1 = 0$) because $2\sqrt{k}L \neq 0$ and therefore its exponential is not 1.

For $k = 0$ the general solution is $F(x) = C_1 x + C_2$ which is also not compatible with boundary conditions unless $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_2 \quad \Longrightarrow F(x) = C_1 x$$

and then

$$0 = F(L) = C_1 L \quad \Leftrightarrow \quad C_1 = 0.$$

It remains the case $k < 0$, in which its convenient to write it in the form $k = -p^2$ for positive real number p, and general solutions of $F'' = -p^2 F$ are:

$$F(x) = A\cos(px) + B\sin(px).$$

We impose the boundary conditions:

$$0 = F(0) = A \qquad \Longrightarrow F(x) = B\sin(px)$$

and

$$0 = F(L) = B\sin(pL) \quad \overset{(\text{if } B\neq 0)}{\Leftrightarrow} \quad pL = n\pi, \quad n \in \mathbb{Z}_{\geq 1}$$

<u>Conclusion:</u> we have a nontrivial solution for each $n \geq 1$, $k = k_n = -\frac{n^2\pi^2}{L^2}$:

$$F_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right).$$

The corresponding equation for $G(t)$ is

$$\dot{G} = -2t\frac{n^2\pi^2c^2}{L^2}G$$

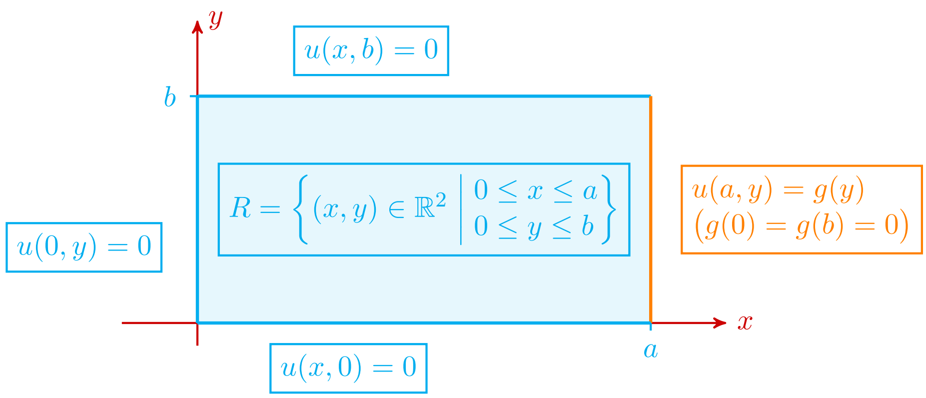
which has general solution

$$G_n(t) = C_n \mathrm{e}^{-\frac{n^2\pi^2c^2}{L^2}t^2}.$$

The conclusion is that for every $n \geq 1$ we have a solution

$$u_n(x,t) = F_n(x)G_n(t) = A_n\mathrm{e}^{-\frac{n^2\pi^2c^2}{L^2}t^2}\sin\left(\frac{n\pi}{L}x\right), \qquad A_n \in \mathbb{R}.$$

4. Adapt the method used to solve the previous Laplace equation in the case in which the only nontrivial initial boundary condition is on the right vertical segment of the rectangle



$$\begin{cases} \Delta u = 0, & (x,y) \in R \\ u(x,0) = u(x,b) = 0, & 0 \leq x \leq a \\ u(0,y) = 0, & 0 \leq y \leq b \\ u(a,y) = g(y), & 0 \leq y \leq b \end{cases}$$

where $g(y)$ is any function with prescribed boundary conditions

$$g(0) = g(b) = 0.$$

Solution:

We just have to make a few changes from the way the equation was solved in the lecture notes. To solve the differential equation $\Delta u = 0$ by separation of variables

$$u(x,y) = F(x)G(y)$$

we still have to impose for some $k \in \mathbb{R}$:

$$\begin{cases} F'' = -kF \\ G'' = kG. \end{cases}$$

We first impose the boundary conditions $u(x,0) = u(x,b) = 0$, which translate into $G(0) = G(b) = 0$. To have nontrivial solutions, we must have $k < 0$. With this condition we solve

$$\begin{cases} G'' = kG \\ G(0) = G(b) = 0 \end{cases} \Leftrightarrow \begin{cases} G(y) = A\cos\left(\sqrt{-k}y\right) + B\sin\left(\sqrt{-k}y\right) \\ G(0) = G(b) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (G(0)=0) & A = 0 \\ (G(b)=0) & \sqrt{-k}b = n\pi \ (n \in \mathbb{Z}_{\geq 1}) \end{cases} \rightsquigarrow G_n(y) = B_n \sin\left(\frac{n\pi}{b}y\right), \quad n \geq 1$$

For these admissible values we found

$$\sqrt{-k} = \frac{n\pi}{b} \quad \rightsquigarrow \quad k = -\left(\frac{n\pi}{b}\right)^2$$

we have solutions of the other differential equation $F'' = -kF$

$$F_n(x) = A_n^* e^{\frac{n\pi}{b}x} + B_n^* e^{-\frac{n\pi}{b}x}$$

and imposing the boundary condition $u(0,y) = 0$ we have $F_n(0) = 0$, that is

$$F_n(x) = 2A_n^* \sinh\left(\frac{n\pi}{b}x\right).$$

Renaming the product of the constants $A_n := B_n \cdot 2A_n^*$ we get

$$u_n(x,y) = F_n(x)G_n(y) = A_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right),$$

and by the superposition principle

$$u(x,y) = \sum_{n=1}^{+\infty} u_n(x,y) = \sum_{n=1}^{+\infty} A_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

is also a solution. We now only have to impose the last boundary condition $u(a,y) = g(y)$ which translates into

$$g(y) = \sum_{n=1}^{+\infty}\left[A_n \sinh\left(\frac{n\pi}{b}a\right)\right]\sin\left(\frac{n\pi}{b}y\right)$$

so that the expressions in the square brackets must be the coefficients of the odd, $2b$-periodic extension of $g(y)$, or equivalently

$$A_n = \frac{2}{b\sinh\left(\frac{n\pi}{b}a\right)} \int_0^b g(y)\sin\left(\frac{n\pi}{b}y\right)\, dy.$$